Linear and Quasi-linear Evolution Equations in Hilbert Spaces

Pascal Cherrier
Albert Milani

Graduate Studies in Mathematics
Volume 135

American Mathematical Society
Linear and Quasi-linear Evolution Equations in Hilbert Spaces
We dedicate this work to our wives,
Annick and Claudia,
whose love and support has sustained us throughout its redaction.

We bow with respect to the memory of our Teachers,
Thierry Aubin and Tosio Kato,
who have been a continuous source of inspiration and dedication.
Magna non sine Difficultate
# Contents

Preface ix

Chapter 1. Functional Framework 1
  §1.1. Basic Notation 1
  §1.2. Functional Analysis Results 4
  §1.3. Hölder Spaces 7
  §1.4. Lebesgue Spaces 9
  §1.5. Sobolev Spaces 13
  §1.6. Orthogonal Bases in $H^m(\mathbb{R}^N)$ 51
  §1.7. Sobolev Spaces Involving Time 60

Chapter 2. Linear Equations 77
  §2.1. Introduction 77
  §2.2. The Hyperbolic Cauchy Problem 78
  §2.3. Proof of Theorem 2.2.1 81
  §2.4. Weak Solutions 104
  §2.5. The Parabolic Cauchy Problem 107

Chapter 3. Quasi-linear Equations 119
  §3.1. Introduction 119
  §3.2. The Hyperbolic Cauchy Problem 122
  §3.3. Proof of Theorem 3.2.1 131
  §3.4. The Parabolic Cauchy Problem 145
Chapter 4. Global Existence

§4.1. Introduction 153
§4.2. Life Span of Solutions 155
§4.3. Non Dissipative Finite Time Blow-Up 159
§4.4. Almost Global Existence 171
§4.5. Global Existence for Dissipative Equations 175
§4.6. The Parabolic Problem 214

Chapter 5. Asymptotic Behavior

§5.1. Introduction 233
§5.2. Convergence $u^{\text{hyp}}(t) \to u^{\text{sta}}$ 234
§5.3. Convergence $u^{\text{par}}(t) \to u^{\text{sta}}$ 241
§5.4. Stability Estimates 244
§5.5. The Diffusion Phenomenon 278

Chapter 6. Singular Convergence

§6.1. Introduction 293
§6.2. An Example from ODEs 295
§6.3. Uniformly Local and Global Existence 301
§6.4. Singular Perturbation 305
§6.5. Almost Global Existence 326

Chapter 7. Maxwell and von Karman Equations

§7.1. Maxwell’s Equations 335
§7.2. von Karman’s Equations 343

List of Function Spaces 361
Bibliography 365
Index 375
1. In these notes we develop a theory of strong solutions to linear evolution equations of the type

\[(0.0.1) \quad \varepsilon u_{tt} + \sigma u_t - a_{ij}(t, x) \partial_i \partial_j u = f(t, x),\]

and their quasi-linear counterpart

\[(0.0.2) \quad \varepsilon u_{tt} + \sigma u_t - a_{ij}(t, x, u, u_t, \nabla u) \partial_i \partial_j u = f(t, x).\]

In (0.0.1) and (0.0.2), \(\varepsilon\) and \(\sigma\) are non-negative parameters; \(u = u(t, x)\), \(t > 0, x \in \mathbb{R}^N\), and summation over repeated indices \(i, j, 1 \leq i, j \leq N\), is understood. In addition, and in a sense to be made more precise, the quadratic form \(\mathbb{R}^N \ni \xi \mapsto \rightarrow a_{ij}(\cdots) \xi^i \xi^j\) is positive definite.

We distinguish the following three cases.

(1) \(\varepsilon > 0\) and \(\sigma = 0\). Then, (0.0.1) and (0.0.2) are hyperbolic equations; in particular, when \(\varepsilon = 1\), they reduce to

\[(0.0.3) \quad u_{tt} - a_{ij} \partial_i \partial_j u = f,\]

and when \(a_{ij}(\cdots) = \delta_{ij}\) (the so-called Kronecker \(\delta\), defined by \(\delta_{ij} = 0\) if \(i \neq j\), and \(\delta_{ij} = 1\) if \(i = j\)), (0.0.3) further reduces to the classical wave equation

\[(0.0.4) \quad u_{tt} - \Delta u = f.\]

(2) \(\varepsilon = 0\) and \(\sigma > 0\). Then, (0.0.1) and (0.0.2) are parabolic equations; in particular, when \(\sigma = 1\), they reduce to

\[(0.0.5) \quad u_t - a_{ij} \partial_i \partial_j u = f,\]
and when $a_{ij} = \delta_{ij}$, (0.0.5) further reduces to the classical heat equation

\[ (0.0.6) \quad u_t - \Delta u = f. \]

(3) $\varepsilon > 0$ and $\sigma > 0$. Then, (0.0.1) and (0.0.2) are dissipative hyperbolic equations; in particular, when $\varepsilon = \sigma = 1$, they reduce to

\[ (0.0.7) \quad u_{tt} + u_t - a_{ij} \partial_i \partial_j u = f, \]

and when $a_{ij} = \delta_{ij}$, (0.0.7) further reduces to the so-called telegraph equation

\[ (0.0.8) \quad u_{tt} + u_t - \Delta u = f. \]

We prescribe that $u$ should satisfy the initial conditions (or Cauchy data)

\[ (0.0.9) \quad u(0, x) = u_0(x), \quad \varepsilon u_t(0, x) = \varepsilon u_1(x), \]

where $u_0$ and $u_1$ are given functions on $\mathbb{R}^N$, and the second condition is vacuous if $\varepsilon = 0$ (that is, in the parabolic case we only prescribe the initial condition $u(0, x) = u_0(x)$).

Our purpose is to show that the Cauchy problems (0.0.1) + (0.0.9) and (0.0.2) + (0.0.9) are solvable in a suitable class of Sobolev spaces; we call the corresponding solutions strong. By this, we mean that the solutions we seek should be functions $t \mapsto u(t)$, which are valued in a Sobolev space $H^r := H^r(\mathbb{R}^N)$, and possess a sufficient number of derivatives, either classical or distributional, so that equations (0.0.1) and (0.0.2) hold for (almost) all $t$ and all $x$. More precisely, when $\varepsilon > 0$ we seek for solutions of (0.0.1) and (0.0.2) in the space

\[ (0.0.10) \quad C^0([0, T]; H^{s+1}) \cap C^1([0, T]; H^s) \cap C^2([0, T]; H^{s-1}), \]

for some $T > 0$, where $s \in \mathbb{N}$ is such that $s > \frac{N}{2} + 1$; this condition implies that strong solutions are also classical. When $\varepsilon = 0$, we seek instead for solutions of (0.0.1) and (0.0.2) in the space

\[ (0.0.11) \quad \{ u \in C([0, T]; H^{s+1}) \mid u_t \in L^2(0, T; H^s) \}. \]

In addition, we want to show that the Cauchy problems (0.0.1) + (0.0.9) and (0.0.2) + (0.0.9) are well-posed, in Hadamard’s sense, in these spaces; that is, that their solutions should be unique and depend continuously on their data $f$, $u_0$ and $u_1$ (of course, the latter only for $\varepsilon > 0$). Finally, we also consider equations with lower order terms, i.e.,

\[ (0.0.12) \quad \varepsilon u_{tt} + \sigma u_t - a_{ij} \partial_i \partial_j u = f + b_i \partial_i u + c u, \]

in particular in the linear case, as well as equations in the divergence form

\[ (0.0.13) \quad \varepsilon u_{tt} + \sigma u_t - \partial_j (a_{ij} \partial_i u) = f + b_i \partial_i u + c u. \]
In the quasi-linear case, equations (0.0.2) will in general have only local solutions; that is, even if the source term $f$ is defined on a given interval $[0, T]$, or on all of $[0, +\infty[$, the solution will be defined only on some interval $[0, \tau]$, with $\tau < T$, and cannot be extended to all of $[0, T]$.

2. Our main goal is to develop a unified treatment of equations (0.0.1) and (0.0.2), both in the hyperbolic (either dissipative, or not) and the parabolic case, following a common constructive method to solve either problem. In the linear case, of course, a unified theory for both hyperbolic and parabolic equations (0.0.1), in a suitable framework of Hilbert spaces, has been presented by Lions and Magenes in their three-volume treatise [101, 102, 103], where they introduced a variety of arguments and techniques to solve fairly general kinds of initial-boundary value problems. The main reason we seek a unified treatment of equations (0.0.1) and (0.0.2) in the quasi-linear case is that this allows us to compare the solutions to the hyperbolic and the parabolic equations, in a number of ways. In particular, when (0.0.2) admits global solutions (that is, defined on all of $[0, +\infty[$), we wish to study their asymptotic behavior as $t \to +\infty$. We assume that the coefficients $a_{ij}$ in (0.0.2) depend only on the first-order derivatives $u_t$ and $\nabla u$, and are interested in the following questions. The first is that of the convergence of the solutions of (0.0.2) to the solution of the stationary equation

$$(0.0.14) \quad -a_{ij}(0, \nabla v) \partial_i \partial_j v = h.$$ 

The second, when $\varepsilon$ and $\sigma > 0$, is the comparison of the asymptotic profiles of the solutions of the dissipative hyperbolic equation (0.0.2) to those of the solutions of the parabolic equation, corresponding to $\varepsilon = 0$. The third question, related to (0.0.2), is the singular perturbation problem, concerning the convergence, as $\varepsilon \to 0$, of solutions $u^\varepsilon$ of the dissipative hyperbolic equation to the solution $u^0$ of the parabolic equation.

3. Linear hyperbolic equations of the type (0.0.12) and (0.0.13), in particular when $\sigma = 0$, have been studied by many authors, who have considered the corresponding Cauchy problem in different settings. An elementary introduction to both kinds of equations can be found in Evans’ textbook [47]; for more advanced and specific results, renouncing to any pretense of a comprehensive list, we refer, e.g., to Friedrichs [51], Kato [72], Mizohata [122], and Ikawa [63], who resort to a solution method based on a semigroup approach, complicated by the fact that the coefficients $a_{ij}$ depend on $t$. The semigroup method has later been successfully applied to quasi-linear equations; see, e.g., Okazawa [130], Tanaka [153, 154, 155], and, for a more abstract approach, Beyer [15]. Other methods can be seen, e.g., in Racke [136], and Sogge [151], based respectively on the Cauchy-Kovalevskaya and the Hahn-Banach theorems. In the solution theory we present in Chapter
we prefer to follow the so-called Faedo-Galerkin method, which is a generalization of the method of separation of variables, and which explicitly constructs the solution to (0.0.12) as the limit of a sequence of functions, each of which solves an approximate version of the problem, determined by its projection onto suitable finite-dimensional subspaces. The results we establish for (0.0.1) when \( \varepsilon > 0 \) are not specifically dependent on the fact that the equation is hyperbolic; in fact, the Faedo-Galerkin method can be readily adapted to obtain strong solutions of the linear parabolic equation. For general references to parabolic equations, both linear and quasi-linear, we refer, e.g., to Ladyzenshkaya, Solonnikov and Uraltseva [86], Amann [6], Pao [131], Lunardi [107], Lieberman [96], and Krylov [83, 84], where these equations are mostly studied in the Hölder spaces \( C^{m+\alpha/2,2m+\alpha}(\bar{Q}) \). For the numerical treatment of equation (2.1.1), we refer, e.g., to Meister and Struckmeier [112].

4. The Cauchy problem for quasi-linear hyperbolic equations such as (0.0.2), as well as their counterpart in divergence form (0.0.13), has been studied by many authors, who have provided local (and, when possible, global, or, at least, almost global) solutions with a number of methods, including a nonlinear version of the Galerkin scheme and various versions of the Moser-Nash algorithm. Renouncing again to any pretense of a comprehensive list, we refer, e.g., to the classical treatise by Courant and Hilbert [38], as well as the more recent works by Kato [72], John [66], Kichenasamy [74], Racke [136], Sogge [151], Hörmander [57], as well as Lax [88], Li Ta-Tsien [91], and Li Ta-Tsien and Wang Li-Ping [93]. In the above context, local solution means a solution defined on some interval \([0, \tau]\); almost global solution means a solution defined on a prescribed interval \([0, T]\), of finite but arbitrary length, possibly subject to some restrictions on the size of the data, depending on \( T \); global solution means a solution also defined on arbitrary intervals \([0, T]\), but with restrictions on the size of the data, if any, independent of \( T \) (thus, these solutions are defined on the entire interval \([0, +\infty]\). Finally, we also consider global bounded solutions; that is, global solutions which remain bounded as \( t \to +\infty \). In Chapter 3, we present a solution method based on a linearization and fixed point method, introduced by Kato [70, 71, 72], in which we apply the results for the linear theory, developed in Chapter 2. As for the linear case, the results we establish for (0.0.2), when \( \varepsilon > 0 \), are not specifically dependent on the fact that the equation is hyperbolic; in fact, the linearization and fixed point method can be adapted to obtain local, strong solutions of the quasi-linear parabolic equation

\[
(0.0.15) \quad u_t - a_{ij}(t, x, u, \nabla u) \partial_i \partial_j u = f,
\]
as well as of the analogous equation in divergence form. Moreover, these methods can also be applied to other types of evolution equations, such as the so-called dispersive equations considered in Tao [156], and Linares and Ponce [97]; these include, among others, the Schrödinger and the Korteweg-de-Vries equations.

5. Not surprisingly, many more results are available on semi-linear hyperbolic and parabolic equations; that is, equations of the form

\begin{equation}
  u_{tt} - \Delta u = f(t, x, u, Du),
\end{equation}

\begin{equation}
  u_t - \Delta u = f(t, x, u, \nabla u).
\end{equation}

Among the many works on this subject, we limit ourselves to cite Strauss [150], Todorova and Yordanov [159], Zheng [168], Quittner and Souplet [133], Cazenave and Haraux [24], and the references therein. Most of these results concern the well-posedness of the Cauchy problem for (0.0.16) or (0.0.17) in a suitable weak sense; strong solutions are then obtained by appropriate regularity theorems, and the asymptotic behavior of such weak solutions can be studied in terms of suitable attracting sets in the phase space; see, e.g., Milani and Koksch, [119]. In fact, we could try to develop a corresponding weak solution theory for hyperbolic and parabolic quasi-linear equations in the conservation form

\begin{equation}
  \varepsilon u_{tt} + \sigma u_t - \text{div}[a(\nabla u)] = f,
\end{equation}

where $a : \mathbb{R}^N \to \mathbb{R}^N$ is monotone. However, there appears to be a striking difference between the hyperbolic and the parabolic situation. For the latter, i.e., when $\varepsilon = 0$ in (0.0.18), existence, uniqueness and well-posedness results for weak solutions, at least when $a$ is strongly monotone, are available; see, e.g., Lions [99, ch. 2, §1], and Brézis [19]. In contrast, when $\varepsilon > 0$ the question of the existence of even a local weak solution to equation (0.0.18) (that is, in the space (0.0.10) with $s = 0$) is, as far as we know, totally open (unless, of course, $a$ is linear).

6. To our knowledge, there are not yet satisfactory answers to the question of finding sharp life-span estimates for problem (0.0.2), at least in the functional framework we consider. On the other hand, rather precise results have long been available, at least for more regular solutions of the homogeneous equation; that is, when $f \equiv 0$ and $u_0 \in H^{s+1} \cap W^{r+1,p}$, $u_1 \in H^s \cap W^{r,p}$, for suitable integers $s \gg \frac{N}{2} + 1$, $r < s$, and $p \in ]1, 2[$. In this case, the situation also depends on the space dimension $N$; more precisely, one obtains global existence of strong solutions if $N \geq 4$, and also if $N = 3$ if the nonlinearity satisfies an additional structural restriction, known as the null condition. The proof of these results is based on supplementing the direct energy estimates used to establish local solutions, with rather
refined decay estimates of the solution to the linear wave equation. We refer to Racke [136], and John [66], for a comprehensive survey of the results of, among many others, John [65], Klainerman [75, 76], Klainerman and Ponce [81], as well as Klainerman [77] and Christodoulou [31], for the null condition.

7. The theory of quasi-linear evolution equations has many important applications. A non exhaustive list would include fluid dynamics (see, e.g., Majda [108], and Nishida [126]); general relativity, and, specifically, the so-called Einstein vacuum equations (Klainerman and Christodoulou [78], and Klainerman and Nicolò [80]); wave maps (Shatah and Struwe [145], and Tao [156]); von Karman type thin plate equations (Cherrier and Milani [26, 27, 29], Chuesov and Lasiecka [33]); control and observability theory (Li [94]). Other applications, specifically of dissipative hyperbolic equations (0.0.2) with $\sigma > 0$ and $\varepsilon$ small, include models of heat equations with delay (Li [92], Cattaneo [21], Jordan, Dai and Mickens [67], Liu [104]), where $\varepsilon$ is a measure of the delay or heat relaxation time; Maxwell’s equations in ferromagnetic materials (Milani [114]), where $\varepsilon$ is a measure of the displacement currents, usually negligible; simple models of laser optic equations (Haus [54]), where $\varepsilon$ is related to measures of low frequencies of the electromagnetic field; traffic flow models (Schochet [140]), where $\varepsilon$ is a measure of the drivers’ response time to sudden disturbances (which, one hopes, should be small); models of random walk systems (Hadeler [53]), where $\varepsilon$ is related to the reciprocal of the turning rates of the moving particles; construction materials with strong internal stress-strain relations, measured by parameters related to the reciprocal of $\varepsilon$ (see, e.g., Banks et al. [9, 10] for the case of a one-dimensional elastomer); and models of time-delayed information propagation in economics (Ahmed and Abdusalam [2]).

8. These notes have their origin in a series of graduate courses and seminars we gave at Fudan University, Shanghai, at the Université Pierre et Marie Curie (Paris VI), the Technische Universität Dresden, and the Pontificia Universidad Católica of Santiago, Chile. Some of the material we cover is relatively well known, although many results, in particular on hyperbolic equations, seem to be somewhat scattered in the literature, and often subordinate to other topics or applications. Other results, in particular on the diffusion phenomenon for quasi-linear hyperbolic waves, appear to be new. Our intention is, in part, to provide an introduction to the theory of quasi-linear evolution equations in Sobolev spaces, organizing the material in a progression that is as gradual and natural as possible. To this end, we have tried to put particular care in giving detailed proofs of the results we present; thus, if successful, our effort should give readers the necessary basis to proceed to the more specialized texts we have indicated above. In this sense, these notes are not meant to serve as an advanced PDEs textbook;
rather, their didactical scope and subject range is restricted to the effort of explaining, as clearly as we are able to, one possible way to study two simple and fundamental examples of evolution equations (hyperbolic, both dissipative or not, and parabolic) on the whole space $\mathbb{R}^N$. In addition, we also hope that these notes may serve as a fairly comprehensive and self-contained reference for researchers in other areas of applied mathematics and sciences, in which, as we have mentioned, the theory of quasi-linear evolution equations has many important applications. We should perhaps mention explicitly the fact that, given the introductory level of these notes, we have limited ourselves to present only those results that can be obtained by resorting to one of the most standard methods for the study of equations (0.0.1) and (0.0.2); namely, that of the a priori, or energy, estimates. Of course, this choice forces us to neglect other methods that are more specific to the type of equation under consideration and which are extensively studied in more specialized texts. For example, we do not cover, but only mention, the theory of Hölder solutions of parabolic quasi-linear equations (0.0.15) (see, e.g., Krylov [83]), or the theory of weak solutions to quasi-linear first-order hyperbolic systems of conservation laws, as in (3.1.8) of Chapter 3 (see, e.g., Alinhac [5], or Serre [143, 144]), and we do not even mention other very specific and highly refined techniques which have been developed and are being developed for the study of these equations, such as, to cite a few, the theory of nonlinear semigroup (see, e.g., Beyer [15]), the methods of pseudo-differential operators (see, e.g., Taylor [157]) and of microlocal analysis (see, e.g., Bony [17]). On the other hand, one can perhaps be surprised by the extent of the results one can obtain, by means of the one and same technique; that is, the energy method. As we have stated, this method has, among others, the advantage of allowing us to present our results in a highly unified way, and to show that, even today, classical analysis allows us to deal in a simple way, by means of standard and well-tested techniques, with relevant questions in the theory of PDEs of evolution, which are still the subject of considerable study.

9. The material of these notes is organized as follows. In Chapter 1 we provide a summary of the main functional analysis results we need for the development of the theory we wish to present. In Chapter 2 we develop a strong solution theory for the Cauchy problem for the linear equation (0.0.12), with existence, uniqueness, regularity, and well-posedness results for both the hyperbolic equation ($\varepsilon = 1$, $\sigma = 0$) and the parabolic one ($\varepsilon = 0$, $\sigma = 1$). In Chapter 3 we construct local in time solutions to the quasi-linear equations (0.0.2) and (0.0.15), by means of a linearization and fixed-point technique, in which we apply the results on the linear equations we established in the previous Chapter. Again, we give existence, uniqueness, regularity, and well-posedness results for both types of equation. In
Chapter 4 we study the question of the extendibility of these local solutions to either a prefixed finite but arbitrary time interval \([0, T]\) (almost global and global existence), or to the whole interval \([0, +\infty]\) (global and global bounded existence). We present an explicit example of blow-up in finite time for solutions of the quasi-linear equation \((0.0.3)\) in one dimension of space, as well as some global and almost global existence results for either equation, when the data \(u_0, u_1\) and \(f\) are sufficiently small. We also present a global existence result for the parabolic equation \((0.0.15)\), for data of arbitrary size. In Chapter 5 we consider the asymptotic behavior, as \(t \to +\infty\), of global, bounded, small solutions of \((0.0.2)\), both dissipative hyperbolic \((\varepsilon = 1)\) and parabolic \((\varepsilon = 0)\), and we prove some results on their convergence to the solution of the stationary equation \((0.0.14)\). In the homogeneous case \(f \equiv 0\), we also establish some stability estimates, on the rate of decay to 0 of the corresponding solutions. We also give a result on the diffusion phenomenon, which consists in showing that, when \(f \equiv 0\), solutions of the hyperbolic equation \((0.0.7)\) (both linear and quasi-linear) asymptotically behave as those of the parabolic equation \((0.0.5)\) of corresponding type. In Chapter 6 we consider a second way in which we can compare the hyperbolic and the parabolic problems; namely, we consider \((0.0.2)\) as a perturbation, for small values of \(\varepsilon > 0\), of the parabolic equation \((0.0.2)\), with \(\varepsilon = 0\). Denoting by \(u^\varepsilon\) and \(u^0\) the corresponding solutions, we study the problem of the convergence \(u^\varepsilon \to u^0\) as \(\varepsilon \to 0\), on compact time intervals. We consider either intervals \([0, T]\) or \([\tau, T]\), \(\tau \in ]0, T[\); that is, including \(t = 0\) or not. In the former case, the convergence is singular, due to the loss of the initial condition on \(u_t\), and we give rather precise estimates, as \(t\) and \(\varepsilon \to 0\), on the corresponding initial layer. We mention in passing that the estimates we establish on the difference \(u^\varepsilon - u^0\) allow us also to deduce a global existence equivalency result between the two types of equations, in the sense that a global solution to the parabolic equation, corresponding to data of arbitrary size, exists, if and only if global solutions to the dissipative hyperbolic equation also exist, corresponding to data of arbitrary size, and \(\varepsilon\) is sufficiently small. We conclude the chapter with a global result for equation \((0.0.2)\), with data of arbitrary size, when \(\varepsilon\) is sufficiently large. Lastly, in Chapter 7, we present two applications of the theory developed in the previous chapters. In the first example, we consider a model for the complete system of Maxwell’s equations, in which the use of suitable electromagnetic potentials allows us to translate the first-order Maxwell’s system into a second-order evolution equation of the type \((0.0.2)\). In this model, the parameters \(\varepsilon\) and \(\sigma\) can be interpreted as a measure, respectively, of the displacement and the eddy currents; in some situations, such as when the equations are considered in a ferro-magnetic medium, displacement currents are negligible with respect to the eddy ones, and this observation leads to
the question of the control of the error introduced in the model when the term $\varepsilon u_{tt}$ is neglected. In related situations, one is interested in periodic phenomena, with relatively low frequencies, thus leading to the question of the existence of solutions on the whole period of time. It is our hope that these questions may be addressed, at least to some extent, by the results of the previous chapters. In the second example, we consider two systems of evolution equations, of hyperbolic and parabolic type, relative to a highly nonlinear elliptic system of von Karman type equations on $\mathbb{R}^{2m}$, $m \geq 2$. These equations generalize the well-known equations of the same name in the theory of elasticity, which correspond to the case $m = 1$, and model the deformation of a thin plate due to both internal and external stresses. For both types of systems we show the existence and uniqueness of local in time strong solutions, which again can be extended to almost global ones if the initial data are small enough. Even though these systems do not fit exactly in the framework of second-order evolution equations for which our theory is developed, their study allows us to show that the unified methods we present can be applied to a much wider class of equations than those of the form (0.0.2).

10. Finally, we mention that an analogous unified theory could be constructed for initial-boundary value problems for equation (0.0.2), in a subdomain $\Omega \subset \mathbb{R}^N$, with $u$ subject to appropriate conditions at the boundary $\partial \Omega$ of $\Omega$, assumed to be adequately smooth. The type of results one obtains is qualitatively analogous, but in a different functional setting for both the data and the solutions. Indeed, the data have to satisfy a number of so-called compatibility conditions at $\{t = 0\} \times \partial \Omega$, which are different in the hyperbolic and parabolic cases, and the integrations by parts that are usually carried out in order to establish the necessary energy estimates (see Chapter 2) would involve boundary terms that do not appear when $\Omega = \mathbb{R}^N$. For example, in our papers $[116, 117]$ we considered the simple case where equation (0.0.2) is studied in a bounded domain, with homogeneous Dirichlet boundary conditions; other results can be found, e.g., in Dafermos and Hrusa [40]. To discuss this topic in a meaningful degree of detail would require a whole new book; here, we limit ourselves to a reference to the above-mentioned papers, and to the literature quoted therein, for a brief overview of the technical issues typically encountered in this situation.

Acknowledgments. In the preparation of these notes, we have benefitted from the generous support of a number of agencies, including grants from the Fulbright Foundation (Pontificia Universidad Católica of Santiago, Chile, 2006), the Alexander von Humboldt Stiftung (Institut für Analysis, Technische Universität Dresden, 2008), and the Deutsche Forschungsgemeinschaft (TU-Dresden, 2010). We are grateful to the departments of mathematics of these institutions for their kind hospitality. We are also greatly
indebted to Professor A. Negro of the University of Turin, Italy, to Profes-
sors G. Walter and H. Volkmer of the University of Wisconsin-Milwaukee,
to Professor R. Picard of the TU-Dresden, and to Professor Zheng Song-
Mu of Fudan University, Shanghai, for their constant encouragement and
important suggestions. Last, but not least, we owe special gratitude to Ms.
Barbara Beeton and Ms. Jennifer Wright Sharp of the American Mathemat-
ical Society Technical Support team, for their invaluable help in solving all
the \TeX-ncial and typographical problems involved in the production of the
final version of this book.

PASCAL CHERRIER
Université Pierre et Marie Curie, Paris

ALBERT MILANI
University of Wisconsin - Milwaukee
List of Function Spaces

We report a list of all the function spaces we have introduced in this book. In the following, \( X \) is a real Banach space, \( \Omega \) is a domain of \( \mathbb{R}^N \), with boundary \( \partial \Omega \), and \( Q = ]a,b[ \times \Omega \) (or \( Q = ]a,b[ \times \mathbb{R}^N \)). When \( \Omega = \mathbb{R}^N \), the explicit reference to \( \Omega \) is omitted; e.g., \( L^p := L^p(\mathbb{R}^N) \). For each space, we indicate either the page where it has been first introduced or a reference where a definition of the space can be found.

\[ \text{AC}([a,b]; X) \text{ Space of absolutely continuous functions } f : [a,b] \to X \text{ (p.62).} \]
\[ C^m(\Omega) \text{ Space of functions } f : \Omega \to \mathbb{R}, \text{ which have continuous derivatives of order up to } m \text{ (p.2).} \]
\[ C^m_b(\Omega) := \{ f \in C^m(\Omega) \mid \max_{0 \leq j \leq m} \sup_{x \in \Omega} |\partial^j f(x)| < +\infty \} \text{ (p.2).} \]
\[ C^m_0(\Omega) := \{ f \in C^m(\Omega) \mid \text{supp}(f) \text{ is compact} \} \text{ (p.2).} \]
\[ C^m(\overline{\Omega}) \text{: Space of functions which are restrictions to } \overline{\Omega} \text{ of functions in } C^m(\mathbb{R}^N) \text{ (p.2).} \]
\[ C^m_b(\overline{\Omega}) \text{: Space of functions which are restrictions to } \overline{\Omega} \text{ of functions in } C^m_b(\mathbb{R}^N) \text{ (p.3).} \]
\[ C^{m,\alpha}(\Omega) := \{ f \in C^m(\Omega) \mid H_\alpha(\partial^m f) < +\infty \}, \text{ } H_\alpha \text{ defined in (1.3.1) (p.7).} \]
List of Function Spaces

\( C^{m,\alpha}(\Omega) := C^{m,\alpha}(\Omega) \cap C_b^m(\Omega) \) (p. 7).

\( C^{m+\alpha/2,2m+\alpha}(Q) := \{ f \in C(Q) \mid \partial_t^k \partial_x^\lambda f \in C(Q), \ 2k + |\lambda| \leq 2m \}, \ H_\alpha(\partial_t^k \partial_x^\lambda f) < +\infty, \ 2k + |\lambda| = 2m \}, \ H_\alpha \) defined in (1.3.4) (p. 8).

\( \tilde{C}_b^m(Q) := \{ f \in C_b(Q) \mid \partial_t^k \partial_x^\lambda f \in C_b(Q), \ 2k + |\lambda| \leq 2m \} \) (p. 8).

\( C^{m+\alpha/2,2m+\alpha}(\overline{Q}) := \tilde{C}_b^m(Q) \cap C^{m+\alpha/2,2m+\alpha}(Q) \) (p. 8).

\( C([a,b]; X) \) : Space of strongly continuous functions \( f : [a,b] \to X \) (p. 60).

\( C^m([a,b]; X) := \{ u \in C([a,b]; X) \mid \partial_t^j u \in C([a,b]; X), \ 0 \leq j \leq m \} \) (p. 64).

\( C^m([a,b]; X, Y) := \{ u \in C([a,b]; X) \mid \partial_t^m u \in C([a,b]; Y) \} \) (p. 64).

\( C([a, +\infty]; X) \) : Space of strongly continuous functions \( f : [a, +\infty] \to X \) (p. 61).

\( C_b([a, +\infty]; X) \) : Space of strongly continuous and bounded functions \( f : [a, +\infty] \to X \) (p. 61).

\( C_b^m([a, +\infty]; X) \) : Space of continuously differentiable functions \( f : [a, +\infty] \to X \), with bounded derivatives, of order up to \( m \) (p. 64).

\( C_b^m([a, +\infty]; X, Y) := \{ u \in C_b([a, +\infty]; X) \mid \partial_t^m u \in C_b([a, +\infty]; Y) \} \) (p. 64).

\( C_w([a,b]; X) \) : Space of weakly continuous functions \( f : [a,b] \to X \) (p. 61).

\( D(\Omega) \) : Space of test functions in \( \Omega \) (Rudin [139, ch. 6]).

\( D'(\Omega) \) : Space of distributions in \( \Omega \) (Rudin [139, ch. 6]).

\( D([a,b]; X) \) : Space of test functions \( f : [a,b] \to X \) (Lions and Magenes [101, ch. 1]).

\( D'([a,b]; X) := \mathcal{L}(D([a,b]); X) \) (p. 67).

\( G_s(T) := H^{s+1} \times H^s \times V_{s-1}(T) \) (p. 122).

\( G_s(\infty) := H^{s+1} \times H^s \times V_{s-1}(\infty) \) (p. 153).

\( H^m(\Omega) = W^{m,2}(\Omega) := \{ u \in L^2 \mid \partial^\alpha u \in L^2, |\alpha| \leq m \} \) (p. 13).

\( \tilde{H}^m(\Omega) := \{ u \in L^\infty(\Omega) \mid \nabla u \in H^{m-1}(\Omega) \} \) (m \geq 1) (p. 14).

\( H^s(\Omega) := (H^{[s]+1}(\Omega), H^{[s]}(\Omega))_{s-[s]} \) (Lions and Magenes [101, ch. 1]).

\( H^s(\mathbb{R}^N) := \{ f \in S' \mid (1 + |\cdot|^2)^{s/2} \hat{f} \in L^2 \} \) (p. 15).

\( H^{1/2}(\partial\Omega) \) : Space of traces on \( \partial\Omega \) of functions in \( H^1(\Omega) \) (p. 16).
$H^0_0(\Omega) := \{ u \in H^1(\Omega) \mid u|_{\partial \Omega} = 0 \}$ (p. 16).

$H^m_0(\Omega) := \{ u \in H^m(\Omega) \mid \frac{\partial^j u}{\partial x^j} = 0, \quad 0 \leq j \leq m - 1 \}$ (p. 16).

$H^m_\Delta(\Omega) := \{ u \in H^m(\Omega) \mid (-\Delta)^k u \in H^1_0(\Omega), \quad 0 \leq k \leq \lfloor \frac{m-1}{2} \rfloor \}$ (p. 32).

$H_r^m := \{ f \in H^m \mid \mu_r f \in H^{m-1}, \quad 1 \leq r \leq N \}$ (p. 53).

$H^m(a, b; X, Y) := W^{m,2}(a, b; X, Y)$ (p. 68).

$H^m(a, b; X) := W^{m,2}(a, b; X, X)$ (p. 68).

$\mathcal{H}^{h,k}(T) := C([0, T]; H^h) \cap C^1([0, T]; H^k)$ (p. 348).

$\mathcal{K} := \{ f \in C^1(\mathbb{R}^m_\geq 0; \mathbb{R} \geq 0) \mid \partial_j f \geq 0, \quad 1 \leq j \leq m \}$ (p. 4).

$\mathcal{K}_0 := \{ f \in \mathcal{K} \mid f(0) = 0 \}$ (p. 4).

$\mathcal{K}_\infty := \{ f \in C(\mathbb{R} \geq 0; \mathbb{R} \geq 0) \mid \lim_{r \to 0} f(r) = +\infty \}$ (p. 4).

$\mathcal{L}(X, Y) := \{ f : X \to Y \mid f \text{ is linear continuous} \}$ (p. 5).

$L^p(\Omega) := \{ f : \Omega \to \mathbb{R} \mid f \text{ is measurable}, \quad \int_\Omega |f(x)|^p \, dx < +\infty \}$ (up to equivalence on sets of measure zero) (p. 9).

$L^p(\Gamma)$: $L^p$ spaces on a $(N - 1)$-dimensional submanifold $\Gamma \subset \mathbb{R}^N$ (p. 9).

$L^p(a, b; X) := \{ f : ]a, b[ \to X \mid f \text{ is strongly measurable}, \quad \int_a^b \|u(t)\|_X^p \, dt < +\infty \}$ (p. 60).

$L^\infty(a, b; X) := \{ f : ]a, b[ \to X \mid f \text{ is strongly measurable}, \quad \sup_{a < t < b} \text{ess} \|u(t)\|_X < +\infty \}$ (p. 60).

$\mathcal{P}_m(T) := \{ u \in C([0, T]; H^{m+1}) \mid Du \in L^2(0, T; H^m) \}$ (p. 107).

$\mathcal{P}_m(\infty) := \{ u \in C([0, +\infty]; H^{m+1}) \mid Du \in L^2_{\text{loc}}(0, +\infty; H^m) \}$ (p. 215).

$\mathcal{P}_{m,b}(\infty) := \{ u \in C_b([0, +\infty]; H^{m+1}) \mid Du \in L^2(0, +\infty; H^m) \}$ (p. 215).

$\mathcal{P}_s(T) := H^1(0, T; H^{s+2}, H^s)$ (p. 323).

$\mathcal{P}^{h,k}(T) := \{ u \in L^2(0, T; H^h) \mid u_t \in L^2(0, T; H^k) \}$ (p. 353).

$\mathcal{S}$: Schwartz’ space of rapidly decreasing functions in $\mathbb{R}^N$ (Rudin [139, ch. 7]).

$\mathcal{S}'$: Space of tempered distributions in $\mathbb{R}^N$ (Rudin [139, ch. 7]).

$V^m(\mathbb{R}^N)$: The completion of $H^m$ with respect to the norm $u \mapsto \|\nabla u\|_{m-1}$ (p. 28).

$V_m(T) := L^2(0, T; H^{m+1}) \cap C([0, T]; H^m)$ (p. 88).
List of Function Spaces

\( V_m(\infty) := L^2_{\text{loc}}(0, +\infty; H^{m+1}) \cap C([0, +\infty[; H^m) \) (p. 153).

\( W^k_0(\Omega) := \begin{cases} L^2(\Omega) , & \text{if } k = 0 , \\ H^k(\Omega) \cap H^1_0(\Omega) , & \text{if } k \geq 1 . \end{cases} \) (p. 92).

\( W^{m,p} := W^{m,p}(\mathbb{R}^N) \) (p. 14).

\( W^{m,p}(a, b; X, Y) := \{ u \in L^p(a, b; X) \mid u^{(m)} \in L^p(a, b; Y) \} \) (p. 68).

\( W^{m,p}(a, b; X) := W^{m,p}(a, b; X, X) \) (p. 68).

\( W^{m,\ell,k}(a, b) := W^{k,2}(a, b; H^m, H^{m-\ell k}) \) (p. 69).

\( W^m_T(Q) := \{ u \in W^{1,2}(0, T; H^m, L^2) \mid u(T, \cdot) = 0 \} \) (p. 75).

\( \tilde{W}^m_T(Q) := \{ u \in W^{1,2}(-T, T; H^m, L^2) \mid u(-T, \cdot) = u(T, \cdot) = 0 \} \) (p. 75).

\( X_m(T) := C([0, T]; H^{m+1}) \cap C^1([0, T]; H^m) \) (p. 79).

\( X_{k,\ell}(T) := C([0, T]; W^{k+1}_0(\Omega)) \cap C^1([0, T]; W^k_0(\Omega)) \) (p. 92).

\( Y_m(T) := \{ u \in X_m(T) \mid u_{tt} \in L^2(0, T; H^{m-1}) \} \) (p. 79).

\( Y^{h,k}(T) := L^2(0, T; H^h) \cap C([0, T]; H^{(h+k)/2}) \) (p. 353).

\( Y^{h,k}(T) := L^2(0, T; \bar{H}^h) \cap C([0, T]; \bar{H}^{(h+k)/2}) \) (p. 353).

\( Z_m(T) := \bigcap_{j=0}^2 C^j([0, T]; H^{m+1-j}) \) (p. 88).

\( Z_m(\infty) := \bigcap_{j=0}^2 C^j([0, +\infty[; H^{m+1-j}) \) (p. 153).

\( Z_{m,b}(\infty) := \bigcap_{j=0}^2 C^j_b([0, +\infty[; H^{m+1-j}) \) (p. 153).
Bibliography


Index

asymptotic profile, xi, 233, 269, 279
attractor, 245

basis
Fourier, 51
orthogonal, 51
orthonormal, 6, 35
regular, 7, 98, 106
total, 6, 90

blow up, 120, 154, 156, 157, 211
boundary conditions, xvii, 30, 90, 233, 337

Cauchy data, x, 77
Cauchy problem, 77, 107, 119, 145
chain of spaces, 22
chain rule, 38
characteristic, 161, 166
commutator
linear, 45
nonlinear, 46
regularization, 46
compatibility conditions, xvii
constituent relations, 336
contraction, 120, 132, 134
convergence
weak, 5
weak*, 5
convolution, 3, 12, 141
(distribution), 50, 105
corrector, 300, 321
covariant derivative, 33
critical time, 156
currents
displacement, 335
eddy, 335
Foucault, 335
diffusion phenomenon, 233, 279, 285
distribution, tempered, 14
duality, 5
eigenfunctions, 90
eigenvalue, 35, 52, 160
electromagnetic, 35, 52, 160
electromagnetic fields, 335
electromagnetic inductions, 335
electromagnetic potentials, 337
ellipticity condition, 78, 122, 338
equation
Bernoulli, 160
conservation form, 121
dispersive, xiii
dissipative, x, 120, 170, 175
divergence form, 77, 104, 107
elliptic, 229, 233
heat, x, 110, 215, 294
hyperbolic, ix, 77, 119
linear, 77
linearized, 119
Maxwell, xvi, 294, 335
Maxwell, quasi-stationary, 338
parabolic, ix, 78, 121
quasi-linear, 119, 121
Riccati, 211
semi-linear, xiii
stationary, xi, 233, 234, 241
telegraph, x
von Karman, xvii, 335, 343
wave, ix
equiconvergence, 52
existence
  almost global, 120, 154
  global, 120
  local, 122
extension, 156
extension operator, 17, 90

Faraday’s law, 336
ferromagnetic media, 335
fixed point, 119, 135, 146, 234
formula
  Duhamel, 177
  Faà di Bruno, 40
  Leibniz, 25
  Parseval, 14, 59
  Rodrigues, 51
Fourier series, 6, 106
Fourier transform, 14, 51, 53
Fourier’s law, 294
function
  absolutely continuous, 62
  Bessel, 52, 179
  Hermite, 51
  Hölder, 7
  Lipschitz, 7
  regularized, 10
  Riemann, 179
  weakly continuous, 61
Galerkin approximation, 89, 90, 106, 108
gauge relation, 338
Gelfand triple, 70
genuine nonlinearity, 160
Hermite functions, 51
Hermite polynomials, 51
imbedding
  compact, 5, 22, 59, 74
  continuous, 5, 22
  Sobolev, 78
inequality
  Gagliardo-Nirenberg, 19
  Gronwall, 86
  Hausdorff-Young, 15
  Hölder, 11
  interpolation, 13, 21
Minkowski, 3
Poincaré, 20, 32
Schwarz, 4, 11
trace, 74
Young, 12
initial conditions, x, 77, 107
initial layer, xvi, 294, 296, 305, 321
intermediate derivative, 68
interpolation, 13, 68
invariant interval, 154
Kronecker δ, ix
Laplace operator, 28, 30, 90, 339
life span, 154, 156, 293, 333
linearization, 119, 132
local coordinates, 9, 16
lower semi-continuity, 6
map, monotone, 336
material laws, 336
maximal time, 155, 156
maximum principle, 121, 155, 226, 335
measure, Lebesgue, 9
microlocal analysis, xv
mollifiers, 9, 315
Moser-Nash iteration, 328
multi-index, 2
  length, 2
normal derivative, 16
null condition, xiv, 120
operator, smoothing effect, 110
orbit, 170
perturbation, 201, 245, 293, 305
phase space, 81, 186, 245, 304
Picard iterations, 135, 150
positively invariant region, 165, 169
prepared data, 296
problem, well-posed, x
product algebra, 27, 78
product estimates, 24
projection, 7, 91
pseudo-differential operators, xv
regularity, elliptic, 31
restriction, 156
restriction operator, 16, 75, 90
Riemann invariant, 161, 169
Riemannian volume element, 9
semiflow, 81, 245
Index

semigroups, xv
singular convergence, 305
singular perturbation, xi
solution
  almost global, xii
  global, xi, xii, 120, 153
  global bounded, 153
  hyperbolic, 233
  local, xi, 120
  parabolic, 233
  stationary, 208, 233
  strong, ix, 77, 107
  weak, 77, 105, 121
solution kernel, 176, 215
solution operator, 81
solutions, bounded, xii
space
  Bochner, 60
  Hölder, 7, 8, 117
  intermediate, 68
  interpolation, 13
  Lebesgue, 9
  parabolic-Hölder, 8
  reflexive, 5
  Schwartz, 14
  Sobolev, 13
stability, 155, 202, 219, 244
strong monotonicity, 338
system, dissipative, 169

theorem
  Alaoglu, 6
  Ampère, 336
  Carathéodory, 92
  extension, 75
  Fubini, 187, 189
  Tonelli, 187
  trace, 69
trace, 15, 16, 20, 68
trace operator, 103
variation of parameters, 177
Selected Titles in This Series

135 Pascal Cherrier and Albert Milani, Linear and Quasi-linear Evolution Equations in Hilbert Spaces, 2012
134 Jean-Marie De Koninck and Florian Luca, Analytic Number Theory, 2012
132 Terence Tao, Topics in Random Matrix Theory, 2012
131 Ian M. Musson, Lie Superalgebras and Enveloping Algebras, 2012
130 Viviana Ene and Jürgen Herzog, Gröbner Bases in Commutative Algebra, 2011
127 Jeffrey Strom, Modern Classical Homotopy Theory, 2011
126 Terence Tao, An Introduction to Measure Theory, 2011
125 Dror Varolin, Riemann Surfaces by Way of Complex Analytic Geometry, 2011
124 David A. Cox, John B. Little, and Henry K. Schenck, Toric Varieties, 2011
123 Gregory Eskin, Lectures on Linear Partial Differential Equations, 2011
122 Teresa Crespo and Zbigniew Hajto, Algebraic Groups and Differential Galois Theory, 2011
121 Tobias Holck Colding and William P. Minicozzi, II, A Course in Minimal Surfaces, 2011
120 Qing Han, A Basic Course in Partial Differential Equations, 2011
119 Alexander Korostelev and Olga Korosteleva, Mathematical Statistics, 2011
118 Hal L. Smith and Horst R. Thieme, Dynamical Systems and Population Persistence, 2011
117 Terence Tao, An Epsilon of Room, I: Real Analysis, 2010
116 Joan Cerdà, Linear Functional Analysis, 2010
114 Joseph J. Rotman, Advanced Modern Algebra, 2010
113 Thomas M. Liggett, Continuous Time Markov Processes, 2010
112 Fredi Tröltzsch, Optimal Control of Partial Differential Equations, 2010
111 Simon Brendle, Ricci Flow and the Sphere Theorem, 2010
110 Matthias Kreck, Differential Algebraic Topology, 2010
109 John C. Neu, Training Manual on Transport and Fluids, 2010
108 Enrique Outerelo and Jesús M. Ruiz, Mapping Degree Theory, 2009
107 Jeffrey M. Lee, Manifolds and Differential Geometry, 2009
106 Robert J. Daverman and Gerard A. Venema, Embeddings in Manifolds, 2009
105 Giovanni Leoni, A First Course in Sobolev Spaces, 2009
104 Paolo Aluffi, Algebra: Chapter 0, 2009
103 Branko Grünbaum, Configurations of Points and Lines, 2009
102 Mark A. Pinsky, Introduction to Fourier Analysis and Wavelets, 2002
101 Ward Cheney and Will Light, A Course in Approximation Theory, 2000
100 I. Martin Isaacs, Algebra, 1994
99 Gerald Teschl, Mathematical Methods in Quantum Mechanics, 2009
97 David C. Ullrich, Complex Made Simple, 2008
96 N. V. Krylov, Lectures on Elliptic and Parabolic Equations in Sobolev Spaces, 2008
95 Leon A. Takhtajan, Quantum Mechanics for Mathematicians, 2008
94 James E. Humphreys, Representations of Semisimple Lie Algebras in the BGG Category O, 2008
93 Peter W. Michor, Topics in Differential Geometry, 2008
SELECTED TITLES IN THIS SERIES

92 I. Martin Isaacs, Finite Group Theory, 2008
91 Louis Halle Rowen, Graduate Algebra: Noncommutative View, 2008
90 Larry J. Gerstein, Basic Quadratic Forms, 2008
89 Anthony Bonato, A Course on the Web Graph, 2008
86 Yulij Ilyashenko and Sergei Yakovenko, Lectures on Analytic Differential Equations, 2008
84 Charalambos D. Aliprantis and Rabee Tourky, Cones and Duality, 2007
83 Wolfgang Ebeling, Functions of Several Complex Variables and Their Singularities, 2007
81 V. V. Prasolov, Elements of Homology Theory, 2007
80 Davar Khoshnevisan, Probability, 2007
79 William Stein, Modular Forms, a Computational Approach, 2007
78 Harry Dym, Linear Algebra in Action, 2007
77 Bennett Chow, Peng Lu, and Lei Ni, Hamilton’s Ricci Flow, 2006
76 Michael E. Taylor, Measure Theory and Integration, 2006
75 Peter D. Miller, Applied Asymptotic Analysis, 2006
74 V. V. Prasolov, Elements of Combinatorial and Differential Topology, 2006
73 Louis Halle Rowen, Graduate Algebra: Commutative View, 2006
72 R. J. Williams, Introduction to the Mathematics of Finance, 2006
71 S. P. Novikov and I. A. Taimanov, Modern Geometric Structures and Fields, 2006
70 Séan Dineen, Probability Theory in Finance, 2005
69 Sebastián Montiel and Antonio Ros, Curves and Surfaces, 2005
68 Luis Caffarelli and Sandro Salsa, A Geometric Approach to Free Boundary Problems, 2005
67 T.Y. Lam, Introduction to Quadratic Forms over Fields, 2005
66 Yuli Eidelman, Vitali Milman, and Antonis Tsolomitis, Functional Analysis, 2004
65 S. Ramanan, Global Calculus, 2005
64 A. A. Kirillov, Lectures on the Orbit Method, 2004
63 Steven Dale Cutkosky, Resolution of Singularities, 2004
62 T. W. Körner, A Companion to Analysis, 2004
61 Thomas A. Ivey and J. M. Landsberg, Cartan for Beginners, 2003
60 Alberto Candel and Lawrence Conlon, Foliations II, 2003
59 Steven H. Weintraub, Representation Theory of Finite Groups: Algebra and Arithmetic, 2003
58 Cédric Villani, Topics in Optimal Transportation, 2003
57 Robert Plato, Concise Numerical Mathematics, 2003
56 E. B. Vinberg, A Course in Algebra, 2003
55 C. Herbert Clemens, A Scrapbook of Complex Curve Theory, 2003
54 Alexander Barvinok, A Course in Convexity, 2002

For a complete list of titles in this series, visit the AMS Bookstore at www.ams.org/bookstore/.
This book considers evolution equations of hyperbolic and parabolic type. These equations are studied from a common point of view, using elementary methods, such as that of energy estimates, which prove to be quite versatile. The authors emphasize the Cauchy problem and present a unified theory for the treatment of these equations. In particular, they provide local and global existence results, as well as strong well-posedness and asymptotic behavior results for the Cauchy problem for quasi-linear equations. Solutions of linear equations are constructed explicitly, using the Galerkin method; the linear theory is then applied to quasi-linear equations, by means of a linearization and fixed-point technique. The authors also compare hyperbolic and parabolic problems, both in terms of singular perturbations, on compact time intervals, and asymptotically, in terms of the diffusion phenomenon, with new results on decay estimates for strong solutions of homogeneous quasi-linear equations of each type.

This textbook presents a valuable introduction to topics in the theory of evolution equations, suitable for advanced graduate students. The exposition is largely self-contained. The initial chapter reviews the essential material from functional analysis. New ideas are introduced along with their context. Proofs are detailed and carefully presented. The book concludes with a chapter on applications of the theory to Maxwell’s equations and von Karman equations.