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Preface

Free boundary problems (FBPs) are considered today as one of the most important directions in the mainstream of the analysis of partial differential equations (PDEs), with an abundance of applications in various sciences and real world problems. In the past two decades, various new ideas, techniques, and methods have been developed, and new important, challenging problems in physics, industry, finance, biology, and other areas have arisen.

The study of free boundaries is an extremely broad topic not only due to the diversity of applications but also because of the variety of the questions one may be interested in, ranging from modeling and numerics to the purely theoretical questions. This breadth presents challenges and opportunities!

A particular direction in free boundary problems has been the study of the regularity properties of the solutions and those of the free boundaries. Such questions are usually considered very hard, as the free boundary is not known a priori (it is part of the problem!) so the classical techniques in elliptic/parabolic PDEs do not apply. In many cases the success is achieved by combining the ideas from PDEs with the ones from geometric measure theory, the calculus of variations, harmonic analysis, etc.

Today there are several excellent books on free boundaries, treating various issues and questions: e.g. [**DL76**], [**KS80**], [**Cra84**], [**Rod87**], [**Fri88**], [**CS05**]. These books are great assets for anyone who wants to learn FBPs and related techniques; however, with the exception of [**CS05**], they date back two decades. We believe that there is an urge for a book where some of the most recent developments and new methods in the regularity of free boundaries can be introduced to the nonexperts and particularly to the graduate students starting their research in the field. This gap in the literature has been partially filled by the aforementioned book of Caffarelli and

Salsa [**CS05**], which treats the Stefan-type free boundary problems (with the Bernoulli gradient condition). Part 3 in [**CS05**], in particular, covers several technical tools that should be known to anyone working in the field of PDEs/FBPs.

Our intention, in this book, was to give a coherent presentation of the study of the regularity properties of the free boundary for a particular type of problems, known as *obstacle-type problems*. The book grew out of the lecture notes for the courses and mini-courses given by the authors at various locations, and hence we believe that the format of the book is most suitable for a graduate course (see the end of the Introduction for suggestions). Notwithstanding this, we have to warn the reader that this book is far from being a complete reference for the regularity theory. We hope that it gives a reasonably good introduction to techniques developed in the past two decades, including those due to the authors and their collaborators.

We thank many colleagues and fellow mathematicians for reading parts of this book and commenting, particularly Mark Allen, Darya Apushkinskaya, Mahmoudreza Bazarganzadeh, Paul Feehan, Nestor Guillen, Erik Lindgren, Norayr Matevosyan, Andreas Minne, Sadna Sajadini, Wenhui Shi, Martin Strömqvist. Our special thanks go to Luis Caffarelli, Craig Evans, Avner Friedman, and Juan-Luis Vázquez for their useful suggestions and advice regarding the book.

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West Lafayette, IN, USA—Stockholm, Sweden—St. Petersburg, Russia

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Notation

Basic notation

c, c_0, C, C_0, C_1, ...	generic constants
$C_{a_1,\dots,a_k}$, $C(a_1,\dots,a_k)$	constants depending (only) on a_1, ..., a_k
$\mathbb{N}$	$\{1,2,3,\dots\}$, the set of natural numbers
$\mathbb{R}$	$(-\infty,\infty)$, the set of real numbers
$\mathbb{R}^n$	$\{x=(x_1,\dots,x_n) : x_i \in \mathbb{R}, i=1,\dots,n\}$, $n\in\mathbb{N}$, the n-dimensional Euclidean space
e_1, e_2, ..., e_n	$e_i=(0,\dots,1,\dots,0)$, with 1 in the i-th position, the standard coordinate vectors in $\mathbb{R}^n$
$x\cdot y$	$\sum_{i=1}^n x_i y_i$, the interior product of $x,y\in\mathbb{R}^n$
$\lvert x\rvert$	$\sqrt{x\cdot x}$ for $x\in\mathbb{R}^n$, the Euclidean norm
x'	$(x_1,\dots,x_{n-1})$ for $x=(x_1,\dots,x_n)\in\mathbb{R}^n$; we also identify $x=(x',x_n)$
$\mathbb{R}^n_+$, $\mathbb{R}^n_-$	$\{x\in\mathbb{R}^n : x_n>0\}$; $\{x\in\mathbb{R}^n : x_n<0\}$
$B_r(x)$	$\{y\in\mathbb{R}^n : \lvert y-x\rvert<r\}$, the open ball in $\mathbb{R}^n$
$B_r^\pm(x)$	$B_r(x)\cap\mathbb{R}^n_\pm$
B_r, $B_r^\pm$	$B_r(0)$, $B_r^\pm(0)$
B'_r	$\{y'\in\mathbb{R}^{n-1} : \lvert y'\rvert<r\}$, ball in $\mathbb{R}^{n-1}$; often identified with $B_r\cap(\mathbb{R}^{n-1}\times\{0\})$
$\mathcal{C}_\delta$	$\{x\in\mathbb{R}^n : x_n>\delta\lvert x'\rvert\}$, for $\delta>0$, cone with axis e_n and opening angle $2\arctan(1/\delta)$
∂E, $\overline{E}$, $\mathrm{Int}(E)$	the boundary, closure, and the interior of the set E in the relevant topology

E^c	$\mathbb{R}^n \setminus E$, the complement of the set $E \subset \mathbb{R}^n$
$\operatorname{dist}(x, E)$	$\inf_{y\in E} \|x-y\|$ for $x \in \mathbb{R}^n$, $E \subset \mathbb{R}^n$
$\operatorname{diam}(E)$	$\sup_{x,y\in E} \|x-y\|$ for $E \subset \mathbb{R}^n$
$\|E\|$	Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$
$H^s(E)$, $s \le 0 \le n$	s-Hausdorff measure for a Borel set E
χ_E	$\begin{cases} 1 & \text{on } E, \\ 0 & \text{on } E^c, \end{cases}$ the characteristic function of $E \subset \mathbb{R}^n$
$A \Subset B$	$\overline{A}$ is compact and contained in B
a^+, a^-	$\max\{a, 0\}$; $\max\{-a, 0\}$ for a number $a \in \mathbb{R}$; for a function $u : E \to \mathbb{R}$, $u^\pm : E \to \mathbb{R}_+$ are given by $u^\pm(x) = u(x)^\pm$
$\operatorname{supp} u$	$\overline{\{u \ne 0\}}$, the support of a function u
$\partial_e u$, u_e	the partial derivative of a function u in direction e, understood in the classical sense or in the sense of distributions
$\partial_{x_i} u$, u_{x_i}	same as $\partial_{e_i} u$, $i = 1, 2, \dots, n$, where e_i are the standard coordinate vectors
$\partial_{x_{i_1}\cdots x_{i_k}} u$, $u_{x_{i_1}\cdots x_{i_k}}$	$\partial_{x_{i_1}} \partial_{x_{i_2}} \cdots \partial_{x_{i_k}} u$, higher order derivatives
$\partial^\beta u$	$\partial_{x_1}^{\beta_1} \cdots \partial_{x_k}^{\beta_k} u$, for multiindex $\beta = (\beta_1, \dots, \beta_n)$, $\beta_i \in \mathbb{N} \cup \{0\}$, partial derivative of order $\|\beta\| = \beta_1 + \cdots + \beta_n$
Δu	$\sum_{i=1}^n \partial_{x_i x_i} u$, the Laplace operator (Laplacian) in $\mathbb{R}^n$
∇u	$(\partial_{x_1} u, \dots, \partial_{x_n} u)$, the gradient of u
$D^2 u$	$(\partial_{x_i x_j} u)_{i,j=1}^n$, the Hessian matrix of u
$D^k u$	for $k \in \mathbb{N}$, $(\partial_{x_{i_1}\cdots x_{i_k}} u)$, the tensor of partial derivatives of k-th order

Function spaces

$C^0(E)$: the Banach space of continuous function $u : E \to \mathbb{R}$ on the set E with the uniform norm

$$\|u\|_{C^0(E)} = \sup_{x\in E} |u(x)|.$$

$C(E)$: same as $C^0(E)$.

$C^{0,\alpha}(E)$, $\alpha \in (0, 1]$: the Banach space of α-Hölder continuous functions of E with the norm

$$\|u\|_{C^{0,\alpha}(E)} = \|u\|_{C^0(E)} + [u]_{\alpha,E}, \quad [u]_{\alpha,E} = \sup \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

The quantity $[u]_{\alpha,E}$ is called the α-Hölder seminorm of u on E. When $\alpha = 1$, $C^{0,1}$ is the space of Lipschitz continuous functions and $[u]_{1,E}$ is called the Lipschitz constant of u on E.

$C^\alpha(E)$, $\alpha \in (0, 1)$: same as $C^{0,\alpha}(E)$; we never use this notation for $\alpha = 1$.

$C^k(D)$, $k \in \mathbb{N}$: for open $D \subset \mathbb{R}^n$, the Fréchet space of functions $u : D \to \mathbb{R}$ with continuous partial derivatives $\partial^\beta u$ in D, $|\beta| \le k$; it has the family of seminorms

$$\|u\|_{C^k(K)} = \sum_{|\beta|\le k} \|\partial^\beta u\|_{C^0(K)} \quad \text{for } K \Subset D.$$

$C^\infty(D)$: the Fréchet space of infinitely differentiable functions in D, intersection of all $C^k(D)$, $k \in \mathbb{N}$.

$C^k_0(D)$, $k \in \mathbb{N} \cup \{0, \infty\}$: a subspace of $C^k(D)$ of functions compactly supported in D, i.e. with $\operatorname{supp} u \Subset D$.

$C^{k,\alpha}(D)$, $k \in \mathbb{N}$, $\alpha \in (0, 1]$: the Fréchet space consisting of functions $u \in C^k(D)$ such that $\partial^\beta u \in C^{0,\alpha}(K)$, $|\beta| = k$, for any $K \Subset D$, with the family of seminorms

$$\|u\|_{C^{k,\alpha}(K)} = \|u\|_{C^k(K)} + \sum_{|\beta|=k} [\partial^\beta u]_{\alpha,K}.$$

$C^k(D \cup F)$, $C^{k,\alpha}(D \cup F)$: for a relatively open $F \subset \partial D$, similar to $C^k(D)$ and $C^{k,\alpha}(D)$ but with the k-th partial derivatives continuous (α-Hölder continuous) up to F and with the respective seminorms for $K \Subset D \cup F$.

$C^{k,\alpha}_{\mathrm{loc}}(D)$, $C^{k,\alpha}_{\mathrm{loc}}(D \cup F)$: same as $C^{k,\alpha}(D)$, $C^{k,\alpha}(D \cup F)$; we use this notation in the instances where we want to emphasise the local nature of the estimates.

$L^p(E)$, $1 \le p < \infty$: for a measurable set $E \subset \mathbb{R}^n$, the Banach space of measurable functions u on E (more precisely, classes of a.e. equivalences) with the finite norm

$$\|u\|_{L^p(E)} = \left(\int_E |u(x)|^p dx \right)^{1/p}.$$

$L^\infty(E)$: the Banach space of essentially bounded measurable functions u with the norm

$$\|u\|_{L^\infty(E)} = \operatorname*{ess\,sup}_{x \in E} |u(x)|.$$

$W^{k,p}(D)$, $k \in \mathbb{N}$, $1 \le p \le \infty$: the Sobolev space, the Banach space of functions $u \in L^p(D)$ with the (distributional) partial derivative $\partial^\beta u \in L^p(D)$, $|\beta| \le k$; the norm is given by

$$\|u\|_{W^{k,p}(D)} = \sum_{|\beta|\le k} \|\partial^\beta u\|_{L^p(D)}.$$

$W^{k,p}_0(D)$, $k \in \mathbb{N}$, $1 \le p \le \infty$: the closure of $C^\infty_0(D)$ in $W^{k,p}(D)$.

$L^p_{\mathrm{loc}}(D)$, $W^{k,p}_{\mathrm{loc}}(D)$, $k \in \mathbb{N}$, $1 \le p \le \infty$: the Fréchet space of measurable functions $u : D \to \mathbb{R}$ such that $u|_{D'} \in L^p(D')$ or, respectively, $W^{k,p}(D')$ for any open $D' \Subset D$.

Notation related to free boundaries

$\delta(\rho, u, x^0)$	the thickness function for the coincidence set for the solution u at x^0, defined as $\frac{1}{\rho}\min\operatorname{diam}(\Omega(u)^c \cap B_\rho(x^0))$
$\delta(\rho, u)$	same as $\delta(\rho, u, 0)$
$\Gamma_\pm(u)$	in Problem **C**, the free boundaries of positive and negative phases, $\partial\Omega^\pm(u) \cap D$
$\Gamma^0(u)$	the set of branch points in Problem **C**, defined as $\Gamma(u) \cap \{\lvert\nabla u\rvert = 0\}$
$\Gamma^*(u)$	the set of nonbranch points in Problem **C**, defined as $\Gamma(u) \cap \{\lvert\nabla u\rvert \neq 0\}$
$\Gamma_\kappa(u)$	the set of free boundary points in Problem **S**, where the blowups have homogeneity κ, i.e. $\{x^0 \in \Gamma(u) : N(0+, u, x^0) = \kappa\}$
$\Lambda(u)$, Λ	the coincidence set; in the classical obstacle problem, the set $\{x \in D : u(x) = \psi(x)\}$; in Problem **A**, the set $\{x \in D : u(x) = \lvert\nabla u(x)\rvert = 0\}$; in Problem **B**, the set $\{x \in D : \lvert\nabla u(x)\rvert = 0\}$; in the thin obstacle problem, the set $\{x \in \mathcal{M} : u(x) = \psi(x)\}$; in Problem **S**, the set $\{x' \in B'_R : u(x', 0) = 0\}$
$\min\operatorname{diam} E$	the minimum diameter of the set E, the infimum of distances between pairs of parallel planes enclosing E
$M(r, u, p)$, $M(r, u, p, x^0)$	Monneau's monotonicity formula for the classical obstacle problem; see p. 138
$M_\kappa(r, u, p)$, $M_\kappa(r, u, p, x^0)$	Monneau-type monotonicity formula for Problem **S**; see p. 189
$N(r, u)$, $N(r, u, x^0)$	Almgren's frequency function (at x^0); see pp. 173, 178
$\Omega(u)$, Ω	the complement of the coincidence set $\Lambda(u)$
$\Omega_\pm(u)$	in Problem **C**, regions of positive and negative phase defined as $\{x \in D : \pm u(x) > 0\}$
$P_R(x^0, M)$	a class of local solutions of Problems **A**, **B**, **C** in $B_R(x^0)$; see p. 65
$P_R(M)$	same as $P_R(0, M)$
$P_R^+(x^0, M)$, $P_R^+(M)$	classes of local solutions of Problems **A**, **B**, **C** in B_R^+ (see p. 154); $P_R^+(M)$ should not be confused with $P_R^+(0, M)$

$P_\infty(M)$	a class of global solutions of Problems **A**, **B**, **C**; see p. 65
$P_\infty^+(M)$	a class of solutions of Problems **A**, **B**, **C** in $\mathbb{R}^n_+$; see p. 154
$\Sigma(u)$	the set of singular points; for Problem **A**, see p. 134; for Problem **S**, see p. 187
$\Sigma_\kappa(u)$	for $\kappa = 2m$, $m \in \mathbb{N}$, the same as $\Gamma_\kappa(u)$
$\Sigma^d(u)$, $\Sigma^d_\kappa(u)$	the set of singular points where the dimension of the singular set equals d; for classical obstacle problem, see p. 141; for Problem **A**, see p. 146; for Problem **S**, see p. 192
$u_{x^0,\lambda}(x)$	a rescaling of solution u; in Problems **A**, **B**, **C** defined as $u_{x^0,\lambda}(x) := \frac{1}{\lambda^2}(u(x^0+\lambda x)$; in Problem **S** defined as $u_{x^0,\lambda}(x) := \frac{u(x^0+\lambda x)}{\left(\lambda^{-(n-1)}\int_{\partial B_\lambda(x^0)} u^2\right)^{1/2}}$
$u_\lambda(x)$	same as $u_{x^0,\lambda}(x)$ when the scaling center x^0 is unambiguously identified from the context
$u^{(\kappa)}_\lambda(x)$, $u^{(\kappa)}_{x^0,\lambda}(x)$	κ-homogeneous rescalings in Problem **S**, defined as $\frac{1}{\lambda^\kappa}u(x^0+\lambda x)$
$W(r,u)$, $W(r,u,x^0)$	Weiss's energy functional (at x^0) in Problems **A**, **B**, **C**; see p. 74
$W_\kappa(r,u)$, $W_\kappa(r,u,x^0)$	Weiss's energy functional (at x^0) in Problem **S**; see p. 188
$\Phi(r,u_+,u_-)$	the Alt-Caffarelli-Friedman functional; see p. 34
$\phi_e(r,u)$	$\Phi(r,(\partial_e u)^+,(\partial_e u)^-)$

Index

The regularity theory of free boundaries flourished during the late 1970s and early 1980s and had a major impact in several areas of mathematics, mathematical physics, and industrial mathematics, as well as in applications. Since then the theory continued to evolve. Numerous new ideas, techniques, and methods have been developed, and challenging new problems in applications have arisen. The main intention of the authors of this book is to give a coherent introduction to the study of the regularity properties of free boundaries for a particular type of problems, known as obstacle-type problems. The emphasis is on the methods developed in the past two decades. The topics include optimal regularity, nondegeneracy, rescalings and blowups, classification of global solutions, several types of monotonicity formulas, Lipschitz, C^1, as well as higher regularity of the free boundary, structure of the singular set, touch of the free and fixed boundaries, and more.

The book is based on lecture notes for the courses and mini-courses given by the authors at various locations and should be accessible to advanced graduate students and researchers in analysis and partial differential equations.

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