

Regularity of Free Boundaries in Obstacle-Type Problems

Arshak Petrosyan
Henrik Shahgholian
Nina Uraltseva

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in Mathematics**

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Contents

Preface	ix
Introduction	1
Suggestions for reading/teaching	5
Chapter 1. Model problems	7
§1.1. Catalog of problems	7
§1.2. Model Problems A , B , C	15
§1.3. $W^{2,p}$ regularity of solutions	17
Notes	22
Exercises	24
Chapter 2. Optimal regularity of solutions	29
§2.1. Optimal regularity in the classical obstacle problem	29
§2.2. ACF monotonicity formula and generalizations	32
§2.3. Optimal regularity in obstacle-type problems	42
§2.4. Optimal regularity up to the fixed boundary	45
§2.5. A counterexample	49
Notes	51
Exercises	53
Chapter 3. Preliminary analysis of the free boundary	57
§3.1. Nondegeneracy	57
§3.2. Lebesgue and Hausdorff measures of the free boundary	61
§3.3. Classes of solutions, rescalings, and blowups	65

§3.4. Blowups	68
§3.5. Weiss-type monotonicity formulas	74
Notes	78
Exercises	79
Chapter 4. Regularity of the free boundary: first results	81
§4.1. Problem A : C^1 regularity of the free boundary near regular points	81
§4.2. Problem B : the local structure of the patches	87
§4.3. Problems A and B : higher regularity of the free boundary	91
§4.4. Problem C : the free boundary near the branch points	92
§4.5. Problem C : real analyticity of Γ^*	95
Notes	96
Exercises	97
Chapter 5. Global solutions	99
§5.1. Classical obstacle problem	100
§5.2. Problems A , B	101
§5.3. Problem C	108
§5.4. Approximation by global solutions	109
Notes	112
Exercises	112
Chapter 6. Regularity of the free boundary: uniform results	115
§6.1. Lipschitz regularity of the free boundary	115
§6.2. $C^{1,\alpha}$ Regularity of the free boundary: Problems A and B	120
§6.3. C^1 regularity of the free boundary: Problem C	124
§6.4. Higher regularity: Problems A and B	128
Notes	131
Exercises	132
Chapter 7. The singular set	133
§7.1. The characterization of the singular set	133
§7.2. Polynomial solutions	135
§7.3. Examples of singularities	136
§7.4. Singular set: classical obstacle problem	138
§7.5. Singular set: Problem A	143
Notes	149

Exercises	149
Chapter 8. Touch with the fixed boundary	153
§8.1. Contact points	153
§8.2. Global solutions in half-spaces	155
§8.3. Behavior of the free boundary near the fixed boundary	159
§8.4. Uniqueness of blowups at contact points	162
Notes	164
Exercises	165
Chapter 9. The thin obstacle problem	167
§9.1. The thin obstacle problem	167
§9.2. $C^{1,\alpha}$ regularity	171
§9.3. Almgren's frequency formula	173
§9.4. Rescalings and blowups	176
§9.5. Optimal regularity	181
§9.6. The regular set	183
§9.7. The singular set	186
§9.8. Weiss- and Monneau-type monotonicity formulas	188
§9.9. The structure of the singular set	192
Notes	196
Exercises	197
Bibliography	201
Notation	211
Basic notation	211
Function spaces	212
Notation related to free boundaries	214
Index	217

Preface

Free boundary problems (FBPs) are considered today as one of the most important directions in the mainstream of the analysis of partial differential equations (PDEs), with an abundance of applications in various sciences and real world problems. In the past two decades, various new ideas, techniques, and methods have been developed, and new important, challenging problems in physics, industry, finance, biology, and other areas have arisen.

The study of free boundaries is an extremely broad topic not only due to the diversity of applications but also because of the variety of the questions one may be interested in, ranging from modeling and numerics to the purely theoretical questions. This breadth presents challenges and opportunities!

A particular direction in free boundary problems has been the study of the regularity properties of the solutions and those of the free boundaries. Such questions are usually considered very hard, as the free boundary is not known a priori (it is part of the problem!) so the classical techniques in elliptic/parabolic PDEs do not apply. In many cases the success is achieved by combining the ideas from PDEs with the ones from geometric measure theory, the calculus of variations, harmonic analysis, etc.

Today there are several excellent books on free boundaries, treating various issues and questions: e.g. [DL76], [KS80], [Cra84], [Rod87], [Fri88], [CS05]. These books are great assets for anyone who wants to learn FBPs and related techniques; however, with the exception of [CS05], they date back two decades. We believe that there is an urge for a book where some of the most recent developments and new methods in the regularity of free boundaries can be introduced to the nonexperts and particularly to the graduate students starting their research in the field. This gap in the literature has been partially filled by the aforementioned book of Caffarelli and

Salsa [CS05], which treats the Stefan-type free boundary problems (with the Bernoulli gradient condition). Part 3 in [CS05], in particular, covers several technical tools that should be known to anyone working in the field of PDEs/FBPs.

Our intention, in this book, was to give a coherent presentation of the study of the regularity properties of the free boundary for a particular type of problems, known as *obstacle-type problems*. The book grew out of the lecture notes for the courses and mini-courses given by the authors at various locations, and hence we believe that the format of the book is most suitable for a graduate course (see the end of the Introduction for suggestions). Notwithstanding this, we have to warn the reader that this book is far from being a complete reference for the regularity theory. We hope that it gives a reasonably good introduction to techniques developed in the past two decades, including those due to the authors and their collaborators.

We thank many colleagues and fellow mathematicians for reading parts of this book and commenting, particularly Mark Allen, Darya Apushkinskaya, Mahmoudreza Bazarganzadeh, Paul Feehan, Nestor Guillen, Erik Lindgren, Norayr Matevosyan, Andreas Minne, Sadna Sajadini, Wenhui Shi, Martin Strömqvist. Our special thanks go to Luis Caffarelli, Craig Evans, Avner Friedman, and Juan-Luis Vázquez for their useful suggestions and advice regarding the book.

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Finally, we thank the Mathematical Sciences Research Institute (MSRI), Berkeley, CA, for hosting a program on *Free Boundary Problems, Theory and Applications* in Spring 2011, where all three of us were at residence and had a fabulous working environment to complete the book.

West Lafayette, IN, USA—Stockholm, Sweden—St. Petersburg, Russia

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Notation

Basic notation

$c, c_0, C, C_0, C_1, \dots$	generic constants
$C_{a_1, \dots, a_k}, C(a_1, \dots, a_k)$	constants depending (only) on a_1, \dots, a_k
\mathbb{N}	$\{1, 2, 3, \dots\}$, the set of natural numbers
\mathbb{R}	$(-\infty, \infty)$, the set of real numbers
\mathbb{R}^n	$\{x = (x_1, \dots, x_n) : x_i \in \mathbb{R}, i = 1, \dots, n\}$, $n \in \mathbb{N}$, the n -dimensional Euclidean space
e_1, e_2, \dots, e_n	$e_i = (0, \dots, 1, \dots, 0)$, with 1 in the i -th position, the standard coordinate vectors in \mathbb{R}^n
$x \cdot y$	$\sum_{i=1}^n x_i y_i$, the interior product of $x, y \in \mathbb{R}^n$
$ x $	$\sqrt{x \cdot x}$ for $x \in \mathbb{R}^n$, the Euclidean norm
x'	(x_1, \dots, x_{n-1}) for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$; we also identify $x = (x', x_n)$
$\mathbb{R}_+^n, \mathbb{R}_-^n$	$\{x \in \mathbb{R}^n : x_n > 0\}$; $\{x \in \mathbb{R}^n : x_n < 0\}$
$B_r(x)$	$\{y \in \mathbb{R}^n : y - x < r\}$, the open ball in \mathbb{R}^n
$B_r^\pm(x)$	$B_r(x) \cap \mathbb{R}_\pm^n$
B_r, B_r^\pm	$B_r(0), B_r^\pm(0)$
B'_r	$\{y' \in \mathbb{R}^{n-1} : y' < r\}$, ball in \mathbb{R}^{n-1} ; often identified with $B_r \cap (\mathbb{R}^{n-1} \times \{0\})$
\mathcal{C}_δ	$\{x \in \mathbb{R}^n : x_n > \delta x' \}$, for $\delta > 0$, cone with axis e_n and opening angle $2 \arctan(1/\delta)$
$\partial E, \overline{E}, \text{Int}(E)$	the boundary, closure, and the interior of the set E in the relevant topology

E^c	$\mathbb{R}^n \setminus E$, the complement of the set $E \subset \mathbb{R}^n$
$\text{dist}(x, E)$	$\inf_{y \in E} x - y $ for $x \in \mathbb{R}^n$, $E \subset \mathbb{R}^n$
$\text{diam}(E)$	$\sup_{x, y \in E} x - y $ for $E \subset \mathbb{R}^n$
$ E $	Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$
$H^s(E)$, $s \leq 0 \leq n$	s -Hausdorff measure for a Borel set E
χ_E	$\begin{cases} 1 & \text{on } E, \\ 0 & \text{on } E^c, \end{cases}$ the characteristic function of $E \subset \mathbb{R}^n$
$A \Subset B$	\bar{A} is compact and contained in B
a^+, a^-	$\max\{a, 0\}$; $\max\{-a, 0\}$ for a number $a \in \mathbb{R}$; for a function $u : E \rightarrow \mathbb{R}$, $u^\pm : E \rightarrow \mathbb{R}_+$ are given by $u^\pm(x) = u(x)^\pm$
$\text{supp } u$	$\overline{\{u \neq 0\}}$, the support of a function u
$\partial_e u$, u_e	the partial derivative of a function u in direction e , understood in the classical sense or in the sense of distributions
$\partial_{x_i} u$, u_{x_i}	same as $\partial_{e_i} u$, $i = 1, 2, \dots, n$, where e_i are the standard coordinate vectors
$\partial_{x_{i_1} \dots x_{i_k}} u$, $u_{x_{i_1} \dots x_{i_k}}$	$\partial_{x_{i_1}} \partial_{x_{i_2}} \dots \partial_{x_{i_k}} u$, higher order derivatives
$\partial^\beta u$	$\partial_{x_1}^{\beta_1} \dots \partial_{x_k}^{\beta_k} u$, for multiindex $\beta = (\beta_1, \dots, \beta_n)$, $\beta_i \in \mathbb{N} \cup \{0\}$, partial derivative of order $ \beta = \beta_1 + \dots + \beta_n$
Δu	$\sum_{i=1}^n \partial_{x_i x_i} u$, the Laplace operator (Laplacian) in \mathbb{R}^n
∇u	$(\partial_{x_1} u, \dots, \partial_{x_n} u)$, the gradient of u
$D^2 u$	$(\partial_{x_i x_j} u)_{i, j=1}^n$, the Hessian matrix of u
$D^k u$	for $k \in \mathbb{N}$, $(\partial_{x_{i_1} \dots x_{i_k}} u)$, the tensor of partial derivatives of k -th order

Function spaces

$C^0(E)$: the Banach space of continuous function $u : E \rightarrow \mathbb{R}$ on the set E with the uniform norm

$$\|u\|_{C^0(E)} = \sup_{x \in E} |u(x)|.$$

$C(E)$: same as $C^0(E)$.

$C^{0, \alpha}(E)$, $\alpha \in (0, 1]$: the Banach space of α -Hölder continuous functions of E with the norm

$$\|u\|_{C^{0, \alpha}(E)} = \|u\|_{C^0(E)} + [u]_{\alpha, E}, \quad [u]_{\alpha, E} = \sup \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

The quantity $[u]_{\alpha, E}$ is called the α -Hölder seminorm of u on E . When $\alpha = 1$, $C^{0, 1}$ is the space of Lipschitz continuous functions and $[u]_{1, E}$ is called the Lipschitz constant of u on E .

$C^\alpha(E)$, $\alpha \in (0, 1)$: same as $C^{0, \alpha}(E)$; we never use this notation for $\alpha = 1$.

$C^k(D)$, $k \in \mathbb{N}$: for open $D \subset \mathbb{R}^n$, the Fréchet space of functions $u : D \rightarrow \mathbb{R}$ with continuous partial derivatives $\partial^\beta u$ in D , $|\beta| \leq k$; it has the family of seminorms

$$\|u\|_{C^k(K)} = \sum_{|\beta| \leq k} \|\partial^\beta u\|_{C^0(K)} \quad \text{for } K \Subset D.$$

$C^\infty(D)$: the Fréchet space of infinitely differentiable functions in D , intersection of all $C^k(D)$, $k \in \mathbb{N}$.

$C_0^k(D)$, $k \in \mathbb{N} \cup \{0, \infty\}$: a subspace of $C^k(D)$ of functions compactly supported in D , i.e. with $\text{supp } u \Subset D$.

$C^{k,\alpha}(D)$, $k \in \mathbb{N}$, $\alpha \in (0, 1]$: the Fréchet space consisting of functions $u \in C^k(D)$ such that $\partial^\beta u \in C^{0,\alpha}(K)$, $|\beta| = k$, for any $K \Subset D$, with the family of seminorms

$$\|u\|_{C^{k,\alpha}(K)} = \|u\|_{C^k(K)} + \sum_{|\beta|=k} [\partial^\beta u]_{\alpha,K}.$$

$C^k(D \cup F)$, $C^{k,\alpha}(D \cup F)$: for a relatively open $F \subset \partial D$, similar to $C^k(D)$ and $C^{k,\alpha}(D)$ but with the k -th partial derivatives continuous (α -Hölder continuous) up to F and with the respective seminorms for $K \Subset D \cup F$.

$C_{\text{loc}}^{k,\alpha}(D)$, $C_{\text{loc}}^{k,\alpha}(D \cup F)$: same as $C^{k,\alpha}(D)$, $C^{k,\alpha}(D \cup F)$; we use this notation in the instances where we want to emphasise the local nature of the estimates.

$L^p(E)$, $1 \leq p < \infty$: for a measurable set $E \subset \mathbb{R}^n$, the Banach space of measurable functions u on E (more precisely, classes of a.e. equivalences) with the finite norm

$$\|u\|_{L^p(E)} = \left(\int_E |u(x)|^p dx \right)^{1/p}.$$

$L^\infty(E)$: the Banach space of essentially bounded measurable functions u with the norm

$$\|u\|_{L^\infty(E)} = \text{ess sup}_{x \in E} |u(x)|.$$

$W^{k,p}(D)$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$: the Sobolev space, the Banach space of functions $u \in L^p(D)$ with the (distributional) partial derivative $\partial^\beta u \in L^p(D)$, $|\beta| \leq k$; the norm is given by

$$\|u\|_{W^{k,p}(D)} = \sum_{|\beta| \leq k} \|\partial^\beta u\|_{L^p(D)}.$$

$W_0^{k,p}(D)$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$: the closure of $C_0^\infty(D)$ in $W^{k,p}(D)$.

$L_{\text{loc}}^p(D)$, $W_{\text{loc}}^{k,p}(D)$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$: the Fréchet space of measurable functions $u : D \rightarrow \mathbb{R}$ such that $u|_{D'} \in L^p(D')$ or, respectively, $W^{k,p}(D')$ for any open $D' \Subset D$.

Notation related to free boundaries

$\delta(\rho, u, x^0)$	the thickness function for the coincidence set for the solution u at x^0 , defined as $\frac{1}{\rho} \min \text{diam}(\Omega(u)^c \cap B_\rho(x^0))$
$\delta(\rho, u)$	same as $\delta(\rho, u, 0)$
$\Gamma_\pm(u)$	in Problem C , the free boundaries of positive and negative phases, $\partial\Omega^\pm(u) \cap D$
$\Gamma^0(u)$	the set of branch points in Problem C , defined as $\Gamma(u) \cap \{ \nabla u = 0\}$
$\Gamma^*(u)$	the set of nonbranch points in Problem C , defined as $\Gamma(u) \cap \{ \nabla u \neq 0\}$
$\Gamma_\kappa(u)$	the set of free boundary points in Problem S , where the blowups have homogeneity κ , i.e. $\{x^0 \in \Gamma(u) : N(0+, u, x^0) = \kappa\}$
$\Lambda(u), \Lambda$	the coincidence set; in the classical obstacle problem, the set $\{x \in D : u(x) = \psi(x)\}$; in Problem A , the set $\{x \in D : u(x) = \nabla u(x) = 0\}$; in Problem B , the set $\{x \in D : \nabla u(x) = 0\}$; in the thin obstacle problem, the set $\{x \in \mathcal{M} : u(x) = \psi(x)\}$; in Problem S , the set $\{x' \in B'_R : u(x', 0) = 0\}$
$\min \text{diam } E$	the minimum diameter of the set E , the infimum of distances between pairs of parallel planes enclosing E
$M(r, u, p), M(r, u, p, x^0)$	Monneau's monotonicity formula for the classical obstacle problem; see p. 138
$M_\kappa(r, u, p), M_\kappa(r, u, p, x^0)$	Monneau-type monotonicity formula for Problem S ; see p. 189
$N(r, u), N(r, u, x^0)$	Almgren's frequency function (at x^0); see pp. 173, 178
$\Omega(u), \Omega$	the complement of the coincidence set $\Lambda(u)$
$\Omega_\pm(u)$	in Problem C , regions of positive and negative phase defined as $\{x \in D : \pm u(x) > 0\}$
$P_R(x^0, M)$	a class of local solutions of Problems A , B , C in $B_R(x^0)$; see p. 65
$P_R(M)$	same as $P_R(0, M)$
$P_R^+(x^0, M), P_R^+(M)$	classes of local solutions of Problems A , B , C in B_R^+ (see p. 154); $P_R^+(M)$ should not be confused with $P_R^+(0, M)$

$P_\infty(M)$	a class of global solutions of Problems A , B , C ; see p. 65
$P_\infty^\pm(M)$	a class of solutions of Problems A , B , C in \mathbb{R}_+^n ; see p. 154
$\Sigma(u)$	the set of singular points; for Problem A , see p. 134; for Problem S , see p. 187
$\Sigma_\kappa(u)$	for $\kappa = 2m$, $m \in \mathbb{N}$, the same as $\Gamma_\kappa(u)$
$\Sigma^d(u)$, $\Sigma_\kappa^d(u)$	the set of singular points where the dimension of the singular set equals d ; for classical obstacle problem, see p. 141; for Problem A , see p. 146; for Problem S , see p. 192
$u_{x^0, \lambda}(x)$	a rescaling of solution u ; in Problems A , B , C defined as $u_{x^0, \lambda}(x) := \frac{1}{\lambda^2}(u(x^0 + \lambda x))$; in Problem S defined as $u_{x^0, \lambda}(x) := \frac{u(x^0 + \lambda x)}{(\lambda^{-(n-1)} \int_{\partial B_\lambda(x^0)} u^2)^{1/2}}$
$u_\lambda(x)$	same as $u_{x^0, \lambda}(x)$ when the scaling center x^0 is unambiguously identified from the context
$u_\lambda^{(\kappa)}(x)$, $u_{x^0, \lambda}^{(\kappa)}(x)$	κ -homogeneous rescalings in Problem S , defined as $\frac{1}{\lambda^\kappa} u(x^0 + \lambda x)$
$W(r, u)$, $W(r, u, x^0)$	Weiss's energy functional (at x^0) in Problems A , B , C ; see p. 74
$W_\kappa(r, u)$, $W_\kappa(r, u, x^0)$	Weiss's energy functional (at x^0) in Problem S ; see p. 188
$\Phi(r, u_+, u_-)$	the Alt-Caffarelli-Friedman functional; see p. 34
$\phi_e(r, u)$	$\Phi(r, (\partial_e u)^+, (\partial_e u)^-)$

Index

- ACF monotonicity formula, 32, 34, 36, 44, 70, 106, 108, 143, 157, 163, 179
 - ACF estimate, 39
 - case of equality, 39, 70, 157
 - for harmonic functions, 32
- Almgren's frequency formula, 173, 175
 - for harmonic functions, 174
 - for Problem **S**, 175
- Almgren's frequency function, 174
- Almgren's monotonicity formula, *see* Almgren's frequency formula
- almost monotonicity formula, 39, 41
- Alt-Caffarelli-Friedman monotonicity formula, *see* ACF monotonicity formula
- American options, perpetual, 170
- Andersson-Weiss counterexample, 49

- balanced energy, 77, 134
- blowdowns, *see* shrinkdowns
- blowups, 3, 4, 66
 - at a point, 66
 - classification
 - energetic, Problem **A**, 78
 - Problems **A,B,C**, 72
 - Problem **S**, 177
 - continuous dependence
 - classical obstacle problem, singular points, 140
 - homogeneity, 76
 - obstacle-type problems, strong, 69
 - Problem **S**, 176
 - Problem **S**, κ -homogeneous, 189
 - uniqueness
 - classical obstacle problem, singular points, 139
 - contact points, 162
 - Problem **A, B**, variable centers, 92
 - Problem **A**, regular points, 85
 - Problem **B**, regular points, 89
 - Problem **S**, singular points, 190
 - uniqueness of type, 72
 - Problems **A, B, C**, 73
 - with variable centers (over a sequence), 66, 92
- boundary Harnack principle, 120, 123, 186
- boundary thin obstacle problem, 169
- branch points, *see* Problem **C**
- Brownian motion, 14

- C^1 touch, 159, 161
- Caffarelli-Jerison-Kenig estimate, *see* CJK estimate
- Calderón-Zygmund estimates, 17, 21
- Cauchy-Kovalevskaya theorem, 10, 25, 188, 200
- characteristic constant, 35, 38
- CJK almost monotonicity formula, *see* CJK estimate
- CJK estimate, 41, 44, 48
- classical obstacle problem, 1, 7, 8
 - $C^{1,1}$ regularity, 31
 - $W^{2,p}$ regularity, 18
 - blowups
 - continuous dependence, singular points, 140
 - uniqueness, singular points, 139
 - global solutions, 100
 - convexity, 100
 - optimal regularity, 29
 - singular set
 - structure, 141

- coercive boundary conditions, 96, 98
 coincidence set, 8
 thin obstacle problem, 168
 complementarity conditions, 168
 complementarity problem, 9
 composite membrane, 14
 cone of monotonicity, 83, 85
 contact points, 153, 154
 cross-shaped singularity, 51
- directional monotonicity, 97
 Problem **A**, 84, 85, 117
 Problem **B**, 88, 89, 118
 Problem **C**, 93, 94
 Problem **S**, 183, 185
 Dirichlet energy, 7
 Dirichlet principle, 7
 Dirichlet problem, 7
- eigenvalue, 14, 36, 38, 135, 143, 146
- filling holes method, 171
 fixed boundary, 153
 C^1 touch, 159, 161
 optimal regularity, up to, 45
 tangential touch, 159
 fractional Laplacian, 170
 free boundary, 1, 9, 16
 $(n - 1)$ -Hausdorff measure, 62
 C^1 regularity
 Problem **A**, regular points, 87
 Problem **B**, regular points, 88
 Problem **C**, branch points, 93
 Problem **C**, closeness to two-half-space solution, 124
 C^1 touch, *see* fixed boundary
 $C^{1,\alpha}$ regularity
 Problems **A**, **B**, 120
 $C^{1,\text{Dini}}$ regularity
 Problem **C**, counterexample, 125
 $C^{2,\alpha}$ regularity
 Problems **A**, **B**, 128
 flatness
 Problem **A**, 83
 higher regularity
 Problems **A**, **B**, 91, 128
 Lebesgue measure, 62
 Lipschitz implies $C^{1,\alpha}$, 122, 124, 183
 Lipschitz regularity
 Problem **A**, regular points, 85, 86
 Problem **A**, thickness condition, 115, 117, 118
 Problem **B**, thickness condition, 118
 Problem **C**, closeness to two-half-space solution, 119
 Problem **S**, regular set, 183
 porosity, 61
- real analyticity
 Problems **A**, **B**, 92, 131
 Problem **C**, nonbranch points, 95
 tangential touch, *see* fixed boundary
 thickness implies Lipschitz, 118
 thin, 168
 touch with fixed boundary, *see* fixed boundary
 boundary
- free boundary points
 classification
 Problems **A**, **B**, **C**, 73
 Problem **S**, 178
 one-phase (positive, negative), 73
 regular, singular, 73
 regular
 Problem **A**, 73, 83, 85–87, 91, 92
 Problem **B**, 73, 88–92
 Problem **S**, 183
 singular, 73, 133, 134, 138, 186
 Schaeffer examples, 136
 two-phase, 74
 branch, nonbranch, 74
- frequency function, *see* Almgren's frequency function
 Friedland-Hayman inequality, 35, 36
- global solutions, 65, 99
 approximations by, 109
 Problems **A**, **B**, 109
 Problem **A**, in half-spaces, 159
 Problem **C**, 111
 Problem **C**, in half-spaces, 160
 classical obstacle problem, 100
 convexity, 100
 in half-spaces, 154, 155
 approximations by, 159
 classification, 155, 156
 Problems **A**, **B**, 101
 compact complement, 103
 unbounded complement, 106
 Problem **C**, 108
 Problem **S**, 177
 Problem **S**, homogeneous, 177, 178, 180
 growth estimate, 181
- half-space solutions, 72
 positive, negative, 72
 harmonic continuation, 10
 Harnack inequality, 31
 hodograph-Legendre transformation
 partial, *see* partial hodograph-Legendre transformation
- homogeneous harmonic polynomials, 33, 180
 homogeneous quadratic polynomials, 69, 135
 Hopf principle, 127

- implicit function theorem, 196
- John's ellipsoid lemma, 110
- κ -homogeneous blowups, 189
- κ -homogeneous rescalings, 189
- Lévy process, α -stable, 170
- local solutions, 65
 - near flat boundaries, 154
- lower-dimensional free boundary, *see* thin free boundary
- membrane, 8
 - composite, 14
 - semipermeable, 169
 - two-phase, 12, 16
- minimal diameter, 109
- minimal homogeneity, 179
- modulus of continuity, 118, 123, 134, 143, 147, 191
- Monneau's monotonicity formula, *see* Monneau-type monotonicity formula
- Monneau-type monotonicity formula
 - classical obstacle problem, 138, 141
 - Problem **S**, 189, 191
- monotone function of one variable, 69
- monotonicity formula
 - ACF (Alt-Caffarelli-Friedman), *see* ACF monotonicity formula
 - ACF estimate, *see* ACF estimate
 - Almgren, *see* Almgren's frequency formula
 - CJK (Caffarelli-Jerison-Kenig) estimate, *see* CJK estimate
 - Monneau, *see* Monneau-type monotonicity formula
 - Weiss-type, *see* Weiss-type monotonicity formula
- Newtonian potential, 10, 18, 25, 103
- no-sign obstacle problem, 11, 16, *see also* Problem **A**
- nonbranch points, *see* Problem **C**
- nondegeneracy, 57
 - of the gradient, 60
 - Problem **A**, 58
 - Problem **B**, 59
 - Problem **C**, 60
 - Problem **S**, singular points, 190
 - up to the boundary, 125, 155, 165
- NTA domains, 122, 132
- obstacle, 8
 - thin, 168
 - zero, 9
- obstacle problem
 - boundary thin, 169
 - classical, 7, 8
 - no-sign, 11, 16
 - thin, *see* thin obstacle problem
 - two-phase, 12, 16
 - unstable, 14, 54
 - non- $C^{1,1}$ solution, 49
 - obstacle-type problems, 17
 - optimal regularity, 42
 - up to the fixed boundary, 45
 - strong blowups, 69
 - optimal pricing, 170
 - optimal regularity
 - classical obstacle problem, 29
 - obstacle-type problems, 42
 - up to the fixed boundary, 45
 - Problem **S**, 181
 - optimal stopping, 14
 - osmosis, 169
 - osmotic pressure, 169
 - OT₁-OT₂** conditions, 17
- partial hodograph-Legendre transformation
 - first order, 129
 - zeroth order, 95
- penalization, 9, 26, 171
- penalized problem, 26, 171
- permeability, 169
- Poisson equation, 8
- polynomial solutions, 72, 135
 - approximations by, 135
 - positive, negative, 72
- Pompeiu problem, 11
 - Pompeiu property, 11
- porosity of free boundary, 61
- porous set, 61
 - locally, 61
- potential theory, 10
- problem
 - A**, 16
 - B**, 16
 - C**, 16
 - S**, 170
 - potential theory, from, 10
 - interior temperature control, 13
 - obstacle-type, *see* obstacle-type problems
 - Pompeiu, 11
 - superconductivity, from, 11, 16
 - two-phase membrane, 12, 16
- Problem **A**, 16
 - blowups
 - classification, 72
 - global solutions, 101
 - nondegeneracy, 58
 - shrinkdowns, 102
 - singular set, 143
 - structure, 146

- Problem **B**, 16
 - blowups
 - classification, 72
 - global solutions, 101
 - nondegeneracy, 59
 - patches, 11, 12, 25
 - local structure, 88
 - shrinkdowns, 102
- Problem **C**, 16
 - blowups
 - classification, 72
 - branch points, 74, 92
 - global solutions, 108
 - nonbranch points, 95
 - nondegeneracy, 60
 - shrinkdowns, 108
- Problem **S**, 170
 - $C^{1,\alpha}$ regularity, 171, 197
 - blowups, 176
 - homogeneity, 177
 - minimal homogeneity, 179
 - classification of free boundary points, 178
 - directional monotonicity, 183, 185
 - growth estimate, 181
 - regular set, 183
 - rescalings, 176
 - singular set, 186
 - structure, 192
- quadratic growth, 30
- quadrature domains, 2, 11
- regular free boundary points, *see* free boundary points
- regular points, *see* free boundary points
- regular set
 - Problem **S**, 183
- regularization, 9, 20, 22, 49
- rescalings, 66
 - Problem **S**, 176
 - Problem **S**, κ -homogeneous, 189
- Schaeffer examples of singular points, 136
- Schauder fixed point theorem, 49
- Schiffer conjecture, 11
- semipermeable membranes, 169
- shrinkdowns, 67
 - Problem **A**, **B**, 102
 - Problem **C**, 108
- sign condition, 1
- Signorini boundary conditions, 169, 171
- Signorini problem, *see* thin obstacle problem
- singular free boundary points, *see* free boundary points
- singular points, *see* free boundary points
- singular set
 - classical obstacle problem, 138
 - Problem **A**, 133, 143
 - Problem **S**, 186
 - structure
 - classical obstacle problem, 141
 - Problem **A**, 146
 - Problem **S**, 192
 - solutions
 - $C^{1,1}$ regularity, 29, 31, 42
 - counterexample, 49
 - up to the boundary, 45
 - $W^{2,p}$ regularity, 17
 - directional monotonicity, *see* directional monotonicity
 - global, *see* global solutions
 - half-space, 72
 - positive, negative, 72
 - in the sense of distributions, 8, 16, 24
 - limits, 67
 - local, 65
 - near fixed boundary
 - C^1 touch, 159
 - tangential touch, 159
 - polynomial, 72
 - positive, negative, 72
 - two-half-space, 72
 - spherical Laplacian, 38
 - stopping time, 14
 - superconductivity problem, 11, 16, *see also* Problem **B**
 - tangential touch, 159
 - temperature control, 13
 - thickness condition
 - Problem **A**, 117, 118
 - Problem **B**, 118
 - thickness function, 109, 134
 - thin free boundary, 15, 168
 - thin obstacle, 168
 - thin obstacle problem, 167, 168, *see also* Problem **S**
 - two-half-space solutions, 72
 - two-phase free boundary points, *see* free boundary points
 - two-phase membrane problem, 12, 16, *see also* Problem **C**
 - $W^{2,p}$ regularity, 22
 - two-phase points, *see* free boundary points
- unstable obstacle problem, 14, 49, 54
- Weiss's energy functional, 74
 - in half-spaces, 158
- Weiss's monotonicity formula, *see* Weiss-type monotonicity formula
- Weiss-type monotonicity formula
 - near fixed boundary, 158, 165

Problems **A**, **B**, **C**, 74–76, 103, 139
Problem **S**, 188, 189
 unstable obstacle problem, 54
Whitney’s extension theorem, 141, 194
Wiener process, 14

zero obstacle, 9
zero thin obstacle, 170

The regularity theory of free boundaries flourished during the late 1970s and early 1980s and had a major impact in several areas of mathematics, mathematical physics, and industrial mathematics, as well as in applications. Since then the theory continued to evolve. Numerous new ideas, techniques, and methods have been developed, and challenging new problems in applications have arisen. The main intention of the authors of this book is to give a coherent introduction to the study of the regularity properties of free boundaries for a particular type of problems, known as obstacle-type problems. The emphasis is on the methods developed in the past two decades. The topics include optimal regularity, nondegeneracy, rescalings and blowups, classification of global solutions, several types of monotonicity formulas, Lipschitz, C^1 , as well as higher regularity of the free boundary, structure of the singular set, touch of the free and fixed boundaries, and more.

The book is based on lecture notes for the courses and mini-courses given by the authors at various locations and should be accessible to advanced graduate students and researchers in analysis and partial differential equations.

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