Ordinary Differential Equations

Qualitative Theory

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Graduate Studies in Mathematics
Volume 137

American Mathematical Society

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http://dx.doi.org/10.1090/gsm/137

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Providence, Rhode Island

This work was originally published in Portuguese by IST Press under the title "Equações Diferenciais: Teoria Qualitativa" by Luis Barreira and Clàudia Valls, © IST Press 2010, Instituto Superior Técnico. All Rights Reserved.

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Translated by the authors.

2010 *Mathematics Subject Classification.* Primary 34-01, 34Cxx, 34Dxx, 37Gxx, 37Jxx.

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Library of Congress Cataloging-in-Publication Data

Barreira, Luis, 1968–
[Equações diferenciais. English]
Ordinary differential equations : qualitative theory / Luis Barreira, Claudia Valls ; translated by the authors.
p. cm. – (Graduate studies in mathematics ; v. 137)
Includes bibliographical references and index.
ISBN 978-0-8218-8749-3 (alk. paper)
1. Differential equations–Qualitative theory. I. Valls, Claudia, 1973– II. Title.

QA372.B31513 2010
515′.352–dc23 2012010848

10 9 8 7 6 5 4 3 2 1 17 16 15 14 13 12

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Preface

The main objective of this book is to give a comprehensive introduction to the qualitative theory of ordinary differential equations. In particular, among other topics, we study the existence and uniqueness of solutions, phase portraits, linear equations and their perturbations, stability and Lyapunov functions, hyperbolicity, and equations in the plane.

The book is also intended to serve as a bridge to important topics that are often left out of a second course of ordinary differential equations. Examples include the smooth dependence of solutions on the initial conditions, the existence of topological and differentiable conjugacies between linear systems, and the Hölder continuity of the conjugacies in the Grobman–Hartman theorem. We also give a brief introduction to bifurcation theory, center manifolds, normal forms, and Hamiltonian systems.

We describe mainly notions, results and methods that allow one to discuss the qualitative properties of the solutions of an equation without solving it explicitly. This can be considered the main aim of the qualitative theory of ordinary differential equations.

The book can be used as a basis for a second course of ordinary differential equations. Nevertheless, it has more material than the standard courses, and so, in fact, it can be used in several different ways and at various levels. Among other possibilities, we suggest the following courses:

a) advanced undergraduate/beginning graduate second course: Chapters 1–5 and 7–8 (without Sections 1.4, 2.5 and 8.3, and without the proofs of the Grobman–Hartman and Hadamard–Perron theorems);

b) advanced undergraduate/beginning graduate course on equations in the plane: Chapters 1–3 and 6–7;

c) advanced graduate course on stability: Chapters 1–3 and 8–9;

d) advanced graduate course on hyperbolicity: Chapters 1–5.

Other selections are also possible, depending on the audience and on the time available for the course. In addition, some sections can be used for short expositions, such as Sections 1.3.2, 1.4, 2.5, 3.3, 6.2 and 8.3.

Other than some basic pre-requisites of linear algebra and differential and integral calculus, all concepts and results used in the book are recalled along the way. Moreover, (almost) everything is proven, with the exception of some results in Chapters 8 and 9 concerning more advanced topics of bifurcation theory, center manifolds, normal forms and Hamiltonian systems. Being self-contained, the book can also serve as a reference or for independent study.

Now we give a more detailed description of the contents of the book. Part 1 is dedicated to basic concepts and linear equations.

- In Chapter 1 we introduce the basic notions and results of the theory of ordinary differential equations, in particular, concerning the existence and uniqueness of solutions (Picard–Lindelöf theorem) and the dependence of solutions on the initial conditions. We also establish the existence of solutions of equations with a continuous vector field (Peano's theorem). Finally, we give an introduction to the description of the qualitative behavior of the solutions in the phase space.
- In Chapter 2 we consider the particular case of (nonautonomous) linear equations and we study their fundamental solutions. It is often useful to see an equation as a perturbation of a linear equation, and to obtain the solutions (even if implicitly) using the variation of parameters formula. This point of view is often used in the book. We then consider the particular cases of equations with constant coefficients and equations with periodic coefficients. More advanced topics include the C^1 dependence of solutions on the initial conditions and the existence of topological conjugacies between linear equations with hyperbolic matrices of coefficients.

Part 2 is dedicated to the study of stability and hyperbolicity.

- In Chapter 3, after introducing the notions of stability and asymptotic stability, we consider the particular case of linear equations, for which it is possible to give a complete characterization of these notions in terms of fundamental solutions. We also consider the particular cases of equations with constant coefficients and equations with periodic coefficients. We then discuss the persistence of asymptotic stability under sufficiently small perturbations of an asymptotically stable linear equation. We also give an introduction to the theory of Lyapunov functions, which sometimes yields the stability of a given solution in a more or less automatic manner.

- In Chapters 4–5 we introduce the notion of hyperbolicity and we study some of its consequences. Namely, we establish two key results on the behavior of the solutions in a neighborhood of a hyperbolic critical point: the Grobman–Hartman and Hadamard–Perron theorems. The first shows that the solutions of a sufficiently small perturbation of a linear equation with a hyperbolic critical point are topologically conjugate to the solutions of the linear equation. The second shows that there are invariant manifolds tangent to the stable and unstable spaces of a hyperbolic critical point. As a more advanced topic, we show that all conjugacies in the Grobman–Hartman theorem are Hölder continuous. We note that Chapter 5 is more technical: the exposition is dedicated almost entirely to the proof of the Hadamard–Perron theorem. In contrast to what happens in other texts, our proof does not require a discretization of the problem or additional techniques that would only be used here. We note that the material in Sections 5.3 and 5.4 is used nowhere else in the book.

In Part 3 we describe results and methods that are particularly useful in the study of equations in the plane.

- In Chapter 6 we give an introduction to index theory and its applications to differential equations in the plane. In particular, we describe how the index of a closed path with respect to a vector field varies with the path and with the vector field. We then present several applications, including a proof of the existence of a critical point inside any periodic orbit, in the sense of Jordan's curve theorem.
- In Chapter 7 we give an introduction to the Poincaré–Bendixson theory. After introducing the notions of α-limit and ω-limit sets, we show that bounded semiorbits have nonempty, compact and connected α-limit and ω-limit sets. Then we establish one of the important results of the qualitative theory of ordinary differential equations in the plane, the Poincaré–Bendixson theorem. In particular, it yields a criterion for the existence of periodic orbits.

Part 4 is of a somewhat different nature and it is only here that not everything is proved. Our main aim is to make the bridge to important topics that are often left out of a second course of ordinary differential equations.

- In Chapter 8 we give an introduction to bifurcation theory, with emphasis on examples. We then give an introduction to the theory of center manifolds, which often allows us to reduce the order of an equation in the study of stability or the existence of bifurcations. We also give an introduction to the theory of normal forms that aims to eliminate through a change of variables all possible terms in the original equation.

- Finally, in Chapter 9 we give an introduction to the theory of Hamiltonian systems. After introducing some basic notions, we describe several results concerning the stability of linear and nonlinear Hamiltonian systems. We also consider the notion of integrability and the Liouville–Arnold theorem on the structure of the level sets of independent integrals in involution. In addition, we describe the basic ideas of the KAM theory.

The book also includes numerous examples that illustrate in detail the new concepts and results as well as exercises at the end of each chapter.

Luis Barreira and Claudia Valls
Lisbon, February 2012

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This textbook provides a comprehensive introduction to the qualitative theory of ordinary differential equations. It includes a discussion of the existence and uniqueness of solutions, phase portraits, linear equations, stability theory, hyperbolicity and equations in the plane. The emphasis is primarily on results and methods that allow one to analyze qualitative properties of the solutions without solving the equations explicitly. The text includes numerous examples that illustrate in detail the new concepts and results as well as exercises at the end of each chapter. The book is also intended to serve as a bridge to important topics that are often left out of a course on ordinary differential equations. In particular, it provides brief introductions to bifurcation theory, center manifolds, normal forms and Hamiltonian systems.

ISBN 978-0-8218-8749-3

9 780821 887493

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