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Semiclassical Analysis

Maciej Zworski

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PREFACE

This book originated with a course I taught at UC Berkeley during the spring of 2003, with class notes taken by my colleague Lawrence C. Evans. Various versions of these notes have been available on-line as the *Evans-Zworski lecture notes on semiclassical analysis* and our original intention was to use them as the basis of a coauthored book. Craig Evans's contributions to the current manuscript can be recognized by anybody familiar with his popular partial differential equations (PDE) text [**E**]. In the end, the scope of the project and other commitments prevented Craig Evans from participating fully in the final stages of the effort, and he decided to withdraw from the responsibility of authorship, generously allowing me to make use of the contributions he had already made. I and my readers owe him a great debt, for this book would never have appeared without his participation.

Semiclassical analysis provides PDE techniques based on the *classical-quantum* (particle-wave) correspondence. These techniques include such well-known tools as geometric optics and the Wentzel–Kramers–Brillouin (WKB) approximation. Examples of problems studied in this subject are high energy eigenvalue asymptotics or effective dynamics for solutions of evolution equations. From the mathematical point of view, semiclassical analysis is a branch of *microlocal analysis* which, broadly speaking, applies *harmonic analysis* and *symplectic geometry* to the study of linear and non-linear PDE.

The book is intended to be a graduate level text introducing readers to semiclassical and microlocal methods in PDE. It is augmented in later chapters with many specialized advanced topics. Readers are expected to have reasonable familiarity with standard PDE theory (as recounted, for example, in Parts I and II of [**E**]), as well as a basic understanding of linear functional analysis. On occasion familiarity with differential forms will also prove useful.

Several excellent treatments of semiclassical analysis have appeared recently. The book [**D-S**] by Dimassi and Sjöstrand starts with the WKB-method, develops the general semiclassical calculus, and then provides high-tech spectral asymptotics. Martinez [**M**] provides a systematic development of FBI transform techniques, with applications to microlocal exponential estimates and to propagation estimates. This text is intended as a more elementary, but much broader, introduction. Except for the general symbol calculus, for which we followed Chapter 7 of [**D-S**], there is little overlap with these other two texts or with the influential books by Helffer [**He**] and by Robert [**R**]. Guillemin and Sternberg [**G-St1**] offer yet another perspective on the subject, very much complementary to that given here. Their notes concentrate on global and functorial aspects of semiclassical analysis, in particular on the theory of Fourier integral operators and on trace formulas.

The approach to semiclassical analysis presented here is influenced by my long collaboration with Johannes Sjöstrand. I would like to thank him for sharing his philosophy and insights over the years. I first learned microlocal analysis from Richard Melrose, Victor Guillemin, and Gunther Uhlmann, and it is a pleasure to acknowledge my debt to them. Discussions of semiclassical physics and chemistry with Stéphane Nonnenmacher, Paul Brumer, William H. Miller, and Robert Littlejohn have been enjoyable and valuable. They have added a lot to my appreciation of the subject.

I am especially grateful to Stéphane Nonnenmacher, Semyon Dyatlov, Claude Zuily, Oran Gannot, Xi Chen, Hans Christianson, Jeff Galkowski, Justin Holmer, Long Jin, Gordon Linoff, and Steve Zelditch for their very careful reading of the earlier versions of this book and for their many valuable comments and corrections.

My thanks also go to Faye Yeager for typing the original lecture notes and to Jonathan Dorfman for TEX advice. Stephen Moye at the AMS provided fantastic help on deeper TEX issues and Arlene O'Sean's excellent copyediting removed many errors and inconsistencies.

I will maintain on my website at the UC Berkeley Mathematics Department `http://math.berkeley.edu/~zworski` a list of errata and corrections, as well as at the American Mathematical Society's website `www.ams.org/bookpages/gsm-138`. Please let me know about any errors you find.

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Maciej Zworski

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This book is an excellent, comprehensive introduction to semiclassical analysis. I believe it will become a standard reference for the subject.

—Alejandro Uribe, University of Michigan

Semiclassical analysis provides PDE techniques based on the *classical-quantum* (particle-wave) correspondence. These techniques include such well-known tools as geometric optics and the Wentzel–Kramers–Brillouin approximation. Examples of problems studied in this subject are high energy eigenvalue asymptotics and effective dynamics for solutions of evolution equations. From the mathematical point of view, semiclassical analysis is a branch of *microlocal analysis* which, broadly speaking, applies *harmonic analysis* and *symplectic geometry* to the study of linear and nonlinear PDE. The book is intended to be a graduate level text introducing readers to semiclassical and microlocal methods in PDE. It is augmented in later chapters with many specialized advanced topics which provide a link to current research literature.

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