Ordinary Differential Equations and Dynamical Systems

Gerald Teschl

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To Susanne, Simon, and Jakob

Contents

Preface

About

When you publish a textbook on such a classical subject the first question you will be faced with is: Why another book on this subject? Everything started when I was supposed to give the basic course on *Ordinary Differential Equations* in Summer 2000. (At that time the course met 5 hours per week.) While there were many good books on the subject available, none of them quite fit my needs. I wanted a concise but rigorous introduction with full proofs that also covered classical topics such as Sturm–Liouville boundary value problems, differential equations in the complex domain, as well as modern aspects of the qualitative theory of differential equations. The course was continued with a second part on *Dynamical Systems and Chaos* in Winter 2000/01, and the notes were extended accordingly. Since then the manuscript has been rewritten and improved several times according to the feedback I got from students over the years when I redid the course. Moreover, since I had the notes on my homepage from the very beginning, this triggered a significant amount of feedback as well, from students who reported typos, incorrectly phrased exercises, etc., to colleagues who reported errors in proofs and made suggestions for improvements, to editors who approached me about publishing the notes. All this interest eventually resulted in a Chinese translation of an earlier version of the book. Moreover, if you google for the manuscript, you can see that it is used at several places worldwide, linked as a reference at various sites, including Wikipedia. Finally, Google Scholar will tell you that it is even cited in several publications. Hence I decided that it was time to turn it into a *real* book.

Content

This book's main aim is to give a self-contained introduction to the field of ordinary differential equations with emphasis on the dynamical systems point of view while still keeping an eye on classical tools as pointed out before.

The first part is what I typically cover in the introductory course for bachelor's level students. Of course it is typically not possible to cover everything and one has to skip some of the more advanced sections. Moreover, it might also be necessary to add some material from the first chapter of the second part to meet curricular requirements.

The second part is a natural continuation beginning with planar examples (culminating in the generalized Poincaré–Bendixson theorem), continuing with the fact that things get much more complicated in three and more dimensions, and ending with the stable manifold and the Hartman–Grobman theorem.

The third and last part gives a brief introduction to chaos, focusing on two selected topics: Interval maps with the logistic map as the prime example plus the identification of homoclinic orbits as a source for chaos and the Melnikov method for perturbations of periodic orbits and for finding homoclinic orbits.

Prerequisites

This book requires only some basic knowledge of calculus, complex functions, and linear algebra. In addition, I have tried to show how a computer system, *Mathematica*[1], can help with the investigation of differential equations. However, the course is not tied to *Mathematica* and any similar program can be used as well.

Updates

The AMS is hosting a Web page for this book at

`http://www.ams.org/bookpages/gsm-140/`

where updates, corrections, and other material may be found, including a link to material on my website:

`http://www.mat.univie.ac.at/~gerald/ftp/book-ode/`

[1] *Mathematica*® is a registered trademark of Wolfram Research, Inc.

There you can also find an accompanying *Mathematica* notebook with the code from the text plus some additional material.

Acknowledgments

I wish to thank my students, Ada Akerman, Kerstin Ammann, Jörg Arnberger, Alexander Beigl, Paolo Capka, Jonathan Eckhardt, Michael Fischer, Anna Geyer, Ahmed Ghneim, Hannes Grimm-Strele, Tony Johansson, Klaus Kröncke, Alice Lakits, Simone Lederer, Oliver Leingang, Johanna Michor, Thomas Moser, Markus Müller, Andreas Németh, Andreas Pichler, Tobias Preinerstorfer, Jin Qian, Dominik Rasipanov, Martin Ringbauer, Reinaldo Garcia Rosario, Simon Rößler, Robert Stadler, Shelby Stanhope, Raphael Stuhlmeier, Gudrun Szewieczek, Gerhard Tulzer, Paul Wedrich, Florian Wisser, and colleagues, Edward Dunne, Klemens Fellner, Giuseppe Ferrero, Ilse Fischer, Delbert Franz, Heinz Hanßmann, Daniel Lenz, Jim Sochacki, and Eric Wahlén, who have pointed out several typos and made useful suggestions for improvements. Finally, I would like to thank the anonymous referees for valuable suggestions that improved the presentation of the material.

If you find any errors or if you have comments or suggestions (no matter how small), please let me know.

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April 2012

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Bibliographical notes

The aim of this section is not to give a comprehensive guide to the literature, but to document the sources from which I have learned the materials and which I have used during the preparation of this text. In addition, I will point out some standard references for further reading.

Chapter 2: Initial value problems

The material in this section is, of course, standard. Classical references are Coddington and Levinson [**7**], Hartman [**14**], Hale [**13**], Ince [**23**], and Walter [**44**]. More modern introductions are Amann [**2**], Arnold [**4**], Hirsch, Smale, and Devaney [**18**], Robinson [**36**], Verhulst [**43**], and Wiggins [**48**].

Further uniqueness results can be found in the book by Walter [**44**] (see the supplement to §12). There you can also find further technical improvements, in particular, for the case alluded to in the remark after Corollary 2.6 (see the second supplement to §10).

More on *Mathematica* in general can be found in the standard documentation [**49**] and in connection with differential equations in [**11**] and [**39**].

General purpose references are the handbooks by Kamke [**24**] and Zwillinger [**50**].

Chapter 3: Linear equations

Again this material is mostly standard and the same references as those for the previous chapter apply. More information in particular on n'th order equations can be found in Coddington and Levinson [**7**], Hartman [**14**], and Ince [**23**].

Chapter 4: Differential equations in the complex domain

Classical references with more information on this topic include Coddington and Levinson [**7**], Hille [**17**], and Ince [**23**]. For a more modern point of view see Ilyashenko and Yakovenko [**21**]. The topics here are also closely connected with the theory of special functions. See Beals and Wong [**5**] for a modern introduction. For a collection of properties of special function a standard reference is the *NIST Handbook of Mathematical Functions* [**30**].

Chapter 5: Boundary value problems

Classical references include Coddington and Levinson [**7**] and Hartman [**14**]. A nice informal treatment (although in German) can be found in Jänich [**22**]. More on Hill's equation can be found in Magnus and Winkler [**28**]. For a modern introduction to singular Sturm–Liouville problems, see the books by Weidmann [**45**], [**46**], my textbook [**42**], and the book by Levitan and Sargsjan [**27**]. A reference with more applications and numerical methods is by Hastings and McLeod [**16**].

Chapter 6: Dynamical systems

Classical references include Chicone [**6**], Guckenheimer and Holmes [**12**], Hasselblat and Katok [**15**], [**25**], Hirsch, Smale, and Devaney [**18**], Palis and de Melo [**33**], Perko [**34**], Robinson [**35**], [**36**], Ruelle [**38**], Verhulst [**43**], and Wiggins [**47**], [**48**]. In particular, [**15**] and [**25**] emphasize ergodic theory, which is not covered here.

More on the connections with Lie groups and symmetries of differential equations, briefly mentioned in Problem 6.5, can be found in the monograph by Olver [**31**].

Chapter 7: Planar dynamical systems

The proof of the Poincaré–Bendixson theorem follows Palis and de Melo [**33**]. More on ecological models can be found in Hofbauer and Sigmund [**19**]. Hirsch, Smale, and Devaney [**18**] and Robinson [**36**] also cover these topics nicely.

Chapter 8: Higher dimensional dynamical systems

More on the Lorenz equation can be found in the monograph by Sparrow [**40**]. The classical reference for Hamiltonian systems is, of course, Arnold's book [**3**] (see also [**4**]) as well as the monograph by Abraham, Marsden, and Ratiu [**1**], which also contains extensions to infinite-dimensional systems. Other references are the notes by Moser [**29**] and the monograph by Wiggins [**47**]. A brief overview can be found in Verhulst [**43**].

Chapter 9: Local behavior near fixed points

The classical reference here is Hartman [**14**]. See also Coddington and Levinson [**7**], Hale [**13**], Robinson [**35**], and Ruelle [**38**].

Chapter 10: Discrete dynamical systems

One of the classical references is the book by Devaney [**8**]. A nice introduction is provided by Holmgren [**20**]. Further references are Hasselblat and Katok [**15**], [**25**] and Robinson [**36**].

Chapter 11: Discrete dynamical systems in one dimension

The classical reference here is Devaney [**8**]. More on the Hausdorff measure can be found in Falconer [**9**]. See also Holmgren [**20**] and Robinson [**36**].

Chapter 12: Periodic solutions

For more information see Chicone [**6**], Robinson [**35**], [**36**], and Wiggins [**47**].

Chapter 13: Chaos in higher dimensional systems

A proof of the Smale–Birkhoff theorem can be found in Robinson [**35**]. See also Chicone [**6**], Guckenheimer and Holmes [**12**], and Wiggins [**47**].

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Glossary of notation

$A_\pm$	...matrix A restricted to $E^\pm(A)$, 264
$B_r(x)$	...open ball of radius r centered at x
$C(U,V)$	...set of continuous functions from U to V
$C_b(U,V)$	...set of bounded continuous functions from U to V
$C(U)$	$= C(U,\mathbb{R})$
$C^k(U,V)$	...set of k times continuously differentiable functions
$\mathbb{C}$	...the set of complex numbers
χ_A	...Characteristic polynomial of A, 103
$d(U)$	...diameter of U, 309
$d(x,y)$	...distance in a metric space
$d(x,A)$	...distance between a point x and a set A, 196
df_x	$= \frac{\partial f}{\partial x}$ Jacobian matrix of a differentiable mapping f at x
$\delta_{j,k}$	...Kronecker delta: $\delta_{j,j} = 1$ and $\delta_{j,k} = 0$ if $j \neq k$
$E^0(A)$	...center subspace of a matrix, 109
$E^\pm(A)$	...(un)stable subspace of a matrix, 109
$\mathrm{Fix}(f)$	$= \{x \mid f(x) = x\}$ set of fixed points of f, 284
$\gamma(x)$	...orbit of x, 192
$\gamma_\pm(x)$	...forward, backward orbit of x, 192
$\Gamma(z)$	...Gamma function, 126
$\mathfrak{H}_0$	...inner product space, 146
$\mathbb{I}$	...identity matrix
I_x	$= (T_-(x), T_+(x))$ maximal interval of existence, 189
$\mathrm{Ker}(A)$	...kernel of a matrix
L_μ	...logistic map, 282
Λ	...a compact invariant set
$M^\pm$	...(un)stable manifold, 258, 322

$\mathbb{N}$	$= \{1, 2, 3, \dots\}$ the set of positive integers
$\mathbb{N}_0$	$= \mathbb{N} \cup \{0\}$
$o(.)$	... Landau symbol
$O(.)$	... Landau symbol
$\Omega(f)$	... set of nonwandering points, 196
$P_\Sigma(y)$	... Poincaré map, 198
$\mathrm{Per}(f)$	$= \{x \mid f(x) = x\}$ set of periodic points of f, 284
$\Phi(t, x_0)$	... flow of a dynamical system, 189
$\Pi(t, t_0)$	... principal matrix of a linear system, 81
$\mathbb{R}$	... the set of reals
$\mathrm{Ran}(A)$	... range of a matrix
σ	... shift map on Σ_N, 305
$\sigma(A)$	... spectrum (set of eigenvalues) of a matrix, 103
Σ_N	... sequence space over N symbols, 304
$\mathrm{sign}(x)$	... $+1$ for $x > 0$ and -1 for $x < 0$; sign function
$T_\pm(x)$	... positive, negative lifetime of x, 192
$T(x)$	... period of x (if x is periodic), 192
T_μ	... tent map, 299
$\omega_\pm(x)$	... positive, negative ω-limit set of x, 193
$W^\pm$	... (un)stable set, 257, 231, 284
$\mathbb{Z}$	$= \{\dots, -2, -1, 0, 1, 2, \dots\}$ the set of integers
z	... a complex number
$\sqrt{z}$	... square root of z with branch cut along $(-\infty, 0)$
z^*	... complex conjugation
$\|.\|$	... norm in a Banach space
$\lvert.\rvert$	... Euclidean norm in $\mathbb{R}^n$ respectively $\mathbb{C}^n$
$\langle ., .. \rangle$	... scalar product in $\mathfrak{H}_0$, 146
(λ_1, λ_2)	$= \{\lambda \in \mathbb{R} \mid \lambda_1 < \lambda < \lambda_2\}$, open interval
$[\lambda_1, \lambda_2]$	$= \{\lambda \in \mathbb{R} \mid \lambda_1 \leq \lambda \leq \lambda_2\}$, closed interval
$\lfloor x \rfloor$	$= \max\{n \in \mathbb{Z} \mid n \leq x\}$, floor function
$\lceil x \rceil$	$= \min\{n \in \mathbb{Z} \mid n \geq x\}$, ceiling function
$a \wedge b$	= cross product in $\mathbb{R}^3$

Index

This book provides a self-contained introduction to ordinary differential equations and dynamical systems suitable for beginning graduate students.

The first part begins with some simple examples of explicitly solvable equations and a first glance at qualitative methods. Then the fundamental results concerning the initial value problem are proved: existence, uniqueness, extensibility, dependence on initial conditions. Furthermore, linear equations are considered, including the Floquet theorem, and some perturbation results. As somewhat independent topics, the Frobenius method for linear equations in the complex domain is established and Sturm–Liouville boundary value problems, including oscillation theory, are investigated.

The second part introduces the concept of a dynamical system. The Poincaré–Bendixson theorem is proved, and several examples of planar systems from classical mechanics, ecology, and electrical engineering are investigated. Moreover, attractors, Hamiltonian systems, the KAM theorem, and periodic solutions are discussed. Finally, stability is studied, including the stable manifold and the Hartman–Grobman theorem for both continuous and discrete systems.

The third part introduces chaos, beginning with the basics for iterated interval maps and ending with the Smale–Birkhoff theorem and the Melnikov method for homoclinic orbits.

The text contains almost three hundred exercises. Additionally, the use of mathematical software systems is incorporated throughout, showing how they can help in the study of differential equations.

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