A Course in Abstract Analysis

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For Ann,
The love of my life, the source of my happiness

Contents

Preface

I am an analyst. I use measure theory almost every day of my life. Yet for most of my career I have disliked it as a stand-alone subject and avoided teaching it. I taught a two-semester course on the subject during the second year after I earned my doctorate and never again until Fall 2010. Then I decided to teach our year long course that had a semester of measure theory followed by a semester of functional analysis, a course designed to prepare first-year graduate students for the PhD Qualifying exam. The spring before the course was to begin, I began to think about how I would present the material. In the process I discovered that with an approach different from what I was used to, there is a certain elegance in the subject.

It seems to me that the customary presentation of basic measure theory has changed little since I took it as a first-year graduate student. In addition, when I wrote my book on functional analysis [**8**], it was premised on students having completed a year long course in measure theory, something that seldom happens now. For these two reasons and because of my newly found appreciation of measure theory, I made the decision that I would write a book. For this project I resolved to look at this subject with fresh eyes, simplifying and streamlining the measure theory, and formulating the functional analysis so it depends only on the measure theory appearing in the same book. This would make for a self-contained treatment of these subjects at the level and depth appropriate for my audience. This book is the culmination of my effort.

What did I formerly find unpleasant about measure theory? It strikes me that most courses on measure theory place too much emphasis on topics I never again encountered as a working analyst. Some of these are natural enough within the framework of measure theory, but they just don't arise in the life of most mathematicians. An example is the question of the measurability of sets. To be sure we need to have our sets measurable, and this comes up in the present book; but when I studied measure theory I spent more time on this topic than I did in the more than 40 years that followed. Simply put, every set and every function I encountered after my first year in graduate school was obviously measurable. Part of my resolve when I wrote this book was to restrict such considerations to what was necessary and simplify wherever I could. Another point in the traditional approach is what strikes me as an overemphasis on pathology and subtleties. I think there are other things on which time is better spent when a student first encounters the subject.

In writing this book I continued to adhere to one of the principles I have tried to adopt in my approach to teaching over the last 20 years or so: start with the particular and work up to the general and, depending on the topic, avoid the most general form of a result unless there is a reason beyond the desire for generality. I believe students learn better this way. Starting with the most general result sometimes saves space and time in the development of the subject, but it does not facilitate learning. To compensate, in many places I provide references where the reader can access the most general form of a result.

Most of the emphasis in this book is on regular Borel measures on a locally compact metric space that is also σ-compact. Besides dealing with the setting encountered most frequently by those who use measure theory, it allows us to bypass a lot of issues. The idea is to start with a positive linear functional on $C(X)$ when X is a compact metric space and use this to generate a measure. The Riemann–Stieltjes integral furnishes a good source of examples. Needless to say, this approach calls for a great deal of care in the presentation. For example, it necessitates a discussion of linear functionals before we begin measure theory, but that is a topic we would encounter in a course like this no matter how we approached measure theory. There is also a bonus to this approach in that it gives students an opportunity to gain facility in manufacturing continuous functions with specified properties, a skill that I have found is frequently lacking after they finish studying topology and measure theory.

Chapter 1 contains the preliminary work. It starts with the Riemann–Stieltjes integral on a bounded interval. Then it visits metric spaces so that all have a common starting point, to provide some handy references, and to present results on manufacturing continuous functions, including partitions of unity, that are needed later. It then introduces topics on normed spaces needed to understand the approach to measure theory. Chapter 2 starts with a positive linear functional on $C(X)$ and shows how to generate a measure space. Then the properties of this measure space are abstracted and the theory of integration is developed for a general measure space, including the usual convergence theorems and the introduction of L^p spaces. Chapter 3 on Hilbert space covers just the basics. There is a later chapter on this subject, but here I just want to present what is needed in the following chapter so as to be sure to cover measure theory in a single semester. Chapter 4 starts by applying the Hilbert space results to obtain the Lebesgue–Radon–Nikodym Theorem. It then introduces complex-valued measures and completes the cycle by showing that when X is a σ-compact locally compact metric space, every bounded linear functional on $C_0(X)$ can be represented as integration with respect to a complex-valued Radon measure. The chapter then develops product measures and closes with a detailed examination of Lebesgue and other measures on Euclidean space, including the Fourier transform. That is the course on measure theory, and I had no difficulty covering it in a semester.

Chapter 5 begins the study of functional analysis by studying linear transformations, first on Banach spaces but quickly focusing on Hilbert space and reaching the diagonalization of a compact hermitian operator. This chapter and the subsequent ones are based on my existing book [**8**]. There are, however, significant differences. *A Course in Functional Analysis* was designed as a one-year course on the subject for students who had completed a year-long study of measure theory as well as having some knowledge of analytic functions. The second half of the present book only assumes the presentation on measures done in the first half and is meant to be covered in a semester. Needless to say, many topics in [**8**] are not touched here. Even when this book does a topic found in [**8**], it is usually treated with less generality and in a somewhat simpler form. I'd advise all readers to use [**8**] as a reference – as I did.

Chapter 6 looks at Banach spaces and presents the three pillars of functional analysis. The next chapter touches on locally convex spaces, but only to the extent needed to facilitate the presentation of duality. It does include, however, a discussion of the separation theorems that follow from the Hahn–Banach Theorem. Chapter 8 treats the relation between a Banach space and its dual space. It includes the Krein–Milman Theorem, which is applied to prove the Stone–Weierstrass Theorem. Chapter 9 returns to operator theory, but this time in the Banach space setting and gets to the Fredholm Alternative. Chapter 10 presents the basics of Banach algebras and lays the groundwork for the last chapter, which is an introduction to C*-algebras. This final chapter includes the functional calculus for normal operators and presents the characterization of their isomorphism classes.

When I taught my course I did not reach the end of the book, though I covered some topics in more detail and generality than they are covered here; I also presented some of the optional sections in this book – those that have a * in the title. Nevertheless, I wanted the readers to have access to the material on multiplicity theory for normal operators, which is one of the triumphs of mathematics. I suspect that with a good class like the one I had and avoiding the starred sections, the entire book could be covered in a year.

Biographies. I have included some biographical information whenever a mathematician's result is presented. (Pythagoras is the lone exception.) There is no scholarship on my part in this, as all the material is from secondary sources, principally what I could find on the web. In particular, I made heavy use of

http://www-history.mcs.st-andrews.ac.uk/history/BiogIndex.html

and Wikipedia. I did this as a convenience for the reader and from my experience that most people would rather have this in front of them than search it out. (A note about web addresses. There are a few others in this book and they were operational when I wrote the manuscript. We are all familiar with the fact that some web sites become moribund with time. If you experience this, just try a search for the subject at hand.)

I emphasize the personal aspects of the mathematicians we encounter along the way, rather than recite their achievements. This is especially so when I discover something unusual or endearing in their lives. I figure many students will see their achievements if they stick with the subject and most students at the start of their education won't know enough mathematics to fully appreciate the accomplishments. In addition I think the students will enjoy learning that these famous people were human beings.

Teaching. I think my job as an instructor in a graduate course is to guide the students as they learn the material, not necessarily to slog through every proof. In the book, however, I have given the details of the most tedious and technical proofs; but when I lecture I frequently tell my class, "Adults should not engage in this kind of activity in public." Students are usually amused at that, but they realize, albeit with my encouragement, that understanding a highly technical argument may be important. It certainly exposes them to a technique. Nevertheless, the least effective way to reach that understanding is to have someone stand in front of a student at a chalkboard and conscientiously go through all the details. The details should be digested by the student in the privacy of his/her office, away from public view.

I also believe in a gradual introduction of new material to the student. This is part of the reason for what I said earlier about going from the particular to the general. This belief is also reflected in making changes in some notation and terminology as we progress. A vivid example of this is the use of the term "measure." Initially it means a positive measure and then in the course of developing the material it migrates to meaning a complex-valued measure. I don't think this will cause problems; in fact, as I said, I think it facilitates learning.

Prerequisites. The reader is assumed to be familiar with the basic properties of metric spaces. In particular the concepts of compactness, connectedness, continuity, uniform continuity, and the surrounding results on these topics are assumed known. I also assume the student has had a good course in basic analysis on the real line. In particular, (s)he should know the Riemann integral and have control of the usual topics appearing in such a course. There are a few other things from undergraduate analysis that are assumed, though usually what appears here doesn't depend so heavily on their mastery.

For students. When I first studied the subject, I regarded it as very difficult. I found the break with ϵ-δ analysis dramatic, calling for a shift in thinking. A year later I wondered what all the fuss was about. So work hard at this, and I can guarantee that no matter how much trouble you have, it will eventually all clear up. Also I leave a lot of detail checking to the reader and frequently insert such things as (Why?) or (Verify!) in the text. I want you to delve into the details and answer these questions. It will check your understanding and give some perspective on the proof. I also strongly advise you to at least read all the exercises. With your schedule and taking other courses, you might not have the time to try to solve them all, but at least read them. They contain additional information. Learning mathematics is not a spectator sport.

Thanks. I have had a lot of help with this book. First my class was great, showing patience when a first draft of an argument was faulty, making comments, and pointing out typos. Specifically Brian Barg, Yosef Berman, Yeyao Hu, Tom Savistky, and David Shoup were helpful; Tanner Crowder was especially so, pointing out a number of typos and gaps. Also William J. Martin, who was an auditor, showed me a proof of Hölder's Inequality using Young's Inequality (though I decided not to use it in the book), and we had several enjoyable and useful discussions. A pair of friends helped significantly. Alejandro Rodríguez-Martínez did a reading of the penultimate draft as did William Ross. Bill, in addition to pointing out typos, made many pedagogical, stylistic, and mathematical comments which influenced the final product. I feel very fortunate to have such friends.

Needless to say, I am responsible for what you see before you.

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List of Symbols

Index

This book covers topics appropriate for a first-year graduate course preparing students for the doctorate degree. The first half of the book presents the core of measure theory, including an introduction to the Fourier transform. This material can easily be covered in a semester. The second half of the book treats basic functional analysis and can also be covered in a semester. After the basics, it discusses linear transformations, duality, the elements of Banach algebras, and C*-algebras. It concludes with a characterization of the unitary equivalence classes of normal operators on a Hilbert space.

The book is self-contained and only relies on a background in functions of a single variable and the elements of metric spaces. Following the author's belief that the best way to learn is to start with the particular and proceed to the more general, it contains numerous examples and exercises.

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