

# A Course in Abstract Analysis

**John B. Conway**

**Graduate Studies  
in Mathematics**

**Volume 141**



**American Mathematical Society**

# A Course in Abstract Analysis



# A Course in Abstract Analysis

John B. Conway

Graduate Studies  
in Mathematics

Volume 141



American Mathematical Society  
Providence, Rhode Island

## EDITORIAL COMMITTEE

David Cox (Chair)  
Daniel S. Freed  
Rafe Mazzeo  
Gigliola Staffilani

2010 *Mathematics Subject Classification*. Primary 28-01, 46-01.

---

For additional information and updates on this book, visit  
[www.ams.org/bookpages/gsm-141](http://www.ams.org/bookpages/gsm-141)

---

### Library of Congress Cataloging-in-Publication Data

Conway, John B., author.

A course in abstract analysis / John B. Conway.

pages ; cm. — (Graduate studies in mathematics ; volume 141)

Includes bibliographical references and index.

ISBN 978-0-8218-9083-7 (alk. paper)

1. Measure theory. 2. Integration, Functional. 3. Functional analysis. I. Title.

QA312.C5785 2012  
515—dc23

2012020947

---

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294 USA. Requests can also be made by e-mail to [reprint-permission@ams.org](mailto:reprint-permission@ams.org).

© 2012 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights  
except those granted to the United States Government.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines  
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 17 16 15 14 13 12

For Ann,  
The love of my life, the source of my happiness



---

# Contents

Preface	xi
Chapter 1. Setting the Stage	1
§1.1. Riemann–Stieltjes integrals	1
§1.2. Metric spaces redux	12
§1.3. Normed spaces	21
§1.4. Locally compact spaces	29
§1.5. Linear functionals	37
Chapter 2. Elements of Measure Theory	41
§2.1. Positive linear functionals on $C(X)$	41
§2.2. The Radon measure space	42
§2.3. Measurable functions	51
§2.4. Integration with respect to a measure	56
§2.5. Convergence theorems	71
§2.6. Signed measures	78
§2.7. $L^p$ -spaces	84
Chapter 3. A Hilbert Space Interlude	93
§3.1. Introduction to Hilbert space	93
§3.2. Orthogonality	98
§3.3. The Riesz Representation Theorem	103
Chapter 4. A Return to Measure Theory	107
§4.1. The Lebesgue–Radon–Nikodym Theorem	107



---

§4.2.	Complex functions and measures	114
§4.3.	Linear functionals on $C(X)$	119
§4.4.	Linear functionals on $C_0(X)$	124
§4.5.	Functions of bounded variation	127
§4.6.	Linear functionals on $L^p$ -spaces	129
§4.7.	Product measures	133
§4.8.	Lebesgue measure on $\mathbb{R}^d$	141
§4.9.	Differentiation on $\mathbb{R}^d$	144
§4.10.	Absolutely continuous functions	151
§4.11.	Convolution*	156
§4.12.	The Fourier transform*	161
Chapter 5.	Linear Transformations	171
§5.1.	Basics	171
§5.2.	Orthonormal basis	175
§5.3.	Isomorphic Hilbert spaces	179
§5.4.	The adjoint	183
§5.5.	The direct sum of Hilbert spaces	190
§5.6.	Compact linear transformations	195
§5.7.	The Spectral Theorem	202
§5.8.	Some applications of the Spectral Theorem*	205
§5.9.	Unitary equivalence*	210
Chapter 6.	Banach Spaces	213
§6.1.	Finite-dimensional spaces	213
§6.2.	Sums and quotients of normed spaces	216
§6.3.	The Hahn–Banach Theorem	220
§6.4.	Banach limits*	226
§6.5.	The Open Mapping and Closed Graph Theorems	228
§6.6.	Complemented subspaces*	232
§6.7.	The Principle of Uniform Boundedness	234
Chapter 7.	Locally Convex Spaces	237
§7.1.	Basics of locally convex spaces	237
§7.2.	Metrizable locally convex spaces*	243
§7.3.	Geometric consequences	244

---

Chapter 8. Duality	251
§8.1. Basics of duality	251
§8.2. The dual of a quotient space and of a subspace	260
§8.3. Reflexive spaces	263
§8.4. The Krein–Milman Theorem	266
§8.5. The Stone–Weierstrass Theorem	270
Chapter 9. Operators on a Banach Space	277
§9.1. The adjoint	277
§9.2. Compact operators	282
Chapter 10. Banach Algebras and Spectral Theory	287
§10.1. Elementary properties and examples	287
§10.2. Ideals and quotients	290
§10.3. Analytic functions	292
§10.4. The spectrum	297
§10.5. The spectrum of an operator	302
§10.6. The spectrum of a compact operator	310
§10.7. Abelian Banach algebras	314
Chapter 11. $C^*$ -Algebras	319
§11.1. Elementary properties and examples	319
§11.2. Abelian $C^*$ -algebras	323
§11.3. Positive elements in a $C^*$ -algebra	327
§11.4. A functional calculus for normal operators	332
§11.5. The commutant of a normal operator	342
§11.6. Multiplicity theory	345
Appendix	355
§A.1. Baire Category Theorem	355
§A.2. Nets	356
Bibliography	359
List of Symbols	361
Index	363



---

# Preface

I am an analyst. I use measure theory almost every day of my life. Yet for most of my career I have disliked it as a stand-alone subject and avoided teaching it. I taught a two-semester course on the subject during the second year after I earned my doctorate and never again until Fall 2010. Then I decided to teach our year long course that had a semester of measure theory followed by a semester of functional analysis, a course designed to prepare first-year graduate students for the PhD Qualifying exam. The spring before the course was to begin, I began to think about how I would present the material. In the process I discovered that with an approach different from what I was used to, there is a certain elegance in the subject.

It seems to me that the customary presentation of basic measure theory has changed little since I took it as a first-year graduate student. In addition, when I wrote my book on functional analysis [8], it was premised on students having completed a year long course in measure theory, something that seldom happens now. For these two reasons and because of my newly found appreciation of measure theory, I made the decision that I would write a book. For this project I resolved to look at this subject with fresh eyes, simplifying and streamlining the measure theory, and formulating the functional analysis so it depends only on the measure theory appearing in the same book. This would make for a self-contained treatment of these subjects at the level and depth appropriate for my audience. This book is the culmination of my effort.

What did I formerly find unpleasant about measure theory? It strikes me that most courses on measure theory place too much emphasis on topics I never again encountered as a working analyst. Some of these are natural enough within the framework of measure theory, but they just don't arise

in the life of most mathematicians. An example is the question of the measurability of sets. To be sure we need to have our sets measurable, and this comes up in the present book; but when I studied measure theory I spent more time on this topic than I did in the more than 40 years that followed. Simply put, every set and every function I encountered after my first year in graduate school was obviously measurable. Part of my resolve when I wrote this book was to restrict such considerations to what was necessary and simplify wherever I could. Another point in the traditional approach is what strikes me as an overemphasis on pathology and subtleties. I think there are other things on which time is better spent when a student first encounters the subject.

In writing this book I continued to adhere to one of the principles I have tried to adopt in my approach to teaching over the last 20 years or so: start with the particular and work up to the general and, depending on the topic, avoid the most general form of a result unless there is a reason beyond the desire for generality. I believe students learn better this way. Starting with the most general result sometimes saves space and time in the development of the subject, but it does not facilitate learning. To compensate, in many places I provide references where the reader can access the most general form of a result.

Most of the emphasis in this book is on regular Borel measures on a locally compact metric space that is also  $\sigma$ -compact. Besides dealing with the setting encountered most frequently by those who use measure theory, it allows us to bypass a lot of issues. The idea is to start with a positive linear functional on  $C(X)$  when  $X$  is a compact metric space and use this to generate a measure. The Riemann–Stieltjes integral furnishes a good source of examples. Needless to say, this approach calls for a great deal of care in the presentation. For example, it necessitates a discussion of linear functionals before we begin measure theory, but that is a topic we would encounter in a course like this no matter how we approached measure theory. There is also a bonus to this approach in that it gives students an opportunity to gain facility in manufacturing continuous functions with specified properties, a skill that I have found is frequently lacking after they finish studying topology and measure theory.

Chapter 1 contains the preliminary work. It starts with the Riemann–Stieltjes integral on a bounded interval. Then it visits metric spaces so that all have a common starting point, to provide some handy references, and to present results on manufacturing continuous functions, including partitions of unity, that are needed later. It then introduces topics on normed spaces needed to understand the approach to measure theory. Chapter 2 starts with a positive linear functional on  $C(X)$  and shows how to generate a measure

space. Then the properties of this measure space are abstracted and the theory of integration is developed for a general measure space, including the usual convergence theorems and the introduction of  $L^p$  spaces. Chapter 3 on Hilbert space covers just the basics. There is a later chapter on this subject, but here I just want to present what is needed in the following chapter so as to be sure to cover measure theory in a single semester. Chapter 4 starts by applying the Hilbert space results to obtain the Lebesgue–Radon–Nikodym Theorem. It then introduces complex-valued measures and completes the cycle by showing that when  $X$  is a  $\sigma$ -compact locally compact metric space, every bounded linear functional on  $C_0(X)$  can be represented as integration with respect to a complex-valued Radon measure. The chapter then develops product measures and closes with a detailed examination of Lebesgue and other measures on Euclidean space, including the Fourier transform. That is the course on measure theory, and I had no difficulty covering it in a semester.

Chapter 5 begins the study of functional analysis by studying linear transformations, first on Banach spaces but quickly focusing on Hilbert space and reaching the diagonalization of a compact hermitian operator. This chapter and the subsequent ones are based on my existing book [8]. There are, however, significant differences. *A Course in Functional Analysis* was designed as a one-year course on the subject for students who had completed a year-long study of measure theory as well as having some knowledge of analytic functions. The second half of the present book only assumes the presentation on measures done in the first half and is meant to be covered in a semester. Needless to say, many topics in [8] are not touched here. Even when this book does a topic found in [8], it is usually treated with less generality and in a somewhat simpler form. I'd advise all readers to use [8] as a reference – as I did.

Chapter 6 looks at Banach spaces and presents the three pillars of functional analysis. The next chapter touches on locally convex spaces, but only to the extent needed to facilitate the presentation of duality. It does include, however, a discussion of the separation theorems that follow from the Hahn–Banach Theorem. Chapter 8 treats the relation between a Banach space and its dual space. It includes the Krein–Milman Theorem, which is applied to prove the Stone–Weierstrass Theorem. Chapter 9 returns to operator theory, but this time in the Banach space setting and gets to the Fredholm Alternative. Chapter 10 presents the basics of Banach algebras and lays the groundwork for the last chapter, which is an introduction to  $C^*$ -algebras. This final chapter includes the functional calculus for normal operators and presents the characterization of their isomorphism classes.

When I taught my course I did not reach the end of the book, though I covered some topics in more detail and generality than they are covered here; I also presented some of the optional sections in this book – those that have a \* in the title. Nevertheless, I wanted the readers to have access to the material on multiplicity theory for normal operators, which is one of the triumphs of mathematics. I suspect that with a good class like the one I had and avoiding the starred sections, the entire book could be covered in a year.

**Biographies.** I have included some biographical information whenever a mathematician’s result is presented. (Pythagoras is the lone exception.) There is no scholarship on my part in this, as all the material is from secondary sources, principally what I could find on the web. In particular, I made heavy use of

<http://www-history.mcs.st-andrews.ac.uk/history/BiogIndex.html>

and Wikipedia. I did this as a convenience for the reader and from my experience that most people would rather have this in front of them than search it out. (A note about web addresses. There are a few others in this book and they were operational when I wrote the manuscript. We are all familiar with the fact that some web sites become moribund with time. If you experience this, just try a search for the subject at hand.)

I emphasize the personal aspects of the mathematicians we encounter along the way, rather than recite their achievements. This is especially so when I discover something unusual or endearing in their lives. I figure many students will see their achievements if they stick with the subject and most students at the start of their education won’t know enough mathematics to fully appreciate the accomplishments. In addition I think the students will enjoy learning that these famous people were human beings.

**Teaching.** I think my job as an instructor in a graduate course is to guide the students as they learn the material, not necessarily to slog through every proof. In the book, however, I have given the details of the most tedious and technical proofs; but when I lecture I frequently tell my class, “Adults should not engage in this kind of activity in public.” Students are usually amused at that, but they realize, albeit with my encouragement, that understanding a highly technical argument may be important. It certainly exposes them to a technique. Nevertheless, the least effective way to reach that understanding is to have someone stand in front of a student at a chalkboard and conscientiously go through all the details. The details should be digested by the student in the privacy of his/her office, away from public view.

I also believe in a gradual introduction of new material to the student. This is part of the reason for what I said earlier about going from the

particular to the general. This belief is also reflected in making changes in some notation and terminology as we progress. A vivid example of this is the use of the term “measure.” Initially it means a positive measure and then in the course of developing the material it migrates to meaning a complex-valued measure. I don’t think this will cause problems; in fact, as I said, I think it facilitates learning.

**Prerequisites.** The reader is assumed to be familiar with the basic properties of metric spaces. In particular the concepts of compactness, connectedness, continuity, uniform continuity, and the surrounding results on these topics are assumed known. I also assume the student has had a good course in basic analysis on the real line. In particular, (s)he should know the Riemann integral and have control of the usual topics appearing in such a course. There are a few other things from undergraduate analysis that are assumed, though usually what appears here doesn’t depend so heavily on their mastery.

**For students.** When I first studied the subject, I regarded it as very difficult. I found the break with  $\epsilon$ - $\delta$  analysis dramatic, calling for a shift in thinking. A year later I wondered what all the fuss was about. So work hard at this, and I can guarantee that no matter how much trouble you have, it will eventually all clear up. Also I leave a lot of detail checking to the reader and frequently insert such things as (Why?) or (Verify!) in the text. I want you to delve into the details and answer these questions. It will check your understanding and give some perspective on the proof. I also strongly advise you to at least read all the exercises. With your schedule and taking other courses, you might not have the time to try to solve them all, but at least read them. They contain additional information. Learning mathematics is not a spectator sport.

**Thanks.** I have had a lot of help with this book. First my class was great, showing patience when a first draft of an argument was faulty, making comments, and pointing out typos. Specifically Brian Barg, Yosef Berman, Yeyao Hu, Tom Savistky, and David Shoup were helpful; Tanner Crowder was especially so, pointing out a number of typos and gaps. Also William J. Martin, who was an auditor, showed me a proof of Hölder’s Inequality using Young’s Inequality (though I decided not to use it in the book), and we had several enjoyable and useful discussions. A pair of friends helped significantly. Alejandro Rodríguez-Martínez did a reading of the penultimate draft as did William Ross. Bill, in addition to pointing out typos, made many pedagogical, stylistic, and mathematical comments which influenced the final product. I feel very fortunate to have such friends.

Needless to say, I am responsible for what you see before you.





---

# Bibliography

- [1] M. B. Abrahamse and T. L. Kriete [1973], “The spectral multiplicity of a multiplication operator,” *Indiana Math. J.*, **22**, pp. 845–857.
- [2] E. Bishop [1961], “A generalization of the Stone–Weierstrass Theorem,” *Pacific J. Math.* **11**, pp. 777–783.
- [3] J. K. Brooks [1971], “The Lebesgue Decomposition Theorem for Measures,” *Amer. Math. Monthly* **78**, pp. 660–662.
- [4] R. B. Burckel [1984], “Bishop’s Stone-Weierstrass theorem,” *Amer. Math. Monthly* **91**, pp. 22–32.
- [5] L. Carleson [1966], “On the convergence and growth of partial sums of Fourier series,” *Acta Mathematica* **116**, pp. 135–157.
- [6] J. B. Conway [1969], “The inadequacy of sequences,” *Amer. Math. Monthly* **76**, pp. 68 – 69.
- [7] J. B. Conway [1978], *Functions of One Complex Variable*, Springer-Verlag, New York.
- [8] J. B. Conway [1990], *A Course in Functional Analysis*, Springer-Verlag, New York.
- [9] M. M. Day [1958], *Normed linear spaces*, Springer-Verlag, Berlin.
- [10] L. de Branges [1959], “The Stone–Weierstrass theorem,” *Proc. Amer. Math. Soc.* **10**, pp. 822–824.
- [11] J. Diestel [1984], *Sequences and Series in Banach Spaces*, Springer-Verlag.
- [12] J. Dixmier [1964], *Les  $C^*$ -Algèbras et leurs Représentations*, Gautiers-Villars, Paris.
- [13] J. Dugundji [1967], *Topology*, Allyn and Bacon.
- [14] Per Enflo [1973], “A counterexample to the approximation problem in Banach spaces,” *Acta Mathematica* **130**, pp. 309–317.
- [15] G. B. Folland [1999], *Real Analysis: Modern Techniques and Their Applications*, Wiley (Hoboken).
- [16] I. Glicksberg [1962], “Measures orthogonal to algebras and sets of antisymmetry,” *Trans. Amer. Math. Soc.* **105**, pp. 415–435 .
- [17] S. Grabiner [1986], “The Tietze extension theorem and the open mapping theorem,” *Amer. Math. Monthly* **93**, pp. 190–191.

- 
- [18] E. Hewitt and K. Stromberg [1975], *Real and Abstract Analysis*, Springer-Verlag, New York.
- [19] R. A. Hunt [1967], “On the convergence of Fourier series, orthogonal expansions and their continuous analogies,” pp. 235–255, *Proc. Conf. at Edwardsville, Ill*, Southern Illinois Univ. Press, Carbondale.
- [20] R. C. James [1951], “A non-reflexive Banach space isometric with its second conjugate space,” *Proc. Nat. Acad. Sci. USA* **37**, pp. 174–177.
- [21] R. V. Kadison and J. R. Ringrose [1997a], *Fundamentals of the Theory of Operator Algebras. Volume I: Elementary Theory*, Amer. Math. Soc., Providence.
- [22] R. V. Kadison and J. R. Ringrose [1997b], *Fundamentals of the Theory of Operator Algebras. Volume II: Advanced Theory*, Amer. Math. Soc, Providence.
- [23] R. V. Kadison and J. R. Ringrose [1998a], *Fundamentals of the Theory of Operator Algebras. Volume III*, Amer. Math. Soc, Providence.
- [24] R. V. Kadison and J. R. Ringrose [1998b], *Fundamentals of the Theory of Operator Algebras. Volume IV*, Amer. Math. Soc, Providence.
- [25] J. L. Kelley [1966], “Decomposition and representation theorems in measure theory,” *Math. Ann.* **163**, pp. 89–94.
- [26] J. L. Kelley [2008], *General Topology*, Ishi Press.
- [27] H. Kestelman [1971], “Mappings with Non-Vanishing Jacobian,” *Amer. Math. Monthly* **78**, pp. 662–663.
- [28] J. Lindenstrauss and L. Tzafriri [1971], “On complemented subspaces problem,” *Israel J. Math.* **5**, pp. 153–156.
- [29] T. J. Ransford [1984], “A short elementary proof of the Bishop–Stone–Weierstrass theorem,” *Math. Proc. Camb. Phil. Soc.* **96**, pp. 309–311.
- [30] W. Rudin [1991], *Functional Analysis*, McGraw-Hill, New York.
- [31] M. E. Taylor [2006], *Measure Theory and Integration*, Amer. Math. Soc., Providence.
- [32] D. E. Varberg [1965], “On absolutely continuous functions,” *Amer. Math. Monthly* **72**, pp. 831–841.
- [33] A. Villani [1984], “On Lusin’s condition for the inverse function,” *Rendiconti Circolo Mat. Palermo* **33**, pp. 331–333.

---

# List of Symbols

$(X, \mathcal{A}, \mu)$ , 57	$[x, y]$ , 29
$2^X$ , 52	$\Sigma$ , 315
$A^*$ , 184	$\ L\ $ , 38
$A^\circ$ , 254	$\ P\ $ , 4
$A^\perp$ , 254	$\ T\ $ , 172
$B(x; r)$ , 13	$\ f\ _\infty$ , 85
$BV[a, b]$ , 3	$\ f\ _p$ , 85
$C(X)$ , 16	$\ x + \mathcal{M}\ $ , 217
$C(X)_+$ , 16	$\alpha_\mu$ , 127
$C_0(X)$ , 30	$\alpha_t$ , 8
$C_b(X)$ , 16	$\bar{z}$ , 93
$C_c(X)$ , 30	$\beta X$ , 126
$C_c^\infty(G)$ , 159	$\bigoplus \mathcal{H}_n$ , 191
$E\Delta F$ , 71	$\bigvee A$ , 27
$E^*$ , 302	$\chi_E$ , 19
$L^1(\mu)$ , 65	$\ell^1$ , 22
$L_s^1(\mu)$ , 63	$\ell^1(w)$ , 28
$L_s^1(\mu)_+$ , 64	$\ell^\infty$ , 22
$L_{\text{loc}}^1$ , 144	$\int f d\alpha$ , 6
$L^p(\mu)$ , 84, 88	$\lambda$ , 50
$L^p(\mathbb{R})$ , 156	$\langle \cdot, \cdot \rangle$ , 94
$L_a$ , 320	$\mu * \nu$ , 161
$M(X, \mathcal{A})$ , 82	$\mu^*(E)$ , 45
$M_\phi$ , 171	$\mu^*(G)$ , 43
$N_\mu$ , 334	$\mu_E$ , 58
$N_e$ , 333	$\mu_\alpha$ , 127
$P_{\mathcal{M}}$ , 101	$\mu_e$ , 333
$S(f, P)$ , 4	$\nu \ll \mu$ , 107
$S_\alpha(f, P)$ , 4	$\nu \perp \mu$ , 108
$T \cong S$ , 211	$\nu_a$ , 109
$T^*$ , 278	$\nu_s$ , 109
$Vf$ , 174	$\omega(f, \delta)$ , 4
$X_\infty$ , 31	$\overline{\sigma}(A)$ , 99

- $\phi(T)$ , 207  
 $\sigma(\mathcal{X}^*, \mathcal{X})$ , 252  
 $\sigma(\mathcal{X}, \mathcal{X}^*)$ , 252  
 $\sigma_p(T)$ , 199  
 $\sigma_{ap}(T)$ , 302  
 $\text{sign}(f)$ , 87  
 $\text{sign } z$ , 93  
 $\tau_a$ , 143  
 $|T|$ , 330  
 $|\mu|$ , 82  
 $|\mu|(E)$ , 117  
 $\widehat{f}$ , 161, 181  
 $\widehat{\mathbb{R}}$ , 53  
 $\widetilde{\alpha}$ , 10  
 $\widetilde{g}$ , 166  
 $^\circ A$ , 254  
 $^\circ B$ , 254  
 $^\perp B$ , 254  
 $^\perp(A^\perp)$ , 254  
 $c_0$ , 22  
 $c_0(w)$ , 28  
 $c_{00}$ , 22  
 $f * g$ , 156  
 $f = \frac{d\nu_a}{d\mu}$ , 111  
 $f \vee g$ , 2  
 $f \wedge g$ , 3  
 $f_\pm$ , 17  
 $h\mu$ , 272  
 $m_T(\lambda)$ , 211  
 $p_x$ , 252  
 $p_{x^*}$ , 252  
 $x \perp y$ , 98  
 $\mathbb{C}$ , 16  
 $\mathbb{F}$ , 16  
 $\mathbb{Q}$ , 33  
 $\mathbb{R}$ , 1  
 $\mathcal{A}'$ , 342  
 $\mathcal{A}_+$ , 327  
 $\mathcal{A}_E$ , 58  
 $\mathcal{A}_\mu$ , 46, 342  
 $\mathcal{B}(\mathcal{X})$ , 171  
 $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ , 171  
 $\mathcal{B}_0(\mathcal{X})$ , 195  
 $\mathcal{B}_0(\mathcal{X}, \mathcal{Y})$ , 195  
 $\mathcal{H} \oplus \mathcal{K}$ , 190  
 $\mathcal{H} \cong \mathcal{K}$ , 179  
 $\mathcal{H}_e$ , 333  
 $\mathcal{M}(X)$ , 53  
 $\mathcal{M}(X, \mathcal{A})$ , 53  
 $\mathcal{M} \leq \mathcal{X}$ , 101  
 $\mathcal{M}_+$ , 55  
 $\mathcal{M}_+(X)$ , 55  
 $\mathcal{M}_+(X, \mathcal{A})$ , 55  
 $\mathcal{P}_\delta$ , 4  
 $S$ , 163  
 $\mathcal{X}^{**}$ , 263  
 $\text{Var}(\alpha)$ , 2  
 $\text{ball } \mathcal{X}$ , 23  
 $\text{cl } A$ , 13  
 $\text{co}(A)$ , 99  
 $\text{diam}$ , 13  
 $\text{dist}(A, B)$ , 21  
 $\text{dist}(x, A)$ , 17  
 $\text{ext } K$ , 266  
 $\text{int } A$ , 13  
 $\ker T$ , 101  
 $\text{ran } T$ , 101  
 $\text{spt } \mu$ , 272

---

# Index

- $\mathcal{A}$ -measurable, 53
- $\mathcal{A}$ -partition, 82
- absolutely continuous, 107
- absolutely continuous function, 152
- adjoining the identity, 289
- adjoint, 184, 278
- affine hyperplane, 246
- affine manifold, 246
- affine map, 269
- affine subspace, 246
- Alaoglu, Leonidas, 256
- Alaoglu's Theorem, 256
- algebra, 16, 287
- algebraically complemented, 232
- almost everywhere, 64
- almost uniformly, 78
- analytic, 293
- annihilator, 254
- approximate identity, 158
- approximate point spectrum, 302
- Arzelà–Ascoli Theorem, 283
- Arzelà, Cesare, 283
- Ascoli, Giulio, 283
- atom, 87
- average, 144
  
- backward shift, 185, 304
- Baire, René-Louis, 355
- Baire Category Theorem, 355
- balanced, 240
- Banach, Stefan, 23
- Banach algebra, 287
- Banach limit, 228
  
- Banach space, 23
- Banach–Steinhaus Theorem, 235
- Banach–Stone Theorem, 280
- basis, 175
- Bessel, Wilhelm, 177
- Bessel's Inequality, 177
- bilateral ideal, 290
- bilateral shift, 180
- bipolar, 254
- Bipolar Theorem, 255
- Borel, Emile, 52
- Borel function, 53
- Borel sets, 52
- bounded, 171, 183
- bounded linear functional, 37
- bounded variation, 1
- Bunyakovsky, Viktor, 94
  
- $C^*$ -identity, 319
- Cantor, Georg, 13
- Cantor function, 61, 62
- Cantor middle-third set, 59
- Cantor ternary set, 59
- Cantor–Lebesgue function, 61
- Cauchy, Augustin-Louis, 94
- Cauchy's Theorem, 295
- Cauchy–Bunyakovsky–Schwarz Inequality, 94
- CBS Inequality, 94
- CGT, 231
- Chain Rule, 113
- Change of Variables Formula, 142
- characteristic function, 19

- Chebyshev's Inequality, 133  
 closed convex hull, 99  
 Closed Graph Theorem (CGT), 230  
 closed half-space, 246  
 closed linear span, 27  
 closed path, 294  
 commutant, 342  
 compact support, 30  
 compact transformation, 195  
 complementary subspaces, 233  
 completion, 27  
 complex measure, 115  
 continuous measure, 88, 113  
 converges in measure, 75  
 convex, 99  
 convex hull, 99  
 convolution, 156, 161  
 countable additivity, 57  
 counting measure, 57
- DCT, 73  
 decreasing, 2  
 diagonalizable, 197  
 differentiating under the integral sign, 158  
 direct sum, 190  
 directed set, 356  
 discrete metric, 29  
 division algebra, 315  
 Dominated Convergence Theorem (DCT), 72  
 double annihilator, 254  
 dual space, 38
- Egorov, Dimitri, 77  
 Egorov's Theorem, 77  
 eigenspace, 199  
 eigenvalue, 199  
 eigenvector, 199  
 equicontinuous, 283  
 equivalent metrics, 32  
 equivalent norms, 214  
 essential range, 301, 339  
 essentially bounded, 84  
 extended real numbers, 53  
 extreme point, 266
- Fatou, Pierre, 72  
 Fatou's Lemma, 72  
 final space, 330  
 finite measure space, 57  
 finite rank, 195
- Fourier, Joseph, 161  
 Fourier coefficient, 181  
 Fourier series, 182  
 Fourier transform, 161, 181  
 Fredholm Alternative, 313  
 Fredholm, Ivar, 313  
 Fubini, Guido, 134  
 Fubini's Theorem, 133, 140  
 Fuglede, Bent, 343  
 Fuglede–Putnam Theorem, 343  
 functional calculus, 207, 325  
 Fundamental Theorem of Calculus, 154
- gauge, 241  
 Gelfand, Israel, 314  
 Gelfand transform, 317  
 Gelfand–Mazur Theorem, 314  
 generators, 317
- Hahn Decomposition Theorem, 80  
 Hahn, Hans, 80  
 Hahn–Banach Theorem (HBT), 221  
 Hardy, G. H., 145  
 Hardy–Littlewood maximal function, 145  
 HBT, 221  
 Hermite, Charles, 186  
 hermitian, 186, 321  
 Hilbert, David, 97  
 Hilbert space, 97  
 Hölder, Otto, 86  
 Hölder's Inequality, 86  
 hyperplane, 220
- ideal, 290  
 idempotent, 191  
 imaginary part, 93, 189  
 IMT, 229  
 increasing, 2  
 indicator function, 43  
 infinite atom, 88  
 initial space, 330  
 inner product, 94  
 integrable, 63, 65, 67  
 integrable function, 82, 116  
 integral operator, 174  
 invariant subspace, 192  
 Inverse Mapping Theorem (IMT), 229  
 Inversion Formula, 165  
 Inversion Theorem, 165  
 invertible, 290  
 involution, 319

- isomorphic, 179
- isomorphism, 179, 230
- Jacobian, 142
- Jordan, Camille, 81
- Jordan decomposition, 81, 116, 123
- kernel, 101, 174
- Krein, Mark, 267
- Krein–Milman Theorem, 267
- L*-measurable, 46
- LCS, 237
- Lebesgue, Henri, 50
- Lebesgue Covering Lemma, 15
- Lebesgue decomposition, 111
- Lebesgue Decomposition Theorem, 111
- Lebesgue Differentiation Theorem, 149
- Lebesgue measurable, 57
- Lebesgue measure, 50, 58
- Lebesgue set, 148
- Lebesgue–Radon–Nikodym Theorem, 108, 116
- left ideal, 290
- left invertible, 290
- left limit, 9
- left regular representation, 320
- left spectrum, 297
- left-continuous, 9
- line segment, 29
- linear functional, 37
- linear manifold, 101
- linear transformation, 171
- Liouville, Joseph, 295
- Liouville’s Theorem, 295
- Lipschitz function, 152
- Littlewood, John E., 145
- locally compact, 29
- locally convex space (LCS), 237
- locally integrable, 144
- Lusin, Nikolai, 120
- maximal ideal, 291
- maximal ideal space, 316
- Maximal Theorem, 146
- maximal vector, 336
- Mazur, Stanisław, 314
- measurable, 46, 53, 75
- measurable function, 114
- measurable rectangle, 140
- measure, 57, 118
- measure space, 57
- mesh, 4
- Milman, David, 267
- modulus of continuity, 4
- Monotone Convergence Theorem, 65
- $\mu$ -measurable, 46
- multiplicity function, 211, 350
- mutually absolutely continuous, 335
- mutually singular, 108
- natural map, 263
- negative part, 67, 81
- negative set, 79
- net, 356
- Nikodym, Otton, 108
- norm, 21, 172
- norm of a linear functional, 38
- normal, 186, 321
- normalization, 10, 127
- normalized, 127
- normed space, 22
- OMT, 229
- one-point compactification, 31
- open half-space, 246
- Open Mapping Theorem (OMT), 228
- operator, 171
- orthogonal, 98
- orthogonal projection, 101
- orthonormal, 175
- outer measure, 43, 45
- pairwise orthogonal, 98
- Parseval, Marc-Antoine, 178
- Parseval’s Identity, 178, 182
- partial isometry, 330, 331
- partition, 1
- partition of unity, 19
- partition of unity subordinate to the cover, 19
- path, 294
- perfect set, 61
- Plancherel, Michel, 168
- Plancherel’s Theorem, 168
- polar, 254
- polar decomposition, 330
- polar identity, 96
- positive, 208, 327
- positive cone, 16
- positive definite, 208
- positive linear functional, 41
- positive measure, 118
- positive part, 67, 81
- positive set, 79



- preannihilator, 254  
 prepolar, 254  
 Principle of Uniform Boundedness (PUB), 234  
 probability measure, 274  
 product measure, 134  
 projection, 101, 191  
 PUB, 234  
 purely atomic measure, 113  
 Putnam, C. Richard, 343  
 Pythagorean Theorem, 98
- quotient map, 217
- radical, 317  
 radius of convergence, 294  
 Radon, Johann, 46  
 Radon measure space, 46  
 Radon–Nikodym derivative, 111  
 Radon–Nikodym Theorem, 111  
 range, 101  
 rapidly decreasing function, 163  
 real part, 93, 189  
 rectifiable, 295  
 reducing subspace, 192  
 refinement, 1  
 reflexive, 263  
 regular Borel measure, 119  
 resolvent equation, 300  
 resolvent set, 297  
 Riemann, G. F. Bernhard, 6  
 Riemann–Lebesgue Lemma, 163, 182  
 Riemann–Stieltjes integral, 6, 129  
 Riesz, Frigyes, 104  
 Riesz Representation Theorem, 104, 123, 124  
 right ideal, 290  
 right invertible, 290  
 right limit, 9  
 right regular representation, 320  
 right spectrum, 297  
 right-continuous, 9
- scalar-valued spectral measure, 336  
 Schauder, Julius, 282  
 Schauder’s Theorem, 282  
 Schwartz, Laurent, 163  
 Schwartz function, 163  
 Schwarz, Hermann, 94  
 second dual, 263  
 self-adjoint, 186  
 semi-inner product, 96
- seminorm, 213  
 separable, 21  
 separated, 246  
 separates the points, 271  
 separation theorem, 246  
 sesquilinear form, 183  
 shrinks nicely, 149  
 $\sigma$ -algebra, 52  
 $\sigma$ -algebra generated, 52  
 $\sigma$ -compact, 30  
 $\sigma$ -finite, 108  
 signed measure, 78, 118  
 simple measurable function, 63  
 Spectral Mapping Theorem, 325  
 spectral radius, 299  
 Spectral Theorem, 202, 206  
 spectrum, 297  
 star-cyclic vector, 334  
 \*-homomorphism, 320  
 \*-isomorphism, 320  
 Steinhaus, Hugo, 235  
 Stieltjes, Thomas Jan, 6  
 Stone, Marshall, 271  
 Stone–Weierstrass Theorem, 270  
 strictly decreasing, 2  
 strictly increasing, 2  
 strictly separated, 246  
 sublinear functional, 221  
 submanifold, 101  
 subspace, 101  
 support, 30  
 support of a measure, 272
- Tietze, Heinrich, 20  
 Tietze Extension Theorem, 20  
 topologically complemented, 233  
 total variation, 2  
 totally bounded, 14  
 totally disconnected, 59  
 trace, 295  
 triangle inequality, 22  
 trigonometric polynomial, 181
- unilateral shift, 180, 303  
 unilateral weighted shift, 185  
 unit ball, 23  
 unit point mass, 88  
 unitarily equivalent, 211  
 unitary, 179, 186, 321  
 Urysohn, Pavel, 17  
 Urysohn’s Lemma, 17

vanishes at infinity, 30  
variation, 82  
Vitali, Giuseppe, 92  
Vitali's Convergence Theorem, 92  
Volterra, Vito, 308  
Volterra kernel, 308  
Volterra operator, 174, 199  
  
weak operator topology, 339  
weak topology, 252  
weak-star topology, 252  
Weierstrass, Karl, 271  
weighted shift, 185

This book covers topics appropriate for a first-year graduate course preparing students for the doctorate degree. The first half of the book presents the core of measure theory, including an introduction to the Fourier transform. This material can easily be covered in a semester. The second half of the book treats basic functional analysis and can also be covered in a semester. After the basics, it discusses linear transformations, duality, the elements of Banach algebras, and  $C^*$ -algebras. It concludes with a characterization of the unitary equivalence classes of normal operators on a Hilbert space.



The book is self-contained and only relies on a background in functions of a single variable and the elements of metric spaces. Following the author's belief that the best way to learn is to start with the particular and proceed to the more general, it contains numerous examples and exercises.

ISBN 978-0-8218-9083-7



GSM/141



For additional information  
and updates on this book, visit

[www.ams.org/bookpages/gsm-141](http://www.ams.org/bookpages/gsm-141)

AMS on the Web  
[www.ams.org](http://www.ams.org)