Lecture Notes on Functional Analysis

With Applications to Linear Partial Differential Equations

Alberto Bressan

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To Wen, Luisa Mei, and Maria Lan

Contents

Preface

The first version of these lecture notes was drafted in 2010 for a course at the Pennsylvania State University. The book is addressed to graduate students in mathematics or other disciplines, who wish to understand the essential concepts of functional analysis and their application to partial differential equations. Most of its content can be covered in a one-semester course at the first-year graduate level.

In writing this textbook, I followed a number of guidelines:

- Keep it short, presenting all the fundamental concepts and results, but not more than that.
- Explain clearly the connections between theorems in functional analysis and familiar results of finite-dimensional linear algebra.
- Cover enough of the theory of Sobolev spaces and semigroups of linear operators as needed to develop significant applications to elliptic, parabolic, and hyperbolic PDEs.
- Include a large number of homework problems and illustrate the main ideas with figures, whenever possible.

In functional analysis one finds a wealth of beautiful results that could be included in a monograph. However, for a textbook of this nature one should resist such a temptation.

After the Introduction, Chapters 2 to 6 cover classical topics in linear functional analysis: Banach spaces, Hilbert spaces, and linear operators. Chapter 4 is devoted to spaces of continuous functions, including the Stone-Weierstrass approximation theorem and Ascoli's compactness theorem. In view of applications to linear PDEs, in Chapter 6 we prove some basic results on Fredholm operators and the Hilbert-Schmidt theorem on compact symmetric operators in a Hilbert space.

Chapter 7 provides an introduction to the theory of semigroups, extending the definition of the exponential function e^{tA} to a suitable class of (possibly unbounded) linear operators. We stress the connection with finite-dimensional ODEs and the close relation between the resolvent operators and backward Euler approximations.

After an introduction explaining the concepts of distribution and weak derivative, Chapter 8 develops the theory of Sobolev spaces. These spaces provide the most convenient abstract framework where techniques of functional analysis can be applied toward the solution of ordinary and partial differential equations.

The first three sections in Chapter 9 describe applications of the previous theory to elliptic, parabolic, and hyperbolic PDEs. Since differential operators are unbounded, it is often convenient to recast a linear PDE in a "weak form", involving only bounded operators on a Hilbert-Sobolev space. This new equation can then be studied using techniques of abstract functional analysis, such as the Lax-Milgram theorem, Fredholm's theory, or the representation of the solution in terms of a series of eigenfunctions.

The last chapter consists of an Appendix, collecting background material. This includes: definition and properties of metric spaces, the contraction mapping theorem, the Baire category theorem, a review of Lebesgue measure theory, mollification techniques and partitions of unity, integrals of functions taking values in a Banach space, a collection of inequalities, and a version of Gronwall's lemma.

These notes are illustrated by 41 figures. Nearly 180 homework problems are collected at the end of the various chapters. A complete set of solutions to the exercises is available to instructors. To obtain a PDF file of the solutions, please contact the author, including a link to your department's web page listing you as an instructor or professor.

It is a pleasure to acknowledge the help I received from colleagues, students, and friends, while preparing these notes. To L. Berlyand, G. Crasta, D. Wei, and others, who spotted a large number of misprints and provided many useful suggestions, I wish to express my gratitude.

Alberto Bressan
State College, July 2012

Summary of Notation

$\mathbb{R}$, the field of real numbers.

$\mathbb{C}$, the field of complex numbers.

$\mathbb{K}$, a field of numbers, either $\mathbb{R}$ or $\mathbb{C}$.

Re z and Im z, the real and imaginary parts of a complex number z.

$\bar{z} = a - ib$, the complex conjugate of the number $z = a + ib \in \mathbb{C}$.

$[a, b]$, a closed interval; $]a, b[$, an open interval; $]a, b]$, $[a, b[$ half-open intervals.

$\mathbb{R}^n$, the n-dimensional Euclidean space.

$\langle \cdot, \cdot \rangle$, scalar product on the Euclidean space $\mathbb{R}^n$.

$|v| \doteq \sqrt{\langle v, v \rangle}$, the Euclidean length of a vector $v \in \mathbb{R}^n$.

$A \setminus B \doteq \{x \in A\,,\ x \notin B\}$, a set-theoretic difference.

$\overline{A}$, the closure of a set A.

∂A, the boundary of a set A.

$\Omega' \subset\subset \Omega$, the closure of Ω' is a compact subset of Ω.

χ_A, the indicator function of a set A. $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A\,, \\ 0 & \text{if } x \notin A\,. \end{cases}$

$f : A \mapsto B$, a mapping from a set A into a set B.

$a \mapsto b = f(a)$, the function f maps the element $a \in A$ to the element $b \in B$.

$\doteq$, equal by definition.

$\iff$, if and only if.

$\mathcal{C}(E) = \mathcal{C}(E, \mathbb{R})$, the vector space of all continuous, real-valued functions on the metric space E.

$\mathcal{C}(E, \mathbb{C})$, the vector space of all continuous, complex-valued functions on the metric space E.

$\mathcal{B}C(E)$, the space of all bounded, continuous, real-valued functions $f : E \mapsto \mathbb{R}$, with norm $\|f\| = \sup_{x \in E} |f(x)|$.

ℓ^1, ℓ^p, ℓ^∞, spaces of sequences of real (or complex) numbers.

$\mathbf{L}^1(\Omega)$, $\mathbf{L}^p(\Omega)$, $\mathbf{L}^\infty(\Omega)$, Lebesgue spaces.

$W^{k,p}(\Omega)$, the Sobolev space of functions whose weak partial derivatives up to order k lie in $\mathbf{L}^p(\Omega)$, for some open set $\Omega \subseteq \mathbb{R}^n$.

$H^k(\Omega) = W^{k,2}(\Omega)$, Hilbert-Sobolev space.

$\mathcal{C}^{k,\gamma}(\Omega)$, the Hölder space of functions $u : \Omega \mapsto \mathbb{R}$ whose derivatives up to order k are Hölder continuous with exponent $\gamma \in]0, 1]$.

$\|\cdot\| = \|\cdot\|_X$, the norm on a vector space X.

$(\cdot, \cdot) = (\cdot, \cdot)_H$, the inner product on a Hilbert space H.

X^*, the dual space of X, i.e., the space of all continuous linear functionals $x^* : X \mapsto \mathbb{K}$.

$\langle x^*, x\rangle \;=\; x^*(x)$, the duality product of $x^* \in X^*$ and $x \in X$.

$x_n \to x$, strong convergence in norm; this means $\|x_n - x\| \to 0$.

$x_n \rightharpoonup x$, weak convergence.

$\varphi_n \overset{*}{\rightharpoonup} \varphi$, weak-star convergence.

$f * g$, the convolution of two functions $f, g : \mathbb{R}^n \mapsto \mathbb{R}$.

$\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$, the gradient of a function $u : \mathbb{R}^n \mapsto \mathbb{R}$.

$D^\alpha \;=\; \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \;=\; \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}$, a partial differential operator of order $|\alpha| \;\doteq\; \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

$\mathrm{meas}(\Omega)$, the Lebesgue measure of a set $\Omega \subset \mathbb{R}^n$.

$\displaystyle ⨍_\Omega f\,dx \;=\; \frac{1}{\mathrm{meas}(\Omega)}\int_\Omega f\,dx$, the average value of f over the set Ω.

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