

# Introduction to 3-Manifolds

**Jennifer Schultens**

**Graduate Studies  
in Mathematics**

**Volume 151**



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Dedicated to Misha and Esther and our extended family



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# Preface

This book grew out of a graduate course on 3-manifolds taught at Emory University in the spring of 2003. It aims to introduce the beginning graduate student to central topics in the study of 3-manifolds. Prerequisites are kept to a minimum but do include some point set topology (see [109]) and some knowledge of general position (see [128]). In a few places, it is worth our while to mention results or proofs involving concepts from algebraic topology or differential geometry. This should not stop the interested reader with no background in algebraic topology or differential geometry from enjoying the material presented here. The sections and exercises involving algebraic topology are marked with a \*, those involving differential geometry with a \*\*.

This book conveys my personal path through the subject of 3-manifolds during a certain period of time (roughly 1990 to 2007). Marty Scharlemann deserves credit for setting me on this path. He remains a much appreciated guide. Other guides include Misha Kapovich, Andrew Casson, Rob Kirby, and my collaborators.

In Chapter 1 we introduce the notion of a manifold of arbitrary dimension and discuss several structures on manifolds. These structures may or may not exist on a given manifold. In addition, if a particular structure exists on a given manifold, it may or may not be presented as part of the information given. In Chapter 2 we consider manifolds of a particular dimension, namely 2-manifolds, also known as surfaces. Here we provide an overview of the classification of surfaces and discuss the mapping class group. Chapter 3 gives examples of 3-manifolds and standard techniques used to study 3-manifolds. In Chapter 4 we catch a glimpse of the interaction of pairs of manifolds, specifically pairs of the form (3-manifold, 1-manifold). Of

particular interest here is the consideration of knots from the point of view of the complement (“Not Knot”). For other perspectives, we refer the reader to the many books, both new and old, mentioned in Chapter 4, that provide a more in-depth study. In Chapter 5 we consider triangulated 3-manifolds, normal surfaces, almost normal surfaces, and how these set the stage for algorithms pertaining to 3-manifolds. In Chapter 6 we cover a subject near and dear to the author’s heart: Heegaard splittings. Heegaard splittings are decompositions of 3-manifolds into symmetric pieces. They can be thought of in many different ways. We discuss key examples, classical problems, and recent advances in the subject of Heegaard splittings. In Chapter 7 we introduce hyperbolic structures on manifolds and complexes and provide a glimpse of how they affect our understanding of 3-manifolds. We include two appendices: one on general position and one on Morse functions. Exercises appear at the end of most sections.

I wish to thank the many colleagues and students who have given me the opportunity to learn and teach. I also wish to thank the institutions that have supported me through the years: University of California, Emory University, Max-Planck-Institut für Mathematik Bonn, Max-Planck-Institut für Mathematik Leipzig, and the National Science Foundation.

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This book grew out of a graduate course on 3-manifolds and is intended for a mathematically experienced audience that is new to low-dimensional topology.

The exposition begins with the definition of a manifold, explores possible additional structures on manifolds, discusses the classification of surfaces, introduces key foundational results for 3-manifolds, and provides an overview of knot theory. It then continues with more specialized topics by briefly considering triangulations of 3-manifolds, normal surface theory, and Heegaard splittings. The book finishes with a discussion of topics relevant to viewing 3-manifolds via the curve complex.

With about 250 figures and more than 200 exercises, this book can serve as an excellent overview and starting point for the study of 3-manifolds.

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