Introduction to Analytic and Probabilistic Number Theory

THIRD EDITION

Gérald Tenenbaum

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Introduction to Analytic and Probabilistic Number Theory

Third Edition

Gérald Tenenbaum

Translated by Patrick D. F. Ion

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À Catherine Jablon,

pour la douceur du jour,
ce bouquet de symboles
dont ta conversation
éclaire les secrets.

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Foreword

Arising, as it does, from advanced lectures given in Bordeaux, Paris and Nancy over the past fifteen years (and for which an earlier English version is available from *Cambridge University Press*), this book is a revised, updated, and expanded version of a volume that appeared in 1990 in the *Publications de l'Institut Élie Cartan*. It was written with the purpose of providing young researchers with a self-contained introduction to the analytic methods of number theory, and their elders with a source of references for a number of fundamental questions. Such an undertaking necessarily involves choices. As these were made, they were generally taken on aesthetic grounds—not to forget the categorical imperatives imposed by ignorance.

The double motivation mentioned above has led to a special usage of the traditional subdivision of chapters into text, notes and exercises. Thus the basic text, while restricted as a rule to assertions that are proved in detail, may also contain additional bibliographic comments when providing a useful background upon first reading. Conversely, the notes often give way to statements, and even proofs, of related results which may safely be omitted on first contact. In a parallel way, the exercises serve a double purpose.[1] Whereas some of them are classically designed to facilitate the mastering of previously introduced concepts, some others lead to actual research results, sometimes unpublished, mainly in Part III. We used to believe, naively, that we could avoid an unfortunate current tendency by producing exercises that could be solved without prodigious ingenuity or technical virtuosity. The numerous requests for solutions received after the publication of the first edition have shown that such a goal might be illusory. Result: the reader

[1] Complete solutions to all exercises from this third edition are available as a companion book published by Belin (Paris).

will find in the solution book, written with the collaboration of my colleague Jie Wu, an attempt to make things right. It remains nonetheless true that open questions are exceptional in the formulations of exercises, and that the results aimed at are usually explicit, with the essential steps set out. This part of the work may thus serve, even without making the effort of solving the problems or consulting the solutions, as an informal repository of references.

The writing of this book has been guided by the constant concern of emphasizing methods more than results—a strategy which we believe to be specifically heuristic. This has led to the somewhat artificial subdivision into three parts, respectively devoted to elementary, complex-analytical and probabilistic methods. It will be easy to criticize this taxonomy: is the method of van der Corput, based on the Poisson summation formula, more elementary than the Selberg–Delange method, which employs complex integration? Why qualify as probabilistic the saddle-point method, whose initial step amounts to an inverse Laplace transform? One could multiply the examples of inconsistency with respect to this or that criterion, and it is obvious that the choices have been made on grounds that can be questioned. Thus, we regard as elementary a method that exclusively employs *real variables*, and we choose to view the saddle-point method as probabilistic as much because it is an ever-present tool in probability theory, as for being a specific method implemented to solve problems in probabilistic number theory... One might as well say at the very outset that the classification at work in this book is anything but a Bourbakist choice. Its ambition is limited to the mere wish that it might, at least for a while, shed some light on the path of the neophyte.

Without aiming at complete originality, the text tries to avoid well-trodden paths. We have reconsidered, when it seemed desirable and indeed possible, the exposition of classical results: either by employing new approaches (such as Nair's method for Chebyshev's estimates), or by occasionally introducing technical simplifications that are invisible in the table of contents, but will hopefully be useful to the active reader.

Certain developments, meanwhile, are innovative. This mainly concerns: some uniform results arising from the Selberg–Delange method (Chapter II.5); the version with explicit remainder of the Ikehara–Ingham theorem (§ II.7.5); the study of the sieve function $\Phi(x, y)$ by the saddle-point method (Chapter III.6). The effective form of Ikehara's theorem turns out to be closely related to the Berry–Esseen inequality—an almost conceptual identity which we continue to find fascinating. Besides, a concern for complementarity with respect to the existing literature (and especially the fine book by Elliott) has influenced some of our decisions, such as the choice

of the method of proof for the theorems of Erdős–Wintner, Erdős–Kac, or Halász—see Chapter III.4. This last result corresponds to an extension of Montgomery's method, developed in a way he suggested.

This second edition, like the first, owes much to all those colleagues and friends who helped me clarify and clean up the manuscript. It is a pleasant duty to express my gratitude here to Michel Balazard, Régis de la Bretèche, Gautami Bhomwik, Paul Erdős, Michel Mendès France, Olivier Ramaré, Jean-Luc Rémy, Imre Ruzsa, Patrick Sargos, András Sárközy, Marijke Wijsmuller, and Jie Wu: as long as the list of errata might turn out to be (and experience has shown this is not just a cliché), it would have been a good deal longer without their help. Finally, I would like to warmly thank Daniel Barlet for his friendly and effective involvement in the process of publication of the text by the *Société mathématique de France.*

Nancy, *March 1995* G.T.

Preface to the third edition

While retaining the same structure and the same expository options, we have extensively expanded the contents of this book for its third edition. This meets a three-fold goal: to take recent advances into account, to flesh out the methodological aspect of the exposition, and to provide basic knowledge or useful supplements for university graduate students, in particular for those preparing for higher teaching diplomas.

Updating with the results from the literature is mostly done in the Notes or Exercises. However, such updates may also be done in new subsections, such as § III.6.5 on Kubilius' model. New proofs of previously included statements are also offered, such as for Tauber's theorem (§ II.7.2) or Halász's (§ III.4.3). Finally, as in the case of the Turán–Kubilius inequality and its friable generalization, the influence of recent results led us to substantially modify the exposition.

Numerous new developments have been inserted in order to preserve general consistency. This essentially concerns: section I.4.7, which is devoted to Selberg's sieve in a little known general form; some applications to small gaps between prime numbers given in the Exercises of the same chapter; the description of Ramanujan's method for the maximal order of the divisor function (Exercise 90); the statements of the Kusmin–Landau inequality (I.6.6) and of van der Corput's general theorem (I.6.10); the inclusion of the explicit formulae of the theory of numbers (§§ II.4.4 and II.8.6); a significant expansion of Chapter II.8, devoted to the distribution of prime numbers in arithmetic progressions; the introduction of Jacobsthal's function and of

the proof of Rankin's theorem on large gaps between consecutive primes (§ III.5.6).

Aside from the inclusion in the Exercises of statements following straightforwardly from the main theorems and of synthetic problems, the new items intended for students and future graduate students concern: the Euler–Maclaurin formula (see the exercises of Chapter I.0); an elementary exposition of the Legendre symbol and the theory of quadratic residues (exercises in Chapter I.1); an introduction to the theory of equidistribution modulo 1 (§ I.6.5); a first treatment of Diophantine approximation and a synthetic exposition of continued fractions (Chapter I.7); as well as a *vade mecum* on the theory of Euler's Gamma function (Chapter II.0).

The description sketched above is obviously too succinct to reflect the numerous correlations between developments arising from various motivations. It is also fails to be exhaustive. The text as a whole has been revised, and whole passages have been rewritten. The presentation is further supported by the addition of one hundred and twenty-five new exercises offering, for some important theorems, variations of proofs, or simplified versions, as in the cases of van der Corput's theorem or of the Erdős–Turán inequality. The initial choices of presentation, however, have not been fundamentally modified.

The author wishes to warmly thank all those who have contributed to an attentive and critical rereading of this almost new manuscript, in particular Joseph Basquin, Régis de la Bretèche, Farrell Brumley, Cécile Dartyge, Kevin Ford, Bruno Martin, Michel Mendès France, Aziz Raouj, Jean-Luc Rémy, Olivier Robert, Anne de Roton, Patrick Sargos, and Jie Wu.

Nancy, *November 2007* G.T.

Preface to the English translation

This translation essentially follows the text of the French edition published in 2008, with many corrections and a few updates. It is a pleasure to express here warm thanks to Edward Dunne for his indestructible commitment to making this book available in English, to Patrick Ion, for his careful translation, and to Nicholas Bingham and Matthew de Courcy-Ireland for their invaluable help.

Nancy, *October 2013* G.T.

Notation

The following notation and conventions will be used freely in the text.

Except in explicitly stated or in special cases clear from context, the letter p, with or without subscript, denotes a prime number. We write $\mathbb{P}$ for the set of all primes.

$a|b$ means: a divides b; $p^\nu \| a$ means: $p^\nu | a$ and $p^{\nu+1} \nmid a$; $a|b^\infty$ means: $p|a \Rightarrow p|b$. We also use the notation $[a,b] := \operatorname{lcm}(a,b)$, and $(a,b) := \gcd(a,b)$.

$P^+(n)$ (resp. $P^-(n)$) denotes the largest (resp. the smallest) prime factor of the integer $n > 1$. By convention $P^+(1) = 1,\ P^-(1) = +\infty$.

The lower and upper integer parts, and the fractional part of the real number x are, respectively, denoted by $\lfloor x \rfloor$, $\lceil x \rceil$ and $\langle x \rangle$.

We put $\|x\| := \min_{n\in\mathbb{Z}} |x-n|$, $x^+ := \max(x,0)$ $(x \in \mathbb{R})$ and use the notation $\mathrm{e}(x) := \mathrm{e}^{2\pi i x}$ $(x \in \mathbb{R})$, $\ln^+ x := \max\{0, \ln x\}$ $(x > 0)$. We write $\ln_k$ for the k-fold iterated logarithm. The notation log is reserved for the complex logarithm, taken, if not otherwise specified, in its principal branch.

When the letter s denotes a complex number, we implicitly define real numbers σ and τ by the relation $s = \sigma + i\tau$.

We use interchangeably Landau's notation $f = O(g)$ and Vinogradov's $f \ll g$ both to mean that $|f| \leqslant C|g|$ for a suitable positive constant C, which may be absolute or depend upon various parameters, in which case the dependence may be indicated in a subscript. Moreover, we write $f \asymp g$ to indicate that $f \ll g$ and $g \ll f$ hold simultaneously. We draw the reader's attention to the fact that we have therefore extended the common use of these symbols to complex-valued functions.

We denote the cardinality of a finite set $\mathcal{A}$ either by $\operatorname{card}\mathcal{A}$, or $|\mathcal{A}|$.

We list below page numbers where various notations in the body of the text are introduced.

$b_r(x)$, B_r, $B_r(x)$	5	$\boldsymbol{\delta}\mathcal{A}$	416	σ_a, σ_c	191
$\mathrm{e}(x)$	73	$\delta(n)$	32	$\sigma_k(n)$	30
$\mathbf{d}\mathcal{A}$	415	$\zeta(s)$	19	$\tau(n)$	30
$j(n)$	34	$\zeta(s,y)$	512	$\tau(n,\vartheta)$	240
$k(n)$	64	$\lambda(n)$	64	$\varphi(n)$	30
$N(T)$	243	$\Lambda(n)$	30	$\Phi(x,y)$	70
$N(x,y)$	198	$\mu(n)$	30	$\chi(n)$,	363
$p_j(n)$	460	ν_N	416	$\chi_0(n)$	364
pp	420	$\xi(s)$	242	$\psi(x)$	36
$S(\mathcal{A},\mathcal{P};y)$	69, 91	$\pi(x)$	11	$\psi(x;a,q)$	370
$v_p(n)$	15	$\pi(x;a,q)$	83	$\Psi(x,y)$	511
$\mathbf{1}(n)$	34	$\varrho(u)$	519	$\omega(n)$, $\Omega(n)$	30
				$\Omega_\pm$	111

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