Expansion in Finite Simple Groups of Lie Type

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In memory of Garth Gaudry, who set me on the road

Contents

Preface

Expander graphs are a remarkable type of graph (or more precisely, a family of graphs) on finite sets of vertices that manage to simultaneously be both sparse (low-degree) and "highly connected" at the same time. They enjoy very strong mixing properties: if one starts at a fixed vertex of an (two-sided) expander graph and randomly traverses its edges, then the distribution of one's location will converge exponentially fast to the uniform distribution. For this and many other reasons, expander graphs are useful in a wide variety of areas of both pure and applied mathematics.

There are now many ways to construct expander graphs, but one of the earliest constructions was based on the *Cayley graphs* of a finite group (or of a finitely generated group acting on a finite set). The expansion property for such graphs turns out to be related to a rich variety of topics in group theory and representation theory, including Kazhdan's property (T), Gowers' notion of a quasirandom group, the sum-product phenomenon in arithmetic combinatorics, and the Larsen-Pink classification of finite subgroups of a linear group. Expansion properties of Cayley graphs have also been applied in analytic number theory through what is now known as the *affine sieve* of Bourgain, Gamburd, and Sarnak, which can count almost prime points in thin groups.

This text is based on the lecture notes from a graduate course on these topics I gave at UCLA in the winter of 2012, as well as from some additional posts on my blog at `terrytao.wordpress.com` on further related topics. The first part of this text can thus serve as the basis for a one-quarter or one-semester advanced graduate course, depending on how much of the optional material one wishes to cover. While the material here is largely self-contained, some basic graduate real analysis (in particular, measure

theory, Hilbert space theory, and the theory of L^p norms), graph theory, and linear algebra (e.g., the spectral theorem for unitary matrices) will be assumed. Some prior familiarity with the classical Lie groups (particularly the special linear group SL_n and the unitary group U_n) and representation theory will be helpful but not absolutely necessary. To follow Section 3.3 (which is optional) some prior exposure to Riemannian geometry would also be useful.

The core of the text is Part 1. After discussing the general theory of expander graphs in the first section, we then specialise to the case of Cayley graphs, starting with the remarkable observation[1] of Margulis linking Kazhdan's property (T) with expansion, and then turning to the more recent observations of Sarnak, Xue, Gamburd, and Bourgain linking the property of finite groups now known as *quasirandomness* with expansion, which is also related to the famous "3/16 theorem" of Selberg. As we will present in this text, this sets up a general "machine" introduced by Bourgain and Gamburd for verifying expansion in a Cayley graph, which in addition to quasirandomness requires two additional ingredients, namely a product theorem and a nonconcentration estimate. These two ingredients are then the focus of the next two sections of this part. The former ingredient uses techniques from arithmetic combinatorics related to the sum-product theorem, as well as estimates of Larsen and Pink on controlling the interaction between finite subgroups of a linear group and various algebraic varieties (such as conjugacy classes or maximal tori). The latter ingredient is perhaps the most delicate aspect of the theory, and often requires a detailed knowledge of the algebraic (and geometric) structure of the ambient group. Finally, we present an application of these ideas to number theory by introducing the basics of sieve theory, and showing how expansion results may be inserted into standard sieves to give new bounds on almost primes in thin groups.

Part 2 contains a variety of additional material that is related to one or more of the topics covered in Part 1, but which can be omitted for the purposes of teaching a graduate course on the subject.

Notation

For reasons of space, we will not be able to define every single mathematical term that we use in this book. If a term is italicised for reasons other than emphasis or for definition, then it denotes a standard mathematical object, result, or concept, which can be easily looked up in any number of references.

[1] This material in Section 2 is not absolutely required for subsequent sections of this part, although it does provide some helpful context for these later sections. Thus, this section may be abridged or even omitted altogether in a lecture course if desired.

(In the blog version of the book, many of these terms were linked to their Wikipedia pages, or other on-line reference pages.)

Given a subset E of a space X, the *indicator function* $1_E : X \to \mathbf{R}$ is defined by setting $1_E(x)$ equal to 1 for $x \in E$ and equal to 0 for $x \notin E$.

The cardinality of a finite set E will be denoted $|E|$. We will use[2] the asymptotic notation $X = O(Y)$, $X \ll Y$, or $Y \gg X$ denote the estimate $|X| \leq CY$ for some absolute constant $C > 0$. In some cases we will need this constant C to depend on a parameter (e.g., d), in which case we shall indicate this dependence by subscripts, e.g., $X = O_d(Y)$ or $X \ll_d Y$. We also sometimes use $X \sim Y$ as a synonym for $X \ll Y \ll X$. If n is a parameter going to infinity, we let $o_{n\to\infty}(1)$ denote a quantity depending on n and bounded in magnitude by $c(n)$ for some quantity $c(n)$ that goes to zero as $n \to \infty$. More generally, given an additional parameter such as k, we let $o_{n\to\infty;k}(1)$ denote a quantity that may depend on both k and n, which is bounded by $c_k(n)$ for some quantity $c_k(n)$ that goes to zero as $n \to \infty$ for each fixed k.

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`terrytao.wordpress.com/category/teaching/254b-expansion-in-groups/`

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[2]Once we deploy the machinery of nonstandard analysis in Section 6, we will use a closely related, but slightly different, asymptotic notation.

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Expander graphs are an important tool in theoretical computer science, geometric group theory, probability, and number theory. Furthermore, the techniques used to rigorously establish the expansion property of a graph draw from such diverse areas of mathematics as representation theory, algebraic geometry, and arithmetic combinatorics. This text focuses on the latter topic in the important case of Cayley graphs on finite groups of Lie type, developing tools such as Kazhdan's property (T), quasirandomness, product estimates, escape from subvarieties, and the Balog–Szemerédi–Gowers lemma. Applications to the affine sieve of Bourgain, Gamburd, and Sarnak are also given. The material is largely self-contained, with additional sections on the general theory of expanders, spectral theory, Lie theory, and the Lang–Weil bound, as well as numerous exercises and other optional material.

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