GRADUATE STUDIES IN MATHEMATICS 170

Colored Operads

Donald Yau

American Mathematical Society

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10 9 8 7 6 5 4 3 2 1 21 20 19 18 17 16

To Eun Soo and Jacqueline

Contents

Part 2. Category Theory

Part 3. Operads and Algebras

Preface

Operads

An *operad* is a mathematical object for organizing operations with multiple, possibly zero, inputs and one output. An operad (Definition 11.6.1)

$$\big(\mathsf{O}, \gamma, \mathbb{1}\big)$$

in a symmetric monoidal category $(\mathsf{M}, \otimes, I)$—for example, M may consist of sets with $\otimes$ the Cartesian product and I the one-point set—consists of

(1) objects $\mathsf{O}(n)$ in M with a right Σ_n-action for all $n \geq 0$, where Σ_n is the symmetric group on n letters;

(2) a *unit* $\mathbb{1} : I \longrightarrow \mathsf{O}(1)$, where I is the $\otimes$-unit in M;

(3) an *operadic composition*

$$\underbrace{\mathsf{O}(n)}_{\substack{n \text{ inputs} \\ 1 \text{ output}}} \otimes \underbrace{\mathsf{O}(k_1) \otimes \cdots \otimes \mathsf{O}(k_n)}_{\substack{k_1 + \cdots + k_n \text{ inputs} \\ n \text{ outputs}}} \xrightarrow{\gamma} \underbrace{\mathsf{O}(k_1 + \cdots + k_n)}_{\substack{k_1 + \cdots + k_n \text{ inputs} \\ 1 \text{ output}}}$$

for all $n \geq 1$, $k_1, \ldots, k_n \geq 0$.

This data is assumed to satisfy some associativity, unity, and equivariance axioms. The key point is that the object $\mathsf{O}(n)$ parametrizes operations with n inputs and 1 output.

The name *operad* was coined by May in [**May72**], where operads were used to study iterated loop spaces. About a decade before [**May72**], Stasheff's study of loop spaces [**Sta63**] already had some of the essential ideas for an operad. At about the same time as the publication of [**May72**], the operadic actions on loop spaces also appeared in the work of Boardman and Vogt [**BV73**], who were using the more general concept of props by Adams and Mac Lane [**Mac65**]. Also, Kelly [**Kel72**] was studying a categorical structure closely related to operads called clubs. It was recognized almost immediately [**Kel05**] that operads are monoids with respect to the circle product and that they could be defined in any bicomplete symmetric monoidal closed categories.

Operads are now standard tools in homotopy theory. Furthermore, they have applications in string topology, algebraic deformation theory, category and higher category theory, homotopical algebras, combinatorics of trees, and vertex operator algebras. Outside of pure mathematics, operads are important in some aspects of mathematical physics, computer science, biology, and other sciences. The appendix entitled *Further Reading* has some relevant references.

Colored Operads

For some recent applications, it is necessary to have a more general form of an operad, called a *colored operad* or a *symmetric multicategory*. Without the symmetric group action, *multicategories* were defined by Lambek [**Lam69**] a few years before [**May72**]. Suppose $\mathfrak{C}$ is a non-empty set whose elements are called colors. A $\mathfrak{C}$*-colored operad* O (Definition 11.2.1) consists of objects

$$\mathsf{O}\binom{d}{c_1,\ldots,c_n} \quad \text{for} \quad d, c_1, \ldots, c_n \in \mathfrak{C}, n \geq 0,$$

parametrizing operations with n inputs indexed by the colors $c_1, \ldots, c_n$ and one output indexed by the color d. There are colored versions of the Σ-action and an operadic composition that is only defined when the colors match. For each color, there is a colored unit. This data is supposed to satisfy colored versions of the operad axioms. So what is called an operad above is a 1-colored operad, where the color set $\mathfrak{C}$ consists of a single color.

Here are a few ways in which colored operads arise.

(1) A small category C—that is, a category with a *set* of objects—is a colored operad O in which the set of objects of C forms the color set $\mathfrak{C}$. The hom-set $\mathsf{C}(x, y)$ is the object $\mathsf{O}\binom{y}{x}$. We will discuss these colored operads in Section 12.3.

(2) Every planar rooted tree T freely generates a colored operad $\Sigma_{\mathsf{p}}(T)$, which we will define in (20.4.2). The colored operad $\Sigma_{\mathsf{p}}(T)$ is important in the study of ∞-operads [**MW07**].

(3) For some applications in algebraic K-theory [**EM06, EM09**], general colored operads are needed.

(4) In the realm of knot theory, a suitably parametrized version of the set of planar tangles is a colored operad [**Jon12**].

(5) Applications in other sciences [**Spi13, Spi14**], such as wiring diagrams, also require general colored operads as opposed to 1-colored operads.

Purpose

This book is an introduction to colored operads and their algebras in symmetric monoidal categories. Various free colored operad functors are discussed in complete detail and in full generality. The reasons for our choices of topics and setting are as follows.

(1) We discuss the more general colored operads instead of 1-colored operads because many recent applications—such as those in ∞-operads, knot theory, and wiring diagrams—require colored operads.

(2) We work at the generality of symmetric monoidal categories because colored operads are most naturally defined on them. Depending on one's intended applications, one may want to work with sets, topological spaces, modules or chain complexes over a commutative ring, or other objects. Symmetric monoidal categories are general enough to include all of these examples and many more.

(3) We discuss free colored operads in detail and in full generality because they are extremely important in several areas, including algebraic deformations, homotopical algebra, higher category theory, and higher algebra.

Audience and Prerequisite

The intended audience of this book includes students and researchers in mathematics, physics, computer science, and other sciences where operads and colored operads are used. Since this book is intended for a broad audience, the mathematical prerequisite is kept to a minimum. Specifically:

(1) The reader is assumed to be familiar with basic concepts of sets and functions, as discussed in, for example, [**Yau13**] (1.1 and 1.2).

(2) The reader is assumed to be comfortable with basic proof techniques, including mathematical induction. Such concepts are covered in most books about the introduction to advanced undergraduate level mathematics, such as [**Vel06, Woh11**].

Some knowledge of permutations and categories is certainly useful but not required. These concepts and many others will be recalled in this book.

In a few instances, we mention some objects—such as topological spaces—that are neither defined nor discussed at length in this book. In those cases, we provide an appropriate reference for the reader to consult.

Features

With a broad audience in mind, here are a few features of this book.

(1) **Motivation**. A lot of space in this book is devoted to motivating definitions and constructions that might be difficult to digest for beginners. Every major concept is thoroughly motivated before it is defined. For example:
   - Section 4.1 provides motivation for collapsing an internal edge in a rooted tree.
   - Section 5.1 provides motivation for grafting of rooted trees.
   - Section 8.1 provides motivation for a monoidal category.
   - Section 13.1 provides motivation for an algebra over a colored operad.
   - Chapters 10, 15, and 17 are entirely devoted to motivating colored operads, partial compositions in a colored operad, and free colored operads, respectively.

   Other such discussion designed to motivate an upcoming definition or construction is clearly marked as *Motivation*.

(2) **Graphical Illustrations**. Rooted trees are a special kind of graphs that play an important role in the theory of colored operads. Part 1 provides a leisurely but rigorous introduction to graphs and rooted trees. There are many figures of graphs and rooted trees throughout this book. They are designed to help the reader visualize the objects being discussed. In total there are more than 100 graphical illustrations. Many of the more complicated definitions and constructions are motivated using these illustrations.

(3) **Exercises**. There are about 150 exercises, collected at the end of almost every chapter. Unless stated otherwise, a text cross-reference to an exercise is to that exercise in the same chapter. For example, the mention of Exercise (2) on page 8 refers to Exercise (2) in Chapter 1. Some of these are routine exercises, but some are more substantial. Many of the longer exercises have hints and outlines. Some of the exercises explore topics that are not treated in the main text. For example, the colored coendomorphism operad and coalgebras over a colored operad are only considered in the exercises in Chapter 13.

Related Literature

There are several excellent monographs about 1-colored operads. Both [**KM95**] and [**LV12**] deal with 1-colored operads in an algebraic setting, namely modules and chain complexes over a commutative ring. The book [**MSS02**] deals with 1-colored operads in a symmetric monoidal category and has ample discussion of applications. Compared to [**KM95, LV12, MSS02**], this book is different in several ways.

(1) The most prominent difference is that our main focus is on *colored* operads, instead of 1-colored operads. Of course, colored operads include 1-colored operads. Whenever we have an important concept about colored operads, we will also state the 1-colored and the colored non-symmetric versions. So everything in this book does apply in the 1-colored case.

(2) This book is designed for a broad audience with no prior knowledge of operads, category theory, or graph theory. Our mathematical prerequisite is minimal, and our discussion goes at a leisurely pace. As a result, we do not go as deeply into the theory as the books [**KM95, LV12, MSS02**]. However, we do discuss free colored operads in complete detail and in full generality in Part 4.

(3) Just like [**MSS02**] but unlike [**KM95, LV12**], we work in the general setting of symmetric monoidal categories. Part 3 of this book is devoted to elementary category theory.

One may use this book as a springboard for more advanced literature on operads, such as [**Fre09, KM95, LV12, MSS02, MT10**]. One may also use this book alongside the monographs [**Spi14, Men15**], both of which discuss applications of colored operads in sets.

Contents

This book is divided into four parts:

Part 1. Graphs and Trees: Chapters 1– 6,

Part 2. Category Theory: Chapters 7–9,

Part 3. Operads and Algebras: Chapters 10–16,

Part 4. Free Colored Operads: Chapters 17–20.

Part 1 and Part 2 can be read independently. Part 3 uses both Part 1 and Part 2, and Part 4 uses all three previous parts. Within each part, the chapters are essentially cumulative. We now provide a brief description of each part and each chapter.

Part 1. Graphs and Trees: Chapters 1– 6.

Rooted trees are a special type of graphs that play several roles in the theory of colored operads. First, they are useful for visualizing definitions and constructions. Second, they provide examples of colored operads, some of which are important in combinatorics and ∞-operads. Furthermore, some constructions, such as the free colored operad functors in Part 4, directly employ rooted trees. Assuming no prior knowledge of graph theory, in Part 1 we develop from scratch the relevant concepts of graphs and rooted trees. The material in Part 1 is used repeatedly in Part 3 and Part 4.

In Chapter 1 we introduce directed graphs with specified inputs and outputs, called directed (m, n)-graphs.

In Chapter 2 we discuss extra structures on graphs, including edge coloring, vertex decoration, input labeling, and incoming edge labeling.

In Chapter 3 we introduce rooted trees, which are special kinds of directed $(m, 1)$-graphs. We discuss several important classes of rooted trees, including exceptional edge, corollas, simple trees, level trees, and linear graphs. All of these rooted trees will be referred to in later chapters.

In Chapter 4 we discuss the construction of collapsing an internal edge in a rooted tree. This construction is important in Part 4 when we discuss the general operadic composition in a colored non-symmetric operad.

In Chapter 5 we discuss grafting of rooted trees and observe that grafting is unital and associative. It is then observed that every rooted tree admits a grafting decomposition into corollas. This decomposition is used in several constructions in later chapters.

In Chapter 6 we discuss how the extra structures on graphs in Chapter 2 are extended to the grafting of two rooted trees.

Part 2. Category Theory: Chapters 7–9.

To learn about colored operads, it is important that one knows a little bit of category theory. The most natural setting on which a colored operad can be defined is a symmetric monoidal category. Moreover, in order to discuss free colored operads in Part 4, we need the concept of adjoint functors. Assuming no prior knowledge of category theory, the main purpose of Part 2 is to discuss some basic category theory so that colored operads, free colored operads, and so forth can be properly discussed in Part 3 and Part 4.

In Chapter 7 we introduce the most basic concepts of category theory, including categories, functors, natural transformations, equivalence, isomorphism of categories, coproducts, products, and adjoint functors. For the purpose of this book, the most important examples of categories are in Example 7.3.14. These are very common categories on which colored operads are defined. They are referred to multiple times in later chapters.

In Chapter 8 we discuss symmetric monoidal categories. These are categories equipped with a form of multiplication, somewhat similar to the tensor product of vector spaces. In the majority of the rest of this book, we work over a symmetric monoidal category satisfying some natural conditions as stated in Assumption 8.8.1.

In Chapter 9 we introduce colored symmetric sequences and colored objects. Every colored operad has an underlying colored symmetric sequence, which captures its equivariant structure. For a fixed non-empty set of colors, colored symmetric sequences form a diagram category. Colored objects are needed to discuss algebras over a colored operad and some forgetful functors about colored operads.

Part 3. Operads and Algebras: Chapters 10–16.

The main purposes of Part 3 are

(1) to introduce colored operads and their algebras in a symmetric monoidal category;
(2) to discuss partial compositions.

These partial compositions provide another way to define a colored operad and are used multiple times in Part 4.

In Chapter 10 we provide motivation for the definition of a colored operad. As a warm-up exercise, first we discuss how the axioms of a category can be understood via linear graphs. Using categories as a model, we then discuss how switching from linear graphs to level trees naturally leads to a colored operad. The main point is that the definition of a colored operad—the operadic composition and the associativity axiom in particular—can be easily visualized using a few pictures of level trees.

In Chapter 11 we first define colored operads in a symmetric monoidal category. Then we construct the change-of-base category adjunction. We also state the special cases of a 1-colored operad, where the color set contains a single element, and of a colored non-symmetric operad, where there is no equivariant structure.

In Chapter 12 we consider colored operads that are concentrated in arity 1. In the 1-colored case, these are monoids. In the general colored case, these are small enriched categories.

In Chapter 13 we define algebras over a colored operad in a symmetric monoidal category and discuss the colored endomorphism operad. The latter provides a different way to define an algebra over a colored operad as a map of colored operads. This second definition of an operadic algebra is useful in applications when one wishes to transfer an operadic algebra structure along a map.

In Chapter 14 we discuss a few examples of algebras over a colored operad, including the initial and the terminal object in the category of algebras. The (colored) operads for monoids, monoid maps, and colored monoids are described in detail.

In Chapter 15 we provide motivation for the partial compositions in a colored operad. The main point is that partial compositions correspond to simple trees. Using simple trees we explain how one can anticipate the definition of the partial compositions.

In Chapter 16 we introduce colored pseudo-operads, which have partial compositions rather than an operadic composition. Partial compositions are in some ways simpler than an operadic composition because the former are binary operations. The main observation is that the two concepts, colored operads and colored pseudo-operads, are in fact equivalent. Near the end of this chapter, we discuss the colored rooted trees operad and the little square operad.

Part 4. Free Colored Operads: Chapters 17–20.

The main purpose of Part 4 is to discuss the free colored operad functors. There are three such functors, depending on which forgetful functor is considered.

In Chapter 17 we provide motivation for the various free colored operad functors. The main point is that these functors are closely related to rooted trees. As a warm-up exercise, we discuss the free monoid functor in detail. The constructions of the free colored operad functors in later chapters follow similar steps as the monoid case.

In Chapter 18 we introduce the general operadic composition in a colored non-symmetric operad. The domain of the general operadic composition is parametrized by a planar rooted tree. The main observation is that the general operadic composition is associative with respect to grafting of rooted trees. This observation is an essential ingredient in the construction of the free colored non-symmetric operad functor.

In Chapter 19 we consider the left adjoint of the forgetful functor from colored non-symmetric operads to colored objects. This left adjoint is called the free colored non-symmetric operad functor. Near the end of this chapter, we discuss the free colored non-symmetric operad generated by a planar rooted tree. This colored operad is an important construction in the theory of ∞-operads.

In Chapter 20 we first consider the left adjoint of the forgetful functor from colored operads to colored non-symmetric operads. This left adjoint is called the symmetrization functor. Next we consider the left adjoint of the forgetful functor from colored operads all the way down to colored objects. This left adjoint is called the free colored operad functor. Near the end of this chapter, we describe the free colored operad generated by a planar rooted tree.

In an appendix entitled *Further Reading*, we list some references about operads, loosely divided into different topics.

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Donald Yau

List of Notations

	Notation	Page	Description
Ch. 1	$\varnothing$	3	empty set
	$\lvert S \rvert$	3	the number of elements in S
	$S^{\times n}$	3	set of n-tuples of elements in S
	$S^{\underline{2}}$	3	set of unordered pairs in S not of the form $\{x, x\}$ for $x \in S$
	$x \longmapsto f(x)$	4	the image of x is $f(x)$
	Id_S, Id	4	identity function
	gf, $g \circ f$	4	composition of functions
	$\cong$	4	bijection/isomorphism
	$S \subseteq T$	4	S is a subset of T
	$T \smallsetminus S$	4	set difference
	$S \cap T$	4	intersection
	$S_1 \times \cdots \times S_n$	4	product
	$\prod_{i=1}^n S_i$	4	product
	$S_1 \amalg \cdots \amalg S_n$	5	coproduct/disjoint union
	$\coprod_{i=1}^n S_i$	5	coproduct/disjoint union
	$x \sim y$	5	x and y are identified
	(V, E)	5	graph with abstract vertices V and edges E
	$\{x, y\}$	5	an edge with abstract end-vertices x and y
	$(x_0, \ldots, x_l)$	6	a path in a graph
	(x, y)	8	an edge with initial vertex x and terminal vertex y
	$\mathsf{in}(v)$	9	set of incoming edges of v
	$\mathsf{out}(v)$	9	set of outgoing edges of v

	Notation	Page	Description
	$(V, E, \mathsf{in}_G, \mathsf{out}_G)$	10	directed (m,n)-graph with inputs in_G and outputs out_G
	Vt_G	11	set of vertices in G
	Int_G	11	set of internal edges in G
	$\uparrow$	16	exceptional edge
Ch. 2	$\{*\}$	20	one-element set
	$[m]$	21	the set $\{1,\ldots,m\}$, empty if $m = 0$
	$(S, \leq)$	22	ordered set
Ch. 3	$\mathsf{inprof}(v)$	31	incoming profile of v
	$\mathsf{inprof}(T)$	32	input profile of T
	C_m	33	m-corolla
	$T^n_m(j)$	35	simple tree
	$\mathsf{lev}(v)$	36	level of v
	$\max(S)$	36	maximum element in a finite set S of integers
	L_k	40	k-level linear graph
Ch. 4	T/e	45	T with internal edge e collapsed
Ch. 5	$T_1 \circ_e T_2$	55	grafting of T_1 and T_2 along e
Ch. 6	$\kappa_1 \circ_e \kappa_2$	76	induced $\mathfrak{C}$-coloring
	$\lambda_1 \circ_e \lambda_2$, $\lambda_1 \circ_j \lambda_2$	78	induced input labeling
	$\Psi_1 \circ_e \Psi_2$	80	induced incoming edge labeling
	β_T	81	canonical vertex labeling of T
	λ_T	82	canonical input labeling of T
Ch. 7	$\mathsf{Ob}(\mathsf{C})$	90	class of objects in a category
	$\mathsf{C}(a,b)$, $\mathsf{C}(a;b)$	90	morphism set with domain a and codomain b
	1_a, Id_a	90	identity morphism of a
	$\circ$	90	composition in a category
	$f : a \longrightarrow b$	91	morphism $f \in \mathsf{C}(a,b)$
	$a \xrightarrow{f} b$	91	morphism $f \in \mathsf{C}(a,b)$
	$\mathbf{0}$	91	empty category
	Σ_n	93	symmetric group on n letters
	Group	93	category of groups
	Ab	93	category of abelian groups

	Notation	Page	Description
	Ring	93	category of rings
	Top	93	category of topological spaces
	$\mathsf{dis}(\mathsf{C})$	94	discrete category associated to C
	C^{op}	96	opposite category of C
	Δ	96	simplicial category
	$\underline{n}$	96	totally ordered set $\{0 < 1 < \cdots < n\}$
	C^S, $\prod_S \mathsf{C}$	97	product category
	$\prod_{s\in S} \mathsf{C}_s$	97	product category
	Id_{C}	98	identity functor on C
	G_{ab}	99	abelianization
	$\mathsf{C} \cong \mathsf{D}$	99	isomorphism of categories
	C^{D}, $\mathsf{Fun}(\mathsf{D},\mathsf{C})$	103	diagram category
	Set	103	category of sets
	$\mathsf{Mod}(R)$	104	category of left R-modules
	$\mathsf{Chain}(R)$	104	category of chain complexes of left R-modules
	Cat	104	category of small categories
	CHau	104	category of compactly generated Hausdorff spaces
	SSet	105	category of simplicial sets
	$\mathbf{n}$	106	groupoid associated to Σ_{n+1}
	Set_n	106	groupoid of totally ordered sets with $n+1$ elements
	$\coprod_{s\in S} x_s$	107	coproduct of objects $\{x_s\}_{s\in S}$
	$\varnothing$	108	initial object in a category
	$\prod_{s\in S} x_s$	109	product of objects $\{x_s\}_{s\in S}$
	$*$	109	terminal object in a category
	$X \times Y$	110	product of two objects
	F_R	113	free left R-module functor
	M^G	118	category of objects in M equipped with a left G-action
Ch. 8	$\otimes$	123	monoidal product
	$X^{\otimes 0}$, $X^{\otimes\varnothing}$	124	empty tensor product
	$\mathbf{1}$	125	discrete category with one object
	$\xi_{X,Y}$	130	symmetry isomorphism
	(i,j)	132	transposition
	$X^{\otimes n}$	133	iterated tensor product
	$\bigotimes_{j=1}^{n} X_j$	133	iterated tensor product
	$\mathsf{Hom}_{\mathsf{M}}$	135	internal hom

	Notation	Page	Description
Ch. 12	Mon	204	category of monoids
	$\mathsf{Mon}(\mathsf{C})$	207	category of monoids in C
	$\mathsf{Mon}^{\mathfrak{C}}(\mathsf{C})$	210	category of $\mathfrak{C}$-colored monoids in C
Ch. 13	$X_{\underline{c}}$	217	$X_{c_1} \otimes \cdots \otimes X_{c_n}$
	$\mathsf{Alg}(\mathsf{O})$	219	category of O-algebras
	$\mathsf{End}(X)$	226	$\mathfrak{C}$-colored endomorphism operad
Ch. 14	$\varnothing^{\mathsf{O}}$	238	initial O-algebra
	$*^{\mathsf{O}}$	238	terminal O-algebra
	As	241	operad for monoids
	$\mathbf{2}$	247	category with two objects and one non-identity morphism
	$\mathsf{As}^{\mathbf{2}}$	249	2-colored operad for monoid maps
	$\mathsf{As}^{\mathfrak{C}}$	255	$\mathfrak{C}^{\times 2}$-colored non-Σ operad for $\mathfrak{C}$-colored monoids in M
	$\mathsf{CMon}(\mathsf{M})$	259	category of commutative monoids
	Com	259	operad for commutative monoids
Ch. 16	$\underline{a} \circ_i \underline{b}$	276	$\circ_i$ of $\mathfrak{C}$-profiles
	$\sigma \circ_i \tau$	276	$\circ_i$-permutation
	$\circ_i$	278	comp-i composition
	$\mathsf{Operad}_{\circ}^{\Sigma(\mathfrak{C})}(\mathsf{M})$	281	category of $\mathfrak{C}$-colored pseudo-operads
	$\mathsf{Operad}_{\circ}^{\Sigma}(\mathsf{M})$	295	category of 1-colored pseudo-operads
	$\mathsf{Operad}_{\circ}^{\Omega(\mathfrak{C})}(\mathsf{M})$	297	category of $\mathfrak{C}$-colored non-Σ pseudo-operads
	$\mathsf{Alg}_{\circ}(\mathsf{O})$	300	category of O-algebras
	$\mathsf{Tree}_{\mathsf{i}}^{\mathfrak{C}}\binom{d}{\underline{c}}$	302	set of isomorphism classes of $\mathfrak{C}$-colored rooted trees with input labeling and profile $\binom{d}{\underline{c}}$
	$\mathsf{Tree}_{\mathsf{i}}^{\mathfrak{C}}$	304	$\mathfrak{C}$-colored rooted trees operad
	$\mathcal{I}$	306	closed interval $[0,1]$
	$\mathcal{I}^2$	306	standard unit square $[0,1]^{\times 2}$
	$\mathcal{J}^2$	306	interior of the standard unit square
	$\mathsf{C}_2(n)$	307	nth space of the little square operad
	C_2	308	the little square operad
	C_n	316	the little n-cube operad

	Notation	Page	Description
Ch. 18	$[T, \kappa, \Psi]$, $[T]$	336	isomorphism class of a $\mathfrak{C}$-colored planar rooted tree
	$\mathsf{Tree}_{\mathsf{p}}^{\mathfrak{C}}\binom{d}{\underline{c}}$	337	set of isomorphism classes of $\mathfrak{C}$-colored planar rooted trees with profile $\binom{d}{\underline{c}}$
	$\mathsf{Tree}_{\mathsf{p}}(n)$	337	set of isomorphism classes of 1-colored planar rooted trees with n inputs
	$X(v)$	337	X-decoration of a vertex v
	$X[T]$	337	X-decoration of $[T]$
	$\gamma_{[T]}$	341	$[T]$-shaped composition
Ch. 19	U^{Ω}	351	forgetful functor from colored non-Σ operads to colored objects
	F^{Ω}	353	free colored non-Σ operad functor
	X_T	372	colored object of a planar rooted tree
	$\Omega_{\mathsf{p}}(T)$	373	free colored non-Σ operad generated by T
Ch. 20	U_1	382	forgetful functor from colored operads to colored non-Σ operads
	F_1	382	symmetrization functor
	U^{Σ}	395	forgetful functor from colored operads to colored objects
	$\mathsf{Tree}_{\mathsf{ip}}^{\mathfrak{C}}\binom{d}{\underline{c}}$	396	set of isomorphism classes of $\mathfrak{C}$-colored planar rooted trees with an input planar labeling and profile $\binom{d}{\underline{c}}$
	F^{Σ}	398	free colored operad functor
	$\Sigma_{\mathsf{p}}(T)$	404	free colored operad generated by T

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List of Main Facts

	Reference	Fact
Part 1	p. 16	A directed connected forest can always be drawn with all the edges pointing upward.
	3.4.1	Up to isomorphisms, the exceptional edge $\uparrow$ is the only rooted tree with no root vertex.
	3.4.3	Up to isomorphisms, corollas are the only rooted trees with one vertex.
	3.5.2	Up to isomorphisms, the simple trees $T_m^n(j)$ are the only rooted trees with two vertices.
	3.7.3	Up to isomorphisms, $\uparrow$ and L_k for $k \geq 1$ are the only linear graphs.
	4.4.1	Collapsing internal edges is associative.
	5.4.1	Grafting is unital.
	5.5.3	Grafting is horizontally associative.
	5.6.3	Grafting is vertically associative.
	5.7.3	Every rooted tree has a grafting decomposition.
	5.7.15	Every rooted tree is an iterated grafting of corollas.
Part 2	7.1.17	A group is a 1-object groupoid.
	7.4.4	Equivalence is strictly weaker than isomorphism.
	7.7.6	Free module is a left adjoint.
	7.7.11	Left adjoints are characterized by a universal property.
	p. 117	Adjunctions can be composed.
	p. 117	Every small non-empty groupoid decomposes into a coproduct of maximal connected subgroupoids.

	Reference	**Fact**
	8.4.1	Every monoidal category is equivalent to a strict monoidal category via strong monoidal functors.
	8.7.6	Set, $\mathsf{Mod}(R)$, $\mathsf{Chain}(R)$, Cat, CHau, and SSet are symmetric monoidal closed categories.
	9.2.11	The permutation category $\Sigma_{[\underline{a}]}$ is the maximal connected subgroupoid of $\Sigma(\mathfrak{C})$ containing $\underline{a}$.
	9.2.11	The groupoid $\Sigma(\mathfrak{C})$ of $\mathfrak{C}$-profiles is the coproduct of the permutation categories.
	9.3.10	The category $\mathsf{Seq}^{\Sigma(\mathfrak{C})}(\mathsf{M})$ of $\mathfrak{C}$-colored symmetric sequences splits as a product of $\mathsf{M}^{\Sigma^{\mathsf{op}}_{[\underline{c}]}\times\{d\}}$.
	9.3.18	The category $\mathsf{Seq}^{\Sigma}(\mathsf{M})$ of 1-colored symmetric sequences splits as a product of $\mathsf{M}^{\Sigma^{\mathsf{op}}_{n}}$.
	p. 156	The forgetful functor from $\mathsf{Seq}^{\Sigma(\mathfrak{C})}(\mathsf{M})$ to $\mathsf{M}^{\mathsf{Prof}(\mathfrak{C})\times\mathfrak{C}}$ has a left adjoint.
Part 3	11.4.1	The category $\mathsf{Operad}^{\Sigma(\mathfrak{C})}(\mathsf{M})$ of $\mathfrak{C}$-colored operads has an initial object.
	11.4.6	$\mathsf{Operad}^{\Sigma(\mathfrak{C})}(\mathsf{M})$ has a terminal object.
	11.5.1	Every symmetric monoidal functor extends to a functor between the respective categories of $\mathfrak{C}$-colored operads.
	12.2.6	A monoid is equivalent to a 1-colored operad concentrated in arity 1.
	12.3.8	A $\mathfrak{C}$-colored monoid is equivalent to a $\mathfrak{C}$-colored operad concentrated in arity 1.
	13.9.1	An O-algebra is equivalent to a map $\mathsf{O} \longrightarrow \mathsf{End}(X)$ of $\mathfrak{C}$-colored operads.
	(13.10.4)	An O-algebra map f is equivalent to a map $\mathsf{O} \longrightarrow \mathsf{End}(f)$ of $\mathfrak{C}$-colored operads.
	14.1.1	$\mathsf{Alg}(\mathsf{O})$ has an initial object.
	14.1.2	$\mathsf{Alg}(\mathsf{O})$ has a terminal object.
	14.2.18	There is a 1-colored operad As whose algebras are monoids.
	14.3.9	There is a 2-colored operad $\mathsf{As}^{\mathbf{2}}$ whose algebras are monoid maps.
	14.4.7	There is a $\mathfrak{C}^{\times 2}$-colored non-Σ operad $\mathsf{As}^{\mathfrak{C}}$ whose algebras are $\mathfrak{C}$-colored monoids.
	p. 259	There is a 1-colored operad Com whose algebras are commutative monoids.

	Reference	Fact
	p. 261	Left modules over a monoid X are equivalent to algebras over the 1-colored operad of X.
	p. 262	Bimodules over a monoid X are equivalent to algebras over a 1-colored operad $\mathsf{O}_{X,X}$.
	p. 263	For a $\mathfrak{C}$-colored operad O and a small category D, there is a colored operad O^{D} whose algebras are D-diagrams of O-algebras.
	p. 263	There is a colored operad of 1-colored operads.
	p. 263	There is a colored operad of $\mathfrak{C}$-colored operads.
	16.4.1	Colored operads are equivalent to colored pseudo-operads.
	16.5.12	1-colored operads are equivalent to 1-colored pseudo-operads.
	16.6.6	Colored non-Σ operads are equivalent to colored non-Σ pseudo-operads.
	16.7.8	O-algebra in its original sense is equivalent to O-algebra when O is regarded as a $\mathfrak{C}$-colored pseudo-operad.
	16.8.9	There is a $\mathfrak{C}$-colored operad $\mathsf{Tree}_{\mathsf{i}}^{\mathfrak{C}}$ in Set whose elements are isomorphism classes of $\mathfrak{C}$-colored rooted trees with an input labeling.
Part 4	17.1.15	There is a free monoid functor.
	18.3.2	The general operadic composition is associative with respect to grafting.
	19.3.7	The forgetful functor from $\mathfrak{C}$-colored non-Σ operads to $(\mathsf{Prof}(\mathfrak{C}) \times \mathfrak{C})$-colored objects admits a left adjoint F^{Ω}.
	(19.5.4)	Every planar rooted tree T generates a colored non-Σ operad $\Omega_{\mathsf{p}}(T)$ in Set.
	(19.5.7)	Each non-empty entry of $\Omega_{\mathsf{p}}(T)$ contains one element.
	20.2.6	The forgetful functor from $\mathfrak{C}$-colored operads to $\mathfrak{C}$-colored non-Σ operads admits a left adjoint F_1, the symmetrization functor.
	20.3.22	The forgetful functor from $\mathfrak{C}$-colored operads to $(\mathsf{Prof}(\mathfrak{C}) \times \mathfrak{C})$-colored objects admits a left adjoint F^{Σ}, the free $\mathfrak{C}$-colored operad functor.
	(20.4.2)	Every planar rooted tree T generates a colored operad $\Sigma_{\mathsf{p}}(T)$ in Set.
	(20.4.4)	Each non-empty entry of $\Sigma_{\mathsf{p}}(T)$ contains one element.

Index

The subject of this book is the theory of operads and colored operads, sometimes called symmetric multicategories. A (colored) operad is an abstract object which encodes operations with multiple inputs and one output and relations between such operations. The theory originated in the early 1970s in homotopy theory and quickly became very important in algebraic topology, algebra, algebraic geometry, and even theoretical physics (string theory). Topics covered include basic graph theory, basic category theory, colored operads, and algebras over colored operads. Free colored operads are discussed in complete detail and in full generality.

The intended audience of this book includes students and researchers in mathematics and other sciences where operads and colored operads are used. The prerequisite for this book is minimal. Every major concept is thoroughly motivated. There are many graphical illustrations and about 150 exercises. This book can be used in a graduate course and for independent study.

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