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To Yansu, Raymond, and Tommy

Contents

Preface

The theory of nonlinear elliptic partial differential equations of the second order has flourished in the past half-century. The pioneering work of de Giorgi in 1957 opened the door to the study of general quasilinear elliptic differential equations. Since then, the nonlinear elliptic differential equation has become a diverse subject and has found applications in science and engineering. In mathematics, the development of elliptic differential equations has influenced the development of the Riemannian geometry and complex geometry. Meanwhile, the study of elliptic differential equations in a geometric setting has provided interesting new questions with fresh insights to old problems.

This book is written for those who have completed their study of the linear elliptic differential equations and intend to explore the fascinating field of nonlinear elliptic differential equations. It covers two classes of nonlinear elliptic differential equations, quasilinear and fully nonlinear, and focuses on two important nonlinear elliptic differential equations closely related to geometry, the mean curvature equation and the Monge-Ampère equation.

This book presents a detailed discussion of the Dirichlet problems for quasilinear and fully nonlinear elliptic differential equations of the second order: quasilinear uniformly elliptic equations in arbitrary domains, mean curvature equations in domains with nonnegative boundary mean curvature, fully nonlinear uniformly elliptic equations in arbitrary domains, and Monge-Ampère equations in uniformly convex domains. Global solutions of these equations are also characterized. The choice of topics is influenced by my personal taste. Some topics may be viewed by others as too advanced for a graduate textbook. Among those topics are the curvature estimates for minimal surface equations, the complex Monge-Ampère equation, and the

generalized solutions of the (real) Monge-Ampère equations. Inclusion of these topics reflects their importance and their connections to many of the most active current research areas.

There is an inevitable overlap with the successful monograph by Gilbarg and Trudinger. This book, designed as a textbook, is more focused on basic materials and techniques. Many results in this book are presented in special forms. For example, the quasilinear and fully nonlinear uniformly elliptic differential equations studied in this book are not in their most general form. The study of these equations serves as a prerequisite to the study of the mean curvature equation and the Monge-Ampère equation, respectively. More notably, our discussion of the Monge-Ampère equations is confined to the *pure* Monge-Ampère equations, instead of the Monge-Ampère type equations.

This book is based on one-semester courses I taught at Peking University in the spring of 2011 and at the University of Notre Dame in the fall of 2011. Part of it was presented in the Special Lecture Series at Peking University in the summer of 2007, in the Summer School in Mathematics at the University of Science and Technology of China in the summer of 2008, and in a graduate course at Beijing International Center of Mathematical Research in the spring of 2010.

During the writing of the book, I benefitted greatly from comments and suggestions of many friends, colleagues, and students in my classes. Chuanqiang Chen, Xumin Jiang, Weiming Shen, and Yue Wang read the manuscript at various stages. Chuanqiang Chen and Jingang Xiong helped write Chapter 8. Bo Guan, Marcus Khuri, Xinan Ma, and Yu Yuan provided valuable suggestions on the arrangement of the book.

It is with pleasure that I record here my gratitude to my thesis advisor, Fanghua Lin, who guided me into the fascinating world of elliptic differential equations more than twenty years ago.

I am grateful to Arlene O'Sean, my editor at the American Mathematical Society, for reading the manuscript and guiding the effort to turn it into a book. Last but not least, I thank Sergei Gelfand at the AMS for his help in bringing the book to press.

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Qing Han

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Nonlinear elliptic differential equations are a diverse subject with important applications to the physical and social sciences and engineering. They also arise naturally in geometry. In particular, much of the progress in the area in the twentieth century was driven by geometric applications, from the Bernstein problem to the existence of Kähler–Einstein metrics.

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This book, designed as a textbook, provides a detailed discussion of the Dirichlet problems for quasilinear and fully nonlinear elliptic differential equations of the second order with an emphasis on mean curvature equations and on Monge–Ampère equations. It gives a user-friendly introduction to the theory of nonlinear elliptic equations with special attention given to basic results and the most important techniques. Rather than presenting the topics in their full generality, the book aims at providing self-contained, clear, and "elementary" proofs for results in important special cases. This book will serve as a valuable resource for graduate students or anyone interested in this subject.

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