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Appendix A

Linear Algebra and Representation Theory

In this appendix we briefly discuss tensors (§A.1), matrix Lie groups (§A.2) and Lie algebras (§A.4), complex vector spaces and complex structures (§A.3), the octonions and the exceptional group G_2 (§A.5), Clifford algebras and spin groups (§A.7), and outline some rudiments of representation theory (§A.6).

Unless otherwise noted, V, W are real vector spaces of dimensions n and m, with bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$. We use the index ranges $1 \leq i, j \leq n$ and $1 \leq s, t \leq m$.

A.1. Dual spaces and tensor products

A map $f : V \to W$ is *linear* if $f(v + v') = f(v) + f(v')$ and $f(kv) = kf(v)$ for all $v, v' \in V$ and $k \in \mathbb{R}$.

Definition A.1.1. The *dual space* of V, denoted by V^*, is the space of all linear maps $f : V \to \mathbb{R}$. It is a vector space under the operations of scalar multiplication and addition of maps.

Exercises A.1.2:
1. Let $\alpha^i \in V^*$ be defined by $\alpha^i(v_j) = \delta^i_j$. Show that $\alpha^1, \ldots, \alpha^n$ is a basis for V^*, called the *dual basis* to $v_1, \ldots, v_n$. In particular, $\dim V^* = n$.
2. Define, in a coordinate-free way, an isomorphism $V \to (V^*)^*$. (Note that these spaces would not necessarily be isomorphic if V were replaced by an infinite-dimensional vector space.)

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Geometric Aside. Given V, one can form the associated projective space $\mathbb{P}V$, which is the space of all lines thorough the origin in V. A nonzero vector $\alpha \in V^*$ determines a codimension-one linear subspace $\ker \alpha \subset V$, so that $\mathbb{P}V^*$ may be interpreted as the space of hyperplanes through the origin in V.

Definition A.1.3. Let $\mathrm{Hom}(V, W)$ denote the space of linear maps from V to W. Like V^*, this is a vector space under the addition of maps. The space of linear maps from V to V, called *endomorphisms*, is denoted by $\mathrm{End}(V)$.

We may think of $\mathrm{Hom}(V, W)$ as the space of W-valued linear functions on V, and when doing so we denote it by $V^* \otimes W$, the *tensor product* of V^* and W. (Taking the tensor product with $W = \mathbb{R}$ is trivial because $V^* \otimes \mathbb{R} = \mathrm{Hom}(V, \mathbb{R}) = V^*$.) Similarly, we define $V \otimes W$ as the space of W-valued linear functions on V^*.

Given $\alpha \in V^*$ and $w \in W$, let $\alpha \otimes w$ denote the element of $\mathrm{Hom}(V, W)$ defined by

$$v \mapsto \alpha(v)w.$$

We call such an element *decomposable*.

Exercises A.1.4:

1. Show that the decomposable elements in $V \otimes W$ span $V \otimes W$. More precisely, show that the nm vectors $\{v_i \otimes w_s\}$ span $V \otimes W$.
2. After having fixed bases, define an explicit isomorphism between $V^* \otimes W$ and the space of $n \times m$ matrices.
3. Show that the decomposable elements $V^* \otimes W$ are exactly those represented by *rank one* matrices. More generally, show that the *rank* of an element of $V \otimes W$ is well-defined and agrees with the rank of the associated matrix (with respect to any choices of bases).
4. Given $f \in \mathrm{Hom}(V, W)$ we define $f^t \in \mathrm{Hom}(W^*, V^*)$, called the *transpose* or *adjoint* of f, by $f^t(\beta)(v) = \beta(f(v))$. (If we write basis and dual basis vectors as columns vectors, the matrix representative of f^t is the transpose of that of f.) Show that the transpose defines a vector space isomorphism $\mathrm{Hom}(V, W) \cong \mathrm{Hom}(W^*, V^*)$.

Definition A.1.5. Let $V_1, \ldots, V_k$ be vector spaces. A function

$$f : V_1 \times \ldots \times V_k \to W \tag{A.1}$$

is *multilinear* if it is linear with respect to addition and scalar multiplication in each factor V_ℓ. We denote this space of multilinear functions by $V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^* \otimes W$. In particular, $V^{*\otimes 2} = V^* \otimes V^*$ is the space of real-valued bilinear forms on V, and $V^{*\otimes 3} = V^* \otimes V^* \otimes V^*$ is the space of trilinear forms, etc.

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The space $V_1^* \otimes \cdots \otimes V_k^*$ is canonically isomorphic to any re-ordering of the factors and we will make free use of this re-ordering isomorphism without mention.

For $\beta_1 \in V_1^*, \ldots, \beta_k \in V_k^*$, define $\beta_1 \otimes \cdots \otimes \beta_k \in V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$ by

$$\beta_1 \otimes \cdots \otimes \beta_k(u_1, \ldots, u_k) = \beta_1(u_1) \cdots \beta_k(u_k)$$

and call an element of $V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$ *decomposable* if it may be written in this way.

Exercises A.1.6:

1. Show that $(\alpha, v) \mapsto \alpha(v)$ is multilinear from $V^* \times V$ to $\mathbb{R}$.
2. Show that the space of multilinear functions (A.1) is a vector space, and determine its dimension.
3. Show that $V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$ is spanned by its decomposable vectors. ⊚
4. Given $\alpha \in V^*$, $\beta \otimes W^*$, let $\alpha \otimes \beta(v \otimes w) = \alpha(v)\beta(w)$. Show that this defines an isomorphism $V^* \otimes W^* \cong (V \otimes W)^*$. Thus, $V \otimes W$ may be thought of as the set of linear maps from V^* to W, the set of linear maps from W^* to V, the set of bilinear maps from $V^* \times W^*$ to $\mathbb{R}$, or as the dual space of $V^* \otimes W^*$.
5. Let $V^{\otimes k}$ denote the k-fold tensor product of V with itself. Show that this is the dual space of $V^{*\otimes k}$.

Remark A.1.7. One may define the *rank* of an element $X \in V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$ to be the minimal number r such that $X = \sum_{u=1}^r z_u$ with each z_u decomposable. It turns out that the rank is quite subtle for $k > 2$. In fact the maximal rank of an element of a triple tensor product is not known even for low-dimensional vector spaces. Such open questions go under the name *Waring problems*. See §12.5 for a geometric generalization.

An open question of importance to computer science is the following: Let A, B, C be vector spaces, and consider the matrix multiplication operator M that composes a linear map from A to B with a linear map from B to C to obtain a linear map from A to C. Letting $V_1 = A^* \otimes B$, $V_2 = B^* \otimes C$, $V_3 = A^* \otimes C$, we have $M \in V_1^* \otimes V_2^* \otimes V_3$. Then the open question is, determine the rank of M. For an overview of this problem, see [**133**].

Symmetric and skew-symmetric tensors. The tensor product $\otimes$ is not symmetric; even in $V \otimes V$, $v_1 \otimes v_2 \neq v_2 \otimes v_1$.

Consider $V^{\otimes 2} = V \otimes V$ with basis $\{v_i \otimes v_j | 1 \leq i, j \leq n\}$. The subspaces defined by

$$\begin{aligned} S^2V &:= \text{span}\{v_i \otimes v_j + v_j \otimes v_i,\ 1 \leq i, j \leq n\} \\ &= \text{span}\{v \otimes v \,|v \in V\} \\ &= \{X \in V \otimes V \mid X(\alpha, \beta) = X(\beta, \alpha)\ \forall \alpha, \beta \in V^*\} \end{aligned}$$

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and

$$\begin{aligned}\Lambda^2 V &:= \operatorname{span}\{v_i\otimes v_j - v_j\otimes v_i,\ 1\le i,j\le n\}\\ &= \operatorname{span}\{v\otimes w - w\otimes v \,|v,w\in V\}\\ &= \{X\in V\otimes V \mid X(\alpha,\beta) = -X(\beta,\alpha)\ \forall \alpha,\beta\in V^*\}\end{aligned}$$

are respectively the spaces of *symmetric* and *skew-symmetric* 2-tensors of V. For arbitrary $v_1, v_2 \in V$, we define $v_1v_2 := \frac{1}{2}(v_1\otimes v_2 + v_2\otimes v_1) \in S^2V$ and

$$v_1 \wedge v_2 := v_1\otimes v_2 - v_2\otimes v_1 \in \Lambda^2 V. \tag{A.2}$$

(Sometimes a factor of $\frac{1}{2}$ is inserted into the definitions of v_1v_2 and $v_1 \wedge v_2$.) More generally,

$$v_1 \wedge \cdots \wedge v_k := \sum_{\sigma\in\mathfrak{S}_k} (\operatorname{sgn}\sigma) v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)}, \tag{A.3}$$

where $\mathfrak{S}_k$ denotes the group of permutations on k elements and $\operatorname{sgn}\sigma = \pm 1$ is the sign of σ. The product $v_1 \wedge \cdots \wedge v_k$ is called the *wedge product* of the vectors $v_1, \dots, v_k$. Let $\mathcal{A} \in \operatorname{End}(\Lambda^k V)$ be defined by linearly extending $v_1 \otimes \ldots \otimes v_k \mapsto v_1 \wedge \ldots \wedge v_k$ to combinations of decomposable vectors.

Exercises A.1.8:

1. Show that the two definitions for S^2V are equivalent, as well as the two definitions for $\Lambda^2 V$. Show that

$$V\otimes V = S^2V \oplus \Lambda^2 V. \tag{A.4}$$

Note that the coordinate-free definition implies that this decomposition is preserved under linear changes of coordinates.

2. Give similar definitions for the totally symmetric k-tensors $S^kV \subset V^{\otimes k}$ and the alternating k-tensors $\Lambda^k V \subset V^{\otimes k}$. ⊚

We often think of S^kV^* as the space of homogeneous polynomials of degree k on V.

3. Given $\alpha \in \Lambda^i V, \beta \in \Lambda^j V$, define $\alpha\wedge\beta \in \Lambda^{i+j}V$ by composing the inclusion $\Lambda^i V \otimes \Lambda^j V \subset V^{\otimes i+j}$ with the linear mapping $\mathcal{A}$ (whose image is $\Lambda^{i+j}V$), divided by the factor $i!j!$. Show that $\beta\wedge\alpha = (-1)^{ij}\alpha\wedge\beta$.

4. Show that $\dim S^kV = \binom{n+k-1}{k}$ and $\dim \Lambda^k V = \binom{n}{k}$. In particular, $\Lambda^n V \simeq \mathbb{R}$, $\Lambda^l V = 0$ for $l > n$, and $S^3V \oplus \Lambda^3 V \neq V^{\otimes 3}$. ⊚

5. In this exercise, we refine the decomposition $V^{\otimes 3} = V \otimes (V\otimes V) = V\otimes S^2V \oplus V\otimes\Lambda^2 V$ induced by (A.4). Let

$$\rho(v_1\otimes v_2\otimes v_3) = \tfrac{1}{3}(v_1\otimes v_2\otimes v_3 + v_3\otimes v_1\otimes v_2 + v_2\otimes v_3\otimes v_1),$$

and extend this to a linear map $\rho : V^{\otimes 3} \to V^{\otimes 3}$.

(a) Show that ρ restricts to be a projection $V\otimes S^2V \to S^3V$ and let $K_S \subset V\otimes S^2V$ be its kernel.

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(b) Similarly, show that ρ restricts to be a projection $V \otimes \Lambda^2 V \to \Lambda^3 V$ and let K_Λ be its kernel.
(c) Conclude that $V^{\otimes 3} = S^3 V \oplus K_S \oplus \Lambda^3 V \oplus K_\Lambda$.
(d) Show that $\dim K_S = \dim K_\Lambda$.

6. Define

$$F_{12} = \{[v \otimes E] \in \mathbb{P}(V \otimes \Lambda^2 V) \mid [E] \in G(2, V) \text{ and } v \wedge E = 0\},$$

where $G(2, V)$, as defined in §1.9, is the Grassmannian of 2-planes through the origin in V. Show that $F_{12} \subset \mathbb{P}(K_\Lambda)$ and that it admits the geometric interpretation as the space of *flags*, $l \subset E \subset V$, where l is a line through the origin and E is a two-plane. More generally, one can define the space $F_{a_1,\dots,a_r}$ of flags $E_1 \subset \dots \subset E_r \subset V$ such that $\dim E_j = a_j$. See §4.2 for more details.

Lemma A.1.9 (Cartan Lemma). *Let $v_1, \dots, v_k$ be linearly independent elements of V and let $w_1, \dots, w_k$ be elements of V such that $w_1 \wedge v_1 + \dots + w_k \wedge v_k = 0$. Then there exist scalars $h_{ij} = h_{ji}, 1 \leq i, j \leq k$, such that $w_i = \sum_j h_{ij} v_j$.*

Exercise A.1.10: Prove the lemma.

Exercise A.1.11: Let $v_1, \dots, v_n \in V$ be a basis of V, and let $R_1, \dots, R_m \in \Lambda^2 V$ be such that $R_j \wedge v^j = 0$. Show that $R_j = \frac{1}{2} R_{jkl} v^k \wedge v^l$ for some scalars R_{jkl} such that $R_{jlk} = -R_{jkl}$ and $R_{jkl} + R_{klj} + R_{ljk} = 0$. What happens when $\dim V > n$? ⊚

Induced linear maps. Given $\alpha \in \mathrm{End}(V)$, define maps $\alpha^{\otimes k} : V^{\otimes k} \to V^{\otimes k}$ induced by α as follows. On decomposable elements, let

$$v_1 \otimes v_2 \otimes \cdots \otimes v_k \mapsto \alpha(v_1) \otimes \alpha(v_2) \otimes \cdots \otimes \alpha(v_k),$$

and extend by linearity. Note that $\alpha^{\otimes k}$ preserves the subspaces $S^k V$ and $\Lambda^k V$.

In particular, the induced map $\alpha^{\otimes n} : \Lambda^n V \to \Lambda^n V$ is multiplication by some scalar. This number is called the *determinant* of α and denoted $\det(\alpha)$. Geometrically, if $P \subset V$ is a parallelepiped of dimension n with one vertex the origin, then $\det(\alpha) = \mathrm{vol}(\alpha(P)) / \mathrm{vol}(P)$, where vol is any volume form compatible with the linear structure.

Exercise A.1.12: Show that $\det(\alpha)$ equals the determinant of the square matrix which represents α with respect to a given basis.

Interior products. For $x \in V$, define the *interior product* $x \lrcorner : \Lambda^{p+1} V^* \to \Lambda^p V^*$ by

$$(x \lrcorner \phi)(v_1, \dots, v_p) := \phi(x, v_1, \dots, v_p), \qquad \phi \in \Lambda^{p+1} V^*.$$

More generally, for $z \in \Lambda^p V$ define $z \lrcorner : \Lambda^q V^* \to \Lambda^{q-p} V^*$.

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Exercise A.1.13: Show that $x\lrcorner$ is the adjoint of the linear map $x\wedge : \Lambda^p V \to \Lambda^{p+1} V$ given by wedging with x.

Induced inner products and the $*$-operator. An inner product $\langle , \rangle$ on V induces an inner product on V^*, as follows: take any orthonormal basis of V and declare the dual basis to be orthonormal. Alternatively, fix an orthonormal basis $e_1, \ldots, e_n$ for V and define $\langle \alpha, \beta \rangle = \sum_i \alpha(e_i)\beta(e_i)$ for $\alpha, \beta \in V^*$.

Exercise A.1.14: Verify that these two definitions of induced inner products are both well-defined and agree.

One may induce inner products on all tensor spaces constructed from V and V^*. For example, the inner product on $V \otimes V$ is given on decomposable vectors by $\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle$ and extended bilinearly. However, since our definition (A.3) for the wedge product produces $k!$ terms, we normalize the inner product on $\Lambda^k V$ by dividing the inner product inherited from $V^{\otimes k}$ by $k!$. With this normalization, the inner product satisfies *Hadamard's inequality*

$$|a \wedge b| \leq |a||b|. \tag{A.5}$$

Let $\Omega \in \Lambda^n V$ be a volume form with unit length with respect to this renormalized inner product. For $\alpha \in \Lambda^k V$, we define $*\alpha \in \Lambda^{n-k} V$ by requiring that

$$\beta \wedge *\alpha = \langle \alpha, \beta \rangle \Omega \qquad \forall \beta \in \Lambda^k V.$$

Exercise A.1.15: If $e_1, \ldots, e_n$ is an orthonormal basis, show that $\Omega = e_1 \wedge \cdots \wedge e_n$ has unit length, and calculate $*e_j$. When $n = 2$, are there vectors v such that $*v = v$?

A.2. Matrix Lie groups

Let $GL(V) \subset \mathrm{End}(V)$ denote the group of all invertible linear maps. Let G be a *Lie group*, i.e., a C^∞-manifold that has a group structure compatible with its differentiable structure. A *(linear) representation* of G is a group homomorphism $\rho : G \to GL(V)$, and the vector space V is called a *G-module.* If V is endowed with a basis, we call the image $\rho(G)$ a *matrix Lie group.*

A subspace $V_1 \subset V$ is a *G-submodule* if $\rho(G)V_1 \subseteq V_1$. A G-module is said to be *irreducible* if it has no proper G-submodules. For example, $V \otimes V$ is not irreducible as a $GL(V)$-module, since by (A.4) both $S^2 V$ and $\Lambda^2 V$ are G-submodules. On the other hand, $S^2 V$ and $\Lambda^2 V$ are irreducible $GL(V)$-modules.

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Suppose $\rho : G \to GL(V)$ and $\rho' : G \to GL(W)$ are representations of G. A linear map $\alpha : V \to W$ is said to be a *G-module homomorphism* if $\alpha(\rho(g)v) = \rho'(g)\alpha(v)$ for all $v \in V$ and $g \in G$. Note that the images and kernels of G-module homomorphisms are G-submodules. Two G-modules V, W are *isomorphic* if there exists G-module isomorphism (i.e., a bijective G-module homomorphism) between them.

Lemma A.2.1 (Schur's Lemma)**.** *Let $\rho_V : G \to GL(V)$, $\rho_W : G \to GL(W)$ be two irreducible representations of G, and let $f : V \to W$ be a G-module homomorphism.*

If f is nonzero, then f is a G-module isomorphism and is the unique such isomorphism up to scalar multiple.

Exercises A.2.2:

1. Prove the lemma. ⊚
2. Show that K_S, K_Λ of Exercise A.1.8.5 are isomorphic $GL(V)$-modules. The standard notation for this module is $S_{21}(V)$. ⊚

Examples. Let $Q \in S^2V^*$ be a quadratic form that is *positive definite*, i.e., $Q(v,v) > 0$ for all $v \in V\backslash\{0\}$. We define the following subgroups of $GL(V)$:

$$\begin{aligned} SL(V) &:= \{g \in V \otimes V^* |\, \det(g) = 1\}, \\ O(V,Q) &:= \{g \in V \otimes V^* | Q(v,w) = Q(gv,gw)\ \forall v,w \in V\}, \\ SO(V,Q) &:= O(V,Q) \cap SL(V). \end{aligned}$$

These are respectively called the *special linear group*, the *orthogonal group*, and the *special orthogonal group.* We often omit reference to Q when it is understood.

Exercise A.2.3: Show that when Q is definite, $SO(V)$ is the connected component of the identity of $O(V)$. ⊚

When $V = \mathbb{R}^n$ with the standard inner product, then we write $SO(n)$ for $SO(V)$. When $V = \mathbb{R}^n$ or $\mathbb{C}^n$, we respectively write $SL(n,\mathbb{R})$ or $SL(n,\mathbb{C})$ (or SL_n when we wish to remain ambiguous) for $SL(V)$.

When $n = 2m$, a 2-form $\omega \in \Lambda^2V^*$ is *nondegenerate* if $\omega \wedge \cdots \wedge \omega \neq 0$, where ω is wedged with itself m times (or, equivalently, if the map $v \mapsto v \lrcorner\, \omega$ is an isomorphism from V to V^*). In this case, ω is called a *symplectic form* on V, and we may define the *symplectic group*

$$Sp(V,\omega) := \{g \in V \otimes V^* \mid \omega(v \wedge w) = \omega(gv \wedge gw)\ \forall v,w \in V\}.$$

Since all nondegenerate 2-forms in $\Lambda^2(\mathbb{R}^{2m})$ are linearly equivalent, we often denote this group by $Sp(m,\mathbb{R})$ (see Exercise A.4.9.8).

Exercises A.2.4:

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1. Show that all of the above are matrix Lie groups, i.e., that they are smooth submanifolds of $GL(V)$, and they are closed under multiplication and inverses.
2. Show that if we write the matrices of $O(V)$ with respect to an orthonormal basis, then

$$O(V) \cong O(n) = \{A \in M_{n\times n} \mid {}^tAA = \mathrm{Id}\}.$$

3. Every matrix in $SO(3)$ represents a rotation fixing a line through the origin in $\mathbb{R}^3$. Is the analogous statement true for matrices in $SO(4)$?
4. A symmetric matrix S is diagonalizable by some $A \in SO(n)$, i.e., ASA^{-1} is diagonal. Show that if $S_1, \ldots, S_k$ are pairwise commuting symmetric matrices, then they are simultaneously diagonalizable by an element of $SO(n)$.
5. Show that $GL(V)$ acts simply transitively on the bases of V, and thus as a manifold $GL(V)$ is isomorphic to the space of bases of V.
6. Similarly, show that $O(V)$ acts simply transitively on the set of orthonormal bases of V.
7. Let V have inner product Q and let $|v| = \sqrt{Q(v,v)}$. Define $CO(V,Q) \subset GL(V)$ (or $CO(V)$, if Q is understood) to be the linear transformations preserving angles, where $\angle(v,w) := Q(v,w)/|v||w|$. Show that

$$CO(V,Q) \cong O(V,Q) \times \{\lambda\,\mathrm{Id} \mid \lambda > 0\}.$$

A.3. Complex vector spaces and complex structures

A *complex structure* on a (real) vector space V is a map $J \in GL(V)$ such that $J \circ J = -\mathrm{Id}$. For example, if we consider $\mathbb{C}^n$ as the real vector space $\mathbb{R}^{2n}$, then multiplication by $i = \sqrt{-1}$ is not multiplication by a scalar, but it is a linear map whose square is $-\mathrm{Id}$. Define $GL(V,J) := \{g \in GL(V) | Jg = gJ\}$.

For any vector space V, we can define its *complexification* $V_{\mathbb{C}} := V \otimes \mathbb{C} = V \oplus iV$, which can be considered as the complex span of the vectors in V. So, if $\dim V = n$, then $V_{\mathbb{C}}$ has dimension n as a vector space over $\mathbb{C}$, while it has dimension $2n$ as a real vector space. (We make this distinction in notation by writing $\dim_{\mathbb{C}} V_{\mathbb{C}} = n$ and $\dim_{\mathbb{R}} V_{\mathbb{C}} = 2n$.) Any $f \in \mathrm{End}(V)$ may be extended $\mathbb{C}$-linearly to a endomorphism of $V_{\mathbb{C}}$. Note that the characteristic polynomial of this extension is the same as that of the original, so eigenvalues come in complex-conjugate pairs. The corresponding eigenvectors are also conjugate under the complex conjugation that fixes $V \subset V_{\mathbb{C}}$.

Exercises A.3.1:

1. If (V,J) is a vector space with a complex structure, show that V must be even-dimensional. ⊚

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2. Show that J has no eigenvectors as an endomorphism of V, but $V_{\mathbb{C}}$ splits as a direct sum of $+i$ and $-i$ eigenspaces of J. We denote these by $V^{(1,0)}$ and $V^{(0,1)}$ respectively.
3. Let $V = \mathbb{R}^{2n}$ with basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ and the *standard complex structure* J defined by $J(e_i) = f_i$ and $J(f_i) = -e_i$. Calculate $V^{(1,0)}$ and $V^{(0,1)}$.
4. Show that there exists a linear isomorphism $\phi : V \to \mathbb{C}^n$ such that $\phi(Jv) = i\phi(v)$ for all $v \in V$. Thus, all vector spaces with complex structures are isomorphic to the standard example.
5. Find a decomposition of $\Lambda^2 V_{\mathbb{C}}$ into three components invariant under the induced action of J. These components are denoted $\Lambda^{(2,0)}V$, $\Lambda^{(1,1)}V$ and $\Lambda^{(0,2)}V$. More generally, decompose $\Lambda^k V_{\mathbb{C}}$ into $k+1$ components that are J-invariant.

Unitary groups and conjugation. Let W be a complex vector space of (complex) dimension n. The invertible complex-linear endomorphisms of W form the group $GL(W) \cong GL(n, \mathbb{C})$. A *Hermitian form* h on W is a map $h : W \times W \to \mathbb{C}$ which is $\mathbb{C}$-linear in its first argument, and satisfies $h(v,w) = \overline{h(w,v)}$. Notice that $\mathsf{Re}(h)$ is a symmetric bilinear form on W (considered as a real vector space), while the imaginary part of h is skew-symmetric. If $\mathsf{Re}(h)$ is an inner product, i.e., $\mathsf{Re}\, h(v,v) > 0$ for all $v \neq 0$, then h is said to be a *Hermitian inner product*. In this case we define

$$\begin{aligned} U(W,h) &:= \{g \in GL(W) \mid \ h(v,w) = h(gv, gw), \forall v, w \in W\} &&\cong U(n), \\ SU(W,h) &:= \{g \in U(W,h) \mid \det_{\mathbb{C}}(g) = 1\} &&\cong SU(n), \end{aligned}$$

respectively the *unitary* and *special unitary groups.* Note that $\det_{\mathbb{C}}(g)$ is the determinant of the $n \times n$ matrix representing g with respect to a $\mathbb{C}$-basis.

Exercises A.3.2:
1. Relate $\det_{\mathbb{C}}(g)$ to the determinant of g when g is considered as an endomorphism of the underlying real vector space.
2. Let $h(v,w) = v \cdot \overline{w}$ for $v, w \in \mathbb{C}^n$, where $v \cdot u$ is the standard dot product. Show that

$$U(\mathbb{C}^n, h) =: U(n) = \{Ug \in GL(n, \mathbb{C}) \mid g^{-1} = \overline{g}^t\}$$

and deduce that $|\det g| = 1$.

Given a vector space V with both an inner product Q and a complex structure J, one can construct a Hermitian inner product whose real part is Q provided that J preserves Q, i.e., $Q(Jv, Jw) = Q(v,w)$ for all $v, w \in V$. In this situation $\omega(v,w) := Q(v, Jw)$ is a symplectic form on V, and we may define a Hermitian inner product by

$$h(v,w) = Q(v,w) + i\omega(v,w).$$

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Exercises A.3.3:
1. Show that J preserves Q if and only if $J \in \mathfrak{so}(Q)$, so that $J \in SO(Q) \cap \mathfrak{so}(Q)$.
2. Show that ω as defined above is also preserved by J.

Note that assuming compatibility, any two of Q, J, ω determine the third.

A.4. Lie algebras

It is usually difficult to explicitly parametrize Lie groups. Fortunately, we rarely need to do this, but we will often need to write out explicit bases for various *Lie algebras* to be defined below.

Let $\mathfrak{gl}(V) = \text{End}(V) = V \otimes V^*$. After a choice of basis, we may identify $\mathfrak{gl}(V)$ with the set of $n \times n$ matrices. Define a skew-symmetric multiplication $[,]$ on $\mathfrak{gl}(V)$ by

$$[X, Y] = XY - YX, \tag{A.6}$$

where XY is the usual matrix multiplication (representing the composition of linear maps). Expanding out, one can verify the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{A.7}$$

for all $X, Y, Z \in \mathfrak{gl}(V)$.

Definition A.4.1. A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a skew-symmetric bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a *bracket*, that satisfies the Jacobi identity (A.7).

Exercise A.4.2: Show that $\mathbb{R}^3$, with bracket given by the cross product, is a Lie algebra.

Example A.4.3. An important class of Lie algebras is $\Gamma(TM)$, the space of smooth vector fields on a C^∞ manifold M. The bracket is $[X, Y]f := X(Yf) - Y(Xf)$, and the algebra is infinite-dimensional (see Definition B.1).

One can make any vector space into a Lie algebra by using a bracket which is identically zero. Such Lie algebras are called *abelian* Lie algebras. Inside a Lie algebra, any subspace that is closed under the bracket is called a *subalgebra*. It is a Lie algebra in its own right, since the Jacobi identity holds by restriction.

A *representation of a Lie algebra* $\mathfrak{g}$ is a linear transformation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ which preserves the brackets. We say such a V is a $\mathfrak{g}$-*module*. Define a representation ad : $\mathfrak{g} \to \text{End}(\mathfrak{g})$ by $\text{ad}(X)Y := [X, Y]$, called the *adjoint representation of* $\mathfrak{g}$. In particular, any finite-dimensional Lie algebra may be realized as a matrix Lie algebra.

Exercise A.4.4: Show that ad is indeed a representation.

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Given a matrix Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, define a map $\exp_V : \mathfrak{g} \to GL(V)$ by $\exp_V(X) := \mathrm{Id}_V + \sum_{j=1}^{\infty} \frac{1}{j!} X^j$, where X^j is the endomorphism $X \in \mathrm{End}(V)$ composed with itself j times. The image is a Lie subgroup of $GL(V)$ (see, e.g., [**174**, Chap. 10] or [**65**, §8.3]). When $V = \mathfrak{g}$ we drop V from the notation, and denote the corresponding group G, the *adjoint form* of the Lie group associated to $\mathfrak{g}$. Different representations give rise to different groups, but all the groups will have the same universal covering group (see, e.g. [**65**, §7.3, §23.1]). One can also go in the other direction, as the following example shows:

Example A.4.5. Let G be a Lie group. A vector field $X \in \Gamma(TG)$ is *left-invariant* if $L_{a*}(X_b) = X_{ab}$ for all $a, b \in G$, where L_{a*} denotes pushforward (see Appendix B) by left-multiplication by a. The reader can verify that the Lie bracket of two left-invariant vector fields is also left-invariant. Thus $\Gamma^L(TG)$, the space of left-invariant vector fields, is a Lie subalgebra of $\Gamma(TG)$.

A left-invariant vector field is determined by its value at just one point (say, at the identity element $e \in G$), since it is given at all other points by pushforward under left-multiplication. Thus, we may identify $\Gamma^L(TG)$ with T_eG. We define $\mathfrak{g} = T_eG \cong \Gamma^L(TG)$ to be the *Lie algebra of* G.

If $G \subseteq GL(V)$ is a matrix Lie group, then $\mathfrak{g} \cong T_{\mathrm{Id}}G \subset \mathfrak{gl}(V) = \mathrm{End}(V)$ is a *matrix Lie algebra*. Any Lie group G has a canonical representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ defined by

$$\mathrm{Ad}(g)X = L_{g*}R_{g^{-1}*}X,$$

called the *adjoint representation*. For matrix Lie groups, $\mathrm{Ad}(g)X = gXg^{-1}$.

Given any representation of a Lie group G, one obtains a representation of the corresponding Lie algebra $\mathfrak{g}$ by differentiation. In particular $\mathrm{Ad} : G \to GL(\mathfrak{g})$ gives rise to a representation $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

Exercise A.4.6: Let $g(t) \in G$ be a curve with $g(0) = \mathrm{Id}$ and $g'(0) = Y \in \mathfrak{g}$. Show that $\frac{d}{dt}\big|_{t=0} \mathrm{Ad}(g(t))X = [Y, X]$. In particular the representation obtained by differentiating Ad is ad.

Remark A.4.7. Where does the Jacobi identity (A.7) come from? One can interpret it as a Leibniz rule. For, if A is an algebra with product indicated by a dot, then a map $D : A \to A$ is a *derivation* if $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in A$. Now take $A = \mathfrak{g}$ with the product given by the bracket and $D = D_X$ by the bracket with $X \in \mathfrak{g}$.

Exercise A.4.8: Verify that $\mathrm{ad}_X : Y \mapsto [X, Y]$ is a derivation exactly because the Jacobi identity holds.

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If $G \subseteq GL(V)$ is a matrix Lie group, and $\mathfrak{g} \subset \mathfrak{gl}(V)$ is the associated matrix Lie algebra, then $\mathfrak{g}$ has an *induced representation* on the dual spaces and tensor products constructed from V. For example, for $X \in \mathfrak{g}$ and $v \in V$, then

$$\rho_{V \otimes V}(X)(v \otimes w) := X(v) \otimes w + v \otimes X(w).$$

For another example, for $\alpha \in V^*$, $\rho_{V^*}(X)(\alpha) := -\alpha \circ X$, where the circle denotes the composition of the maps $X : V \to V$ and $\alpha : V \to \mathbb{R}$.

Exercises A.4.9:

1. Consider

$$\mathfrak{sl}(V) = \{X \in \mathfrak{gl}(V) |\, \operatorname{tr} X = 0\}, \tag{A.8}$$

$$\mathfrak{so}(V, Q) = \{X \in \mathfrak{gl}(V) | Q(Xv, w) = -Q(v, Xw)\}, \tag{A.9}$$

$$\mathfrak{gl}(V, J) = \{X \in \mathfrak{gl}(V) | X \circ J - J \circ X = 0\}. \tag{A.10}$$

Verify that each of these are Lie algebras, i.e., they are closed under the bracket (A.6). Show these are isomorphic to the Lie algebras associated to $G = SL(V), G = SO(V, Q)$ and $G = GL(V, J)$ respectively.

When V is $\mathbb{R}^n$ with its standard basis, we write $\mathfrak{sl}(V) = \mathfrak{sl}(n, \mathbb{R})$ or $\mathfrak{sl}_n$, and similarly if Q is the standard quadratic form represented by the identity matrix we write $\mathfrak{so}(V, Q) = \mathfrak{so}(n, \mathbb{R})$ or $\mathfrak{so}(n)$.

If V is a complex vector space, we similarly write $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$ etc. and similarly for the corresponding Lie groups.

2. Define a surjective group homomorphism $SL(2, \mathbb{C}) \to SO(3, \mathbb{C})$ by observing that $\mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}^3$ as vector spaces, and the determinant of a 2×2 matrix polarizes to a quadratic form. What is the kernel of this map?

3. Show that if $X \in \mathfrak{so}(2n + 1)$, then $\det(X) = 0$. Show that if $X \in \mathfrak{so}(2n)$, then the eigenvalues of X are purely imaginary and come in complex conjugate pairs, and thus $\det(X)$ is nonnegative.

4. Show that when $\dim V$ is of dimension $2n$, there is a smooth positive function $\operatorname{Pfaff}(X)$, called the *Pfaffian*, such that $\det(X) = \operatorname{Pfaff}(X)^2$. If x_{ij} are the entries of X, then

$$\operatorname{Pfaff}(X) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} (\operatorname{sgn} \sigma) x_{\sigma(1)\sigma(2)} x_{\sigma(3)\sigma(4)} \cdots x_{\sigma(2n-1)\sigma(2n)}.$$

5. A quadratic form $Q \in S^2 V^*$ is *nondegenerate* if the map $v \mapsto v \lrcorner Q$ is an isomorphism $V \to V^*$. Recall that Q is represented with respect to a basis $v_1, \dots, v_n$ by a symmetric matrix with entries $Q_{ij} = Q(v_i, v_j)$. We say Q has *signature* (p, q) if its matrix has p positive eigenvalues and q negative eigenvalues. If Q is nondegenerate, show that there is a basis such that Q is represented by

$$\begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix},$$

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where Id_k denotes the $k \times k$ identity matrix. We write $SO(p,q)$ and $\mathfrak{so}(p,q)$ for the corresponding orthogonal groups and Lie algebras.

Show that if Q has signature $(n,0)$, and we choose bases such that $Q = \mathrm{Id}_n$, then $\mathfrak{so}(V,Q) \cong \{X \mid X + X^t = 0\}$.

6. If Q has signature $(3,1)$, we could take bases such that

$$Q = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \text{or} \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

If we use the first basis, we have

$$\mathfrak{so}(3,1) \simeq \left\{ \begin{pmatrix} 0 & x_1^2 & x_1^3 & x_1^4 \\ x_1^2 & 0 & -x_2^3 & -x_2^4 \\ x_1^3 & x_2^3 & 0 & -x_3^4 \\ x_1^4 & x_2^4 & x_3^4 & 0 \end{pmatrix} \mid x_1^2, x_1^3, \ldots \in \mathbb{R} \right\}.$$

Find the matrix presentation of $\mathfrak{so}(3,1)$ using the second choice for Q.

7. Similarly, given a complex structure on $\mathbb{R}^4$, two choices of bases yield

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Find the matrix presentation of $\mathfrak{gl}(2,\mathbb{C}) \subset \mathfrak{gl}(4,\mathbb{R})$ using these two choices for J.

8. Two choices of symplectic form on $V \simeq \mathbb{R}^{2n}$ are

$$\omega = dx^1 \wedge dx^{n+1} + \cdots + dx^n \wedge dx^{2n},$$
$$\omega = dx^1 \wedge dx^2 + \cdots + dx^{2n-1} \wedge dx^{2n}.$$

Find the two corresponding matrix presentations of $\mathfrak{sp}(n,\mathbb{R})$.

9. The symmetric bilinear form $B(X,Y) := \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ on a Lie algebra is called the *Killing form.* Compute the signature of the Killing forms for $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$, $\mathfrak{gl}(2,\mathbb{C})$ and $\mathfrak{sp}(2,\mathbb{R})$.

Example A.4.10. Let $\mathfrak{su}(2)$ denote the Lie algebra of $SU(2)$. Taking $e_1, e_2 \in \mathbb{C}^2$ as a unitary basis gives

$$\mathfrak{su}(2) = \left\{ \frac{1}{2} \begin{pmatrix} ix & -y+iz \\ y+iz & -ix \end{pmatrix} \middle| \, x,y,z \in \mathbb{R} \right\} \subset \mathfrak{gl}(2,\mathbb{C}).$$

Let $S^4(\mathbb{C}^2)$ have basis $v_1 = e_1^4,\ v_2 = 2e_1^3 e_2,\ v_3 = \sqrt{6} e_1^2 e_2^2,\ v_4 = 2e_1 e_2^3,\ v_5 = e_2^4$. Then the induced representation of $\mathfrak{gl}(2,\mathbb{C})$ on $S^4(\mathbb{C}^2)$ restricts to give

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the following representation of $\mathfrak{su}(2)$:

$$\frac{1}{2}\begin{pmatrix} ix & -y+iz \\ y+iz & -ix \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 2ix & -y+iz & 0 & 0 & 0 \\ y+iz & ix & \sqrt{\frac{3}{2}}(-y+iz) & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}(y+iz) & 0 & \sqrt{\frac{3}{2}}(-y+iz) & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}(y+iz) & -ix & -y+iz \\ 0 & 0 & 0 & y+iz & -2ix \end{pmatrix}.$$

Exercise A.4.11: Take an orthonormal basis for $\mathbb{R}^3$, and compute the induced orthonormal basis of $V = S^2(\mathbb{R}^3)$. Write out the matrix for the representation $\mathfrak{so}(3) \to \mathfrak{gl}(V)$ resulting from the induced representation of $\mathfrak{gl}(3, \mathbb{R})$.

A.5. Division algebras and the simple group G_2

In addition to the orthogonal and symplectic groups, there is just one more series of groups that can be defined by preserving a generic tensor of some type on V. The reason is that most tensor spaces have dimension greater than that of $GL(V)$. For example, S^3V has dimension $\frac{(n+2)(n+1)n}{6} > n^2$, where $n = \dim V$, so the subgroup $G \subset GL(V)$ preserving a generic element of S^3V must be zero-dimensional, and similarly for higher symmetric powers.

In fact, the only potential examples are $\Lambda^3 V$ for dimensions $n = 6, 7$ or 8. (When $n = 5$, $\Lambda^3\mathbb{R}^5 \cong \Lambda^2\mathbb{R}^{5*}$, so one gets nothing new.) We expect the corresponding groups to have dimension 16, 14 and 8 respectively.

In the case $n = 8$, note that $V = \mathfrak{sl}(3)$ (over either $\mathbb{R}$ or $\mathbb{C}$) has dimension eight and there is a natural 3-form induced by the bracket $[,] : V \times V \to V$. Identifying V with V^* via the Killing form, we obtain an element $\phi \in V^{*\otimes 3}$.

Exercise A.5.1: Show that $\phi \in \Lambda^3 V^*$.

By its invariant definition, ϕ is preserved by $SL(3)$, and a little more work shows that ϕ is generic. By counting dimensions, we see that $\mathfrak{sl}(3)$ coincides with $\operatorname{der}(\phi)$, the Lie algebra of $\operatorname{Aut}(\phi) = G$.

In the case $n = 7$, first note that a generic $\phi \in \Lambda^3 V^*$ and a volume form $\Omega \in \Lambda^7 V^*$ determine a bilinear form $\beta(v, w)$ defined by

$$(v \lrcorner \phi) \wedge (w \lrcorner \phi) \wedge \phi = \beta(v, w)\Omega.$$

Exercise A.5.2: Show that β is symmetric, and nondegenerate, so we rename it $Q = \beta$. Thus $G(\phi) \subseteq CO(V, Q)$. (We obtain a conformal group, because Q is *a priori* only well-defined up to the choice of volume form.) ◎

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Over the reals, two types of signature are possible, one of which is definite. Say we are in the case where the signature is definite; then, by comparing homogeneity, one can see that Q is in fact well-defined. A little more calculation (see [**91**, Thm. 6.80]) shows that one obtains a new simple Lie group of dimension 14, which is named G_2.

G_2 and the octonions. The compact form of the group G_2, corresponding to Q definite, is also the automorphism group of the *octonions*. (According to [**91**], its presentation as the group preserving a generic 3-form in seven variables was discovered by Bryant, long after other presentations.)

There are only four normed division algebras over $\mathbb{R}$: $\mathbb{R}$ itself, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the *octonions* $\mathbb{O}$ (see, e.g., [**91**]).

To understand the octonions, we need to review the quaternions. Recall that $\mathbb{H}$ is a normed division algebra that is a four-dimensional vector space over $\mathbb{R}$. Elements of $\mathbb{H}$ may be written as $x = x^0 + x^1 e_1 + x^2 e_2 + x^3 e_3$ where $x^j \in \mathbb{R}$ and the symbols e_j satisfy the multiplication rule ${e_j}^2 = -1$ and either of the following equivalent multiplication rules in Figure 1.

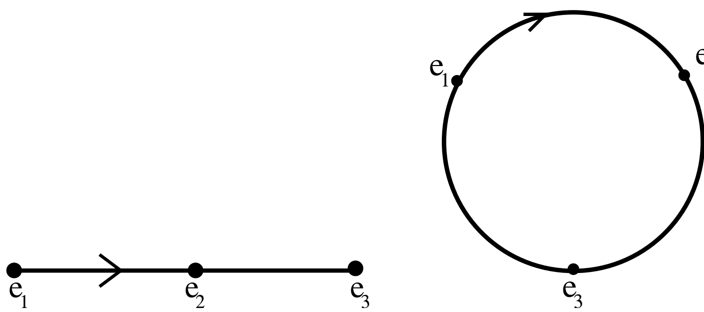

Figure 1. Products are positive if one multiplies with the arrow (e.g., $e_1 e_2 = e_3$), negative against (e.g. $e_3 e_2 = -e_1$)

The octonions (also known as *Cayley numbers*) form an eight-dimensional vector space over $\mathbb{R}$, in which elements may be written as $x = x^0 + x^1 e_1 + \cdots + x^7 e_7$ with $x^j \in \mathbb{R}$ and the symbols e_j satisfying ${e_j}^2 = -1$ and the multiplication rules in Figure 2.

Suppose $\mathbb{A}$ is one of the four normed division algebras. Define its automorphism group

$$\mathrm{Aut}(\mathbb{A}) := \{g \in GL(\mathbb{A}) \mid (gu)(gv) = g(uv)\, \forall u, v \in \mathbb{A}\}.$$

Then $\mathrm{Aut}(\mathbb{A})$ is respectively $\{\mathrm{Id}\}$, $\mathbb{Z}_2$, $SO(3,\mathbb{R})$, or the compact form of G_2.

To partially see the last assertion, take $V = \mathsf{Im}\mathbb{O} = \{e_1, \ldots, e_7\}$ and define the form ϕ in terms of octonionic multiplication:

$$\phi(x,y,z) = -\tfrac{1}{2}\mathsf{Re}[x(yz) - z(yx)], \qquad x, y, z \in \mathsf{Im}\mathbb{O}. \tag{A.11}$$

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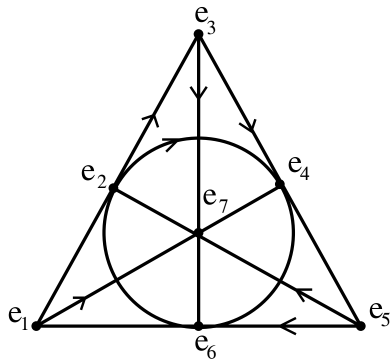

Figure 2. Multiplication rules for the basis for $\mathsf{Im}\mathbb{O}$

The split form of G_2, corresponding to Q of signature $(4,3)$, is the automorphism group of the split octonions [**91**], and also occurs as the automorphism group of a very important Pfaffian system on a 5-manifold [**36**]. (The action on the 5-manifold is not linear; the lowest-dimensional linear representation of G_2 has dimension seven.)

Exercises A.5.3:

1. Prove the *Moufang identities*: for $a, b, c \in \mathbb{O}$,

$$\begin{aligned} ((ab)a)c &= a(b(ac)), \\ c(a(ba)) &= ((ca)b)a, \\ (ab)(ca) &= a(bc)a. \end{aligned} \tag{A.12}$$

2. Use the Moufang identities to verify that $\phi(x,y,z)$ of (A.11) is skew-symmetric in x, y, z.

3. Similarly, verify that the *associator*

$$[a,b,c] := \tfrac{1}{2}[(ab)c - a(bc)], \qquad a,b,c \in \mathbb{O},$$

is skew-symmetric in a, b, c.

4. Calculate ϕ in terms of coordinates $x^1, \dots, x^7$ on $\mathbb{R}^7 = \mathsf{Im}\mathbb{O}$. ⊚

5. Show that the Lie algebra $\mathfrak{g}_2 = \operatorname{der}(\phi)$ has a matrix presentation

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} \kappa & -{}^t A \\ A & \rho(\kappa \oplus \tau) \end{pmatrix} \middle| \, \kappa, \tau \in \mathfrak{so}(3) \right\}, \tag{A.13}$$

where ρ is the isomorphism $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \cong \mathfrak{so}(4)$ and $a_{13} = a_{31} + a_{42}$, $a_{23} = a_{41} - a_{32}$, $a_{33} = a_{22} - a_{11}$, and $a_{43} = -a_{21} - a_{12}$.

6. Determine the classical group(s) preserving a generic $\phi \in \Lambda^3\mathbb{R}^6$. (Over $\mathbb{C}$ there is a unique generic 3-form up to $GL(6,\mathbb{C})$-equivalence, but over $\mathbb{R}$ there can be several.)

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A.6. A smidgen of representation theory

This section contains a brief overview of the rudiments of representation theory. For proofs and more complete statements of what follows, see any of the standard texts such as [**19, 65, 98, 113**].

Why representation theory? In general, when a group G (or Lie algebra $\mathfrak{g}$) acts on a vector space V, one would like to know the decomposition of V into irreducible G-modules, if such a decomposition exists. The utility of having such decompositions will become obvious as you read through this book. For now, consider the following motivating examples:

Example A.6.1. Let (M^n, g) be a Riemannian manifold. Say $c : M \to \mathbb{R}_+$ is a smooth function and consider the new Riemannian metric cg. How does the curvature tensor (see Chapter 3) change under such a change of metric? Is any aspect of it unchanged? Since the group of rotations which preserve the metric pointwise is also unchanged, one might suspect that the answer involves the action of these rotations on the curvature.

Representation theory helps us split up the curvature tensor into irreducible pieces and see how each piece changes. In particular, in §11.1 we will show that there is a piece that doesn't change. This was first discovered by H. Weyl when he was investigating Einstein's theory of general relativity. Weyl's study was what motivated him to make his fundamental contributions to representation theory (see [**93**] for more details).

Example A.6.2. Consider the *rational normal curve* $C_d \subset \mathbb{P}^d = \mathbb{P}(S^d(\mathbb{C}^2))$, which is the image of $\mathbb{CP}^1$ under $[x, y] \mapsto [x^d, x^{d-1}y, x^{d-2}y^2, \ldots, y^d]$, or, without coordinates, $[v] \mapsto [v^d]$ for $v \in \mathbb{C}^2 \backslash \{0\}$ (see Chapter 4). What are the linear changes of coordinates in $\mathbb{C}^{d+1}$ that leave C_d invariant? The set of all such changes forms a subgroup of $GL(d + 1, \mathbb{C})$ which is isomorphic to $GL(2, \mathbb{C})$. The action may be seen explicitly by $g.[v_1 \cdots v_d] = [(gv_1) \cdots (gv_d)]$.

Exercise A.6.3: Show that the representation $\rho : GL(2, \mathbb{C}) \to GL(d+1, \mathbb{C})$ described above is irreducible. ⊚

We will outline how to understand representations in one easy case, when the Lie algebra of G is *simple*. A Lie algebra $\mathfrak{g}$ (or a Lie group G) is called *reductive* if all $\mathfrak{g}$-modules decompose into a direct sum of irreducible $\mathfrak{g}$-modules, *simple* if it has no nontrivial ideals, and *semi-simple* if it is the direct sum of simple Lie algebras. Semi-simple Lie algebras are reductive, but not all reductive Lie algebras are semi-simple. For example, $\mathfrak{sl}(V)$ is simple while $\mathfrak{gl}(V) = \mathfrak{sl}(V) \oplus \{\lambda \,\mathrm{Id}\}$ is reductive. However, it turns out that all reductive Lie algebras are either semi-simple or a direct sum of a

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semi-simple and an abelian Lie algebra. (The Killing form of a Lie algebra is nondegenerate if and only if it is semi-simple; see, e.g., [**98**, §5.1].)

We discuss irreducible representations of simple Lie algebras. The irreducible representations of a semi-simple Lie algebra are just the tensor products of the irreducible representations of its simple components. Thanks to the action of $\mathfrak{g}$ on itself, without loss of generality, we may assume that $\mathfrak{g}$ is a matrix Lie algebra, i.e., $\mathfrak{g} \subseteq \mathfrak{gl}(V)$.

Simple Lie algebras are best studied via a certain kind of abelian subalgebra they contain. We decompose a given $\mathfrak{g}$-module V first with respect to commuting matrices $A_1, \dots, A_r$ that span such a subalgebra.

Exercise A.6.4: Let $A_1, \dots, A_r$ be $n \times n$ matrices that commute. Show that if each A_j is diagonalizable, then $A_1, \dots, A_r$ are simultaneously diagonalizable. ⊚

If a matrix A is diagonalizable, then V decomposes into eigenspaces for A and there is an eigenvalue associated to each eigenspace. Now let $\mathfrak{t} = \{A_1, \dots, A_r\} \subset \mathfrak{g}$ be the subspace spanned by simultaneously diagonalizable $A_1, \dots, A_r$. Then V decomposes into simultaneous eigenspaces for all $A \in \mathfrak{t}$. For each eigenspace V_j define a function $\lambda_j : \mathfrak{t} \to \mathbb{R}$ such that $\lambda_j(A)$ is the eigenvalue of A associated to the eigenspace V_j. Note that λ_j is a *linear* map, so we may think of $\lambda_j \in \mathfrak{t}^*$.

If there are p distinct eigenspaces of V, then these λ_j give p elements of $\mathfrak{t}^*$ which are called the *weights* of V. The dimension of V_j is called the *multiplicity* of λ_j in V. The decomposition $V = \bigoplus_j V_j$ is called the *weight space decomposition* of V.

Exercise A.6.5: Show that the only irreducible representations of an abelian Lie algebra $\mathfrak{t}$ are one-dimensional. ⊚

Now, back to our simple Lie algebra $\mathfrak{g}$. There always exists a maximal simultaneously diagonalizable (and thus abelian) subalgebra $\mathfrak{t} \subset \mathfrak{g}$, unique up to conjugation by G, called a *maximal torus*. We define the *rank* of $\mathfrak{g}$ to be the dimension of a maximal torus. Amazingly, irreducible representations of $\mathfrak{g}$ are completely determined up to equivalence by how $\mathfrak{t}$ acts. More precisely, suppose V is an irreducible $\mathfrak{g}$-module. Then as a $\mathfrak{t}$-module V admits a weight space decomposition. If two irreducible $\mathfrak{g}$ modules V, W have the same weights (with the same multiplicities) as $\mathfrak{t}$-modules, then they are isomorphic as $\mathfrak{g}$-modules.

Let $\mathfrak{g}$ act on itself by the adjoint action. Then as a $\mathfrak{t}$-module, we have the weight space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

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where $R \subset \mathfrak{t}^*$ is some finite subset whose nonzero members are called the *roots* of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ is the eigenspace associated to each root. (Of course, $\mathfrak{g}_0 = \mathfrak{t}$ since $\mathfrak{t}$ is maximal.) Another amazing fact is that the eigenspaces $\mathfrak{g}_\alpha$ for $\alpha \neq 0$ are all one-dimensional.

Remark A.6.6 (Why the word "root"?)**.** Consider the adjoint representation ad : $\mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ restricted to $\mathfrak{t}$. The roots of the characteristic polynomial $p_\lambda(X) = \det(\operatorname{ad}(X) - \lambda \operatorname{Id}_{\mathfrak{g}})$ are the eigenvalues of X. By varying X one obtains linear functions on $\mathfrak{t}$ which are the roots of $\mathfrak{g}$.

Exercise A.6.7: In fact, weights were originally called "generalized roots" when the theory was being developed. They too are roots of a characteristic polynomial—which one?

Let's look at some of our favorite simple Lie algebras:

Example A.6.8 ($\mathfrak{g} = \mathfrak{sl}(n)$)**.** Here the rank of $\mathfrak{g}$ is $n-1$, and we may take

$$\mathfrak{t} = \left\{ T(x^1, \dots, x^n) = \begin{pmatrix} x^1 & & \\ & \ddots & \\ & & x^n \end{pmatrix} \mid x^1 + \cdots + x^n = 0 \right\}.$$

Let $e^i_j = v^i \otimes v_j$. Then we have $T(x)(e^i_j) = (x^i - x^j)e^i_j$ for $i \neq j$ (no sums here). So the roots are $x^i - x^j$ and $\mathfrak{g}_{x^i - x^j} = \{e^i_j\}$.

Example A.6.9 ($\mathfrak{g} = \mathfrak{so}(2n)$)**.** The rank is n, and we may take

$$\mathfrak{t}(x^1, \dots, x^n) = \begin{pmatrix} 0 & x^1 & & & \\ -x^1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x^n \\ & & & -x^n & 0 \end{pmatrix}.$$

Exercise A.6.10: Determine the roots and the root space decomposition for $\mathfrak{so}(2n)$.

Remark A.6.11. The classification of complex simple Lie algebras $\mathfrak{g}$ (due to Killing and Cartan) is based on classifying the possible *root systems*, the collection of roots for $\mathfrak{g}$. The rules for such systems are rather strict and concise. For the record, we need a subset $R \subset \mathfrak{t}^*$ such that

(1) R must span $\mathfrak{t}^*$, and moreover if $\mathfrak{g}^*$ is complex, R lies in an $\mathbb{R}$-linear (in fact $\mathbb{Q}$-linear) subspace;

(2) for each $\alpha \in R$, reflection in the hyperplane perpendicular to α must map R to R;

(3) for all $\alpha, \beta \in R$, the quantity $2\langle \beta, \alpha \rangle / \langle \alpha, \alpha \rangle$ must be an integer; and

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(4) for all $\alpha \in R$, $2\alpha \notin R$.

(The inner product $\langle\, ,\rangle$ on $\mathfrak{t}$ is minus one times the restriction of the Killing form.) The above holds when $\mathfrak{g}$ is semi-simple. When $\mathfrak{g}$ is simple, R also cannot be split up into two separate root systems in complementary subspaces. The classification over the reals involves a modification of the above: for each complex simple Lie algebra, there can be several different real forms.

If $\mathfrak{g}$ is a simple algebra, then to each irreducible representation of $\mathfrak{g}$ is associated a set of weights (points in $\mathfrak{t}^*$), each with some multiplicity. Not every set of points with multiplicity in $\mathfrak{t}^*$ corresponds to an irreducible representation. In the first place, the admissible points lie on a lattice, called the *weight lattice*. The weight lattice is the set of $\ell \in \mathfrak{t}^*$ such that $\langle \ell, \alpha' \rangle \in \mathbb{Z}$ for all $\alpha' \in L_{R'}$, where $L_{R'}$ is the lattice generated by the *co-roots* $\alpha' = \frac{2}{\langle \alpha,\alpha \rangle}\alpha$ for $\alpha \in R$.

If one fixes an appropriate order on the weight lattice, the *highest weight* of an irreducible representation (which necessarily has multiplicity one) determines all other weights, along with their multiplicities. (In fact, the other weights are obtained by translating the highest weight by the negative roots.) A vector in V is called a *weight vector* if it is in an eigenspace for the torus and a *highest weight vector* if the corresponding weight determining the eigenvalues is a highest weight.

The weight lattice has $r = \dim \mathfrak{t}^*$ generators, so once one fixes a set of generators $\omega_1, \dots, \omega_r$, the irreducible representations correspond to r-tuples of nonnegative integers $(\ell_1, \dots, \ell_r)$ which determine a highest weight $\ell = \ell_1\omega_1 + \cdots + \ell_r\omega_r$.

Example A.6.12. When $\mathfrak{g} = \mathfrak{sl}(n) = \mathfrak{sl}(V)$, the weight lattice is generated by $x^1, x^1 + x^2, \dots, x^1 + \cdots + x^{n-1}$. These are the highest weights of the irreducible representations $V, \Lambda^2 V, \dots, \Lambda^{n-1} V$ respectively.

Exercise A.6.13: Verify the last statement by computing the action of $\mathfrak{t}$ on the highest weight vector $e_1 \wedge \cdots \wedge e_k$ of $\Lambda^k V$.

Remark A.6.14. Even when G is not simple, studying weights of a representation (i.e., weights under the maximal torus of a maximal semi-simple subgroup of G) can yield useful information; see §9.8.

A.7. Clifford algebras and spin groups

Let V be a real or complex vector space with a nondegenerate quadratic form $Q \in S^2V^*$. Given any linear subspace $L \subset V$, one can define its Q-orthogonal complement $L^\perp = \{w \in V \mid Q(v,w) = 0\, \forall v \in L\}$. If $Q|_L$ is nondegenerate, then $V = L \oplus L^\perp$. In that case, for all $v \in V$ we may write

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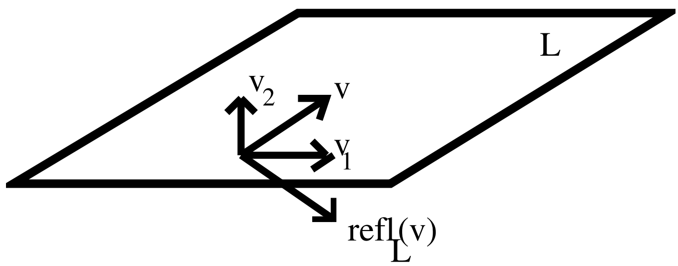
$v = v_1 + v_2$ for $v_1 \in L$, $v_2 \in L^\perp$, and we define the *reflection* of v in L by $\mathrm{refl}_L(v) = v_1 - v_2$.



Recall that $O(V,Q)$ is the subgroup of $GL(V)$ preserving Q, and $SO(V,Q) = O(V,Q) \cap SL(V)$.

Theorem A.7.1 (Cartan-Dieudonné [**91**])**.** *The group $O(V,Q)$ is generated by reflections in lines, and $SO(V,Q)$ is generated by compositions of even numbers of reflections. More precisely, $O(V,Q) = \{\mathrm{refl}_{l_1} \circ \cdots \circ \mathrm{refl}_{l_k} \mid l_j \in \mathbb{P}V\}$ where we may assume $k \leq n$, and similarly for $SO(V,Q)$ except k must be even.*

To define $\mathrm{Spin}(V,Q)$, the connected and simply connected group with Lie algebra $\mathfrak{so}(V,Q)$, we will need to generalize the notion of a reflection.

We first remark that $V^{\otimes} := \bigoplus_{j=0}^{\infty} V^{\otimes j}$ has a natural structure as a graded algebra, called the *tensor algebra* of V. Here we use the convention $V^{\otimes 0} = \mathbb{R}$. Earlier we defined $\Lambda^k V \subset V^{\otimes k}$. It may also be defined as a quotient space. More generally, define the *exterior algebra* of V by $\Lambda^\bullet V := V^{\otimes}/\langle x \otimes y + y \otimes x\rangle$, where $\langle x \otimes y + y \otimes x\rangle$ denotes the ideal generated by expressions of the form $x \otimes y + y \otimes x$ with $x, y \in V$.

Note that Q induces a quadratic form on $\Lambda^\bullet V$ which we also denote by Q. The *exterior product* in $\Lambda^\bullet V$, $(x,y) \mapsto x \wedge y$, may be interpreted as follows: Let $\hat{G}(i,V) \subset \Lambda^i V$ denote the cone over the Grassmannian. If $x \in \hat{G}(i,V)$, $y \in \hat{G}(j,V)$, then $x \wedge y \in \hat{G}(i+j,V) \subset \Lambda^{i+j} V$ represents the $(i+j)$-plane spanned by x and y in the following sense: If $x \in V$ and $y \in \hat{G}(j,V)$, then $x \wedge y$ is analogous to the component of x in $y^\perp$. If $||y||_Q = 1$, then $||x \wedge y||_Q = ||\,\mathrm{proj}_{y^\perp}(x)||_Q$. (Note that we do not need Q to define $x \wedge y$.)

Let $x \lrcorner\, y$ be defined to be the Q-adjoint of $x \wedge y$, that is, $Q(x \lrcorner\, y, z) = Q(y, x \wedge z)$ for all $x, y, z \in \Lambda^\bullet V$. For example, if $x, y \in V$, then $x \lrcorner y = Q(x,y)$. If $x \in V$ and $y \in \hat{G}(j,V)$, then $x \lrcorner\, y$ is analogous to the component of x in y, in the sense that if y has unit length, then $||x \lrcorner\, y||_Q = ||\,\mathrm{proj}_y(x)||_Q$.

Exercise A.7.2: Show that $x \lrcorner\, (x \lrcorner\, y) = 0$ for all $x, y \in \Lambda^\bullet V$.

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Consider, for $x, y \in \Lambda^\bullet V$, the bilinear operation defined by

$$x \copyright y := x \wedge y - x \lrcorner y,$$

which can be thought of as the "generalized reflection" of x in y.

Definition A.7.3. Let V be a vector space with a quadratic form Q. The *Clifford algebra* of (V, Q) is $Cl(V, Q) := (\Lambda^\bullet V, \copyright)$.

Exercise A.7.4: Show that we have an isomorphism of algebras

$$Cl(V, Q) \cong V^{\otimes} / \langle x \otimes y + y \otimes x - 2Q(x, y) \rangle.$$

Lemma A.7.5 (Fundamental Lemma of Clifford algebras)**.** *Let V be a vector space with a quadratic form Q and let $\mathcal{A}$ be an associative algebra with unit. If $\phi : V \to \mathcal{A}$ is a mapping such that for all $x, y \in V$*

$$\phi(x)\phi(y) + \phi(y)\phi(x) = 2Q(x, y)\,\mathrm{Id}_{\mathcal{A}},$$

then ϕ has a unique extension to an algebra homomorphism $\hat{\phi} : Cl(V, Q) \to \mathcal{A}$.

For a proof, see, e.g., [**91**].

Exercise A.7.6: Show that the hypotheses of the lemma are equivalent to ϕ satisfying $\phi(x)^2 = 2||x||_Q^2 \,\mathrm{Id}_{\mathcal{A}}$ for all $x \in V$.

In $Cl(V, Q) = (\Lambda^\bullet V, \copyright)$, the degree of a form is no longer well-defined, but there is still a notion of parity. Let $Cl^{even}(V, Q), Cl^{odd}(V, Q) \subset Cl(V, Q)$ denote the corresponding even and odd subspaces. We have the canonical isomorphisms of $\mathbb{Z}_2$-graded vector spaces (but *not* algebras) $Cl^{even}(V, Q) \cong \Lambda^{even} V$, $Cl^{odd}(V, Q) \cong \Lambda^{odd} V$.

Exercise A.7.7: Verify that the parity is well-defined.

Definition A.7.8. Let $Cl^*(V, Q) \subset Cl(V, Q)$ denote the invertible elements. Let

$$\mathrm{Pin}(V, Q) := \{a \in Cl^*(V, Q) \mid a = u_1 \copyright \cdots \copyright u_r, u_j \in V, Q(u_j, u_j) = 1\},$$
$$\mathrm{Spin}(V, Q) := \{a \in \mathrm{Pin}(V, Q) \mid r \text{ is even}\}.$$

Exercises A.7.9:

Given $a = u_1 \copyright \cdots \copyright u_r \in Cl(V, Q)$, let $\tilde{a} = (-1)^r u_r \copyright \cdots \copyright u_1$.

1. Show that $a \mapsto \tilde{a}$, extended linearly to all of $CL(V, Q)$, is a well-defined involution of the algebra $Cl(V, Q)$, i.e., it is an algebra homomorphism whose square is minus the identity.
2. For $v \in V$, show that $a \copyright v \copyright \tilde{a} \in V$.
3. Using the tilde involution, we define a representation $\rho : \mathrm{Spin}(V, Q) \to GL(V)$ by $\rho(a)v := a \copyright v \copyright \tilde{a}$. Show that ρ is a 2-to-1 surjective group homomorphism $\mathrm{Spin}(V, Q) \to SO(V, Q)$. ⊚

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Clifford algebras as matrix algebras. From now on, assume $\dim V = 2m$, and if we work over $\mathbb{R}$, assume Q has signature (m, m).

We have defined $Cl(V, Q)$ as $\Lambda^\bullet V$ with an exotic multiplication, but in fact, as an algebra, $Cl(V, Q)$ is something familiar, as we now show. Fix $U, U' \subset V$ such that $Q|_U = Q|_{U'} = 0$, and $V = U + U'$. (Note that this implies $\dim U = \dim U'$, $U \cap U' = 0$, $U^\perp = U$, and $V = U \oplus U'$.) Thus for all $v \in V$ we may uniquely write $v = x + y$ with $x \in U$, $y \in U'$. Define a mapping

$$\phi : V \to \mathrm{End}(\Lambda^\bullet U)$$

by, for $v \in V$ and $u \in \Lambda^\bullet U$,

$$\phi(v)(u) = \sqrt{2}(x \wedge u - y \lrcorner u).$$

We calculate

$$\begin{aligned}\phi(v)^2 u &= 2(x \wedge (x \wedge u - y \lrcorner u) - y \lrcorner (x \wedge u - y \lrcorner u))\\ &= x \wedge x \wedge u - x \wedge (y \lrcorner u) - y \lrcorner (x \wedge u) - y \lrcorner (y \lrcorner u)\\ &= 2Q(x, y)u = ||v||_Q^2 u.\end{aligned}$$

Thus the fundamental lemma applies and we obtain an algebra map $\hat{\phi} : Cl(V, Q) \to \mathrm{End}(\Lambda^\bullet U)$.

Exercises A.7.10:

1. $\hat{\phi}$ is a bijection, and thus, as an algebra, $Cl(V, Q) \cong \mathrm{End}(\Lambda^\bullet U)$.
2. Moreover, show that we obtain algebra isomorphisms

$$\begin{aligned}Cl^{even}(V, Q) &\cong \mathrm{End}(\Lambda^{even} U) \oplus \mathrm{End}(\Lambda^{odd} U),\\ Cl^{odd}(V, Q) &\cong [(\Lambda^{even} U)^* \otimes \Lambda^{odd} U] \oplus [(\Lambda^{odd} U)^* \otimes \Lambda^{even} U].\end{aligned}$$

3. Finally, show that $\mathrm{Spin}(V, Q)$ preserves $\mathrm{End}(\Lambda^{even} U)$ and $\mathrm{End}(\Lambda^{odd} U)$, which are often called the space of *positive* (resp. *negative*) *spinors*.
4. If V is a complex vector space with $\dim V > 7$, show that $Cl(V, Q)$ cannot act nontrivially on $\mathbb{C}^n$ for $n < 8$. If $\dim V \leq 7$, determine the values of n such that $Cl(V)$ acts nontrivially on $\mathbb{C}^n$. ⊚

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