GRADUATE STUDIES IN MATHEMATICS 175

Cartan for Beginners

Differential Geometry via Moving Frames and Exterior Differential Systems, Second Edition

Thomas A. Ivey
Joseph M. Landsberg

American Mathematical Society

https://doi.org/10.1090//gsm/175

Cartan for Beginners

GRADUATE STUDIES IN MATHEMATICS 175

Cartan for Beginners

Differential Geometry via Moving Frames and Exterior Differential Systems, Second Edition

Thomas A. Ivey
Joseph M. Landsberg

American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification.* Primary 35N10, 37K35, 53A05, 53A20, 53A30, 53A55, 53C10, 53C15, 58A15, 58A17.

For additional information and updates on this book, visit
www.ams.org/bookpages/gsm-175

Library of Congress Cataloging-in-Publication Data

Names: Ivey, Thomas A. (Thomas Andrew), 1963– | Landsberg, J. M.

Title: Cartan for beginners : differential geometry via moving frames and exterior differential systems / Thomas A. Ivey, Joseph M. Landsberg.

Description: Second edition. | Providence, Rhode Island : American Mathematical Society, [2016] | Series: Graduate studies in mathematics ; volume 175 | Includes bibliographical references and index.

Identifiers: LCCN 2016025601 | ISBN 9781470409869 (alk. paper)

Subjects: LCSH: Geometry, Differential. | Exterior differential systems. | AMS: Partial differential equations – Overdetermined systems – Overdetermined systems with variable coefficients. msc | Dynamical systems and ergodic theory – Infinite-dimensional Hamiltonian systems – Lie-Bäcklund and other transformations. msc | Differential geometry – Classical differential geometry – Surfaces in Euclidean space. msc | Differential geometry – Classical differential geometry – Projective differential geometry. msc | Differential geometry – Classical differential geometry – Conformal differential geometry. msc | Differential geometry – Classical differential geometry – Differential invariants (local theory), geometric objects. msc | Differential geometry – Global differential geometry – G-structures. msc | Differential geometry – Global differential geometry – General geometric structures on manifolds (almost complex, almost product structures, etc.). msc

Classification: LCC QA641 .I89 2016 | DDC 516.3/6–dc23 LC record available at `https://lccn.loc.gov/2016025601`

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1   21 20 19 18 17 16

Contents

Introduction

Preface to the Second Edition

We were very happy with the reception the first edition received after its appearance in 2003. In the years since, numerous readers have contacted us with questions, corrections, and comments on the exposition in the first edition. The book has been used in graduate classes at Texas A& M seven times by the second author, who would like to thank the students for their substantial input, especially the students from the 2012 and 2015 classes where preliminary versions of Chapters 3 and 11 were used. Thanks to our readers and students we have implemented many changes to improve and correct the exposition. While many people have helped, we owe a special debt to Colleen Robles, Matt Stackpole, Pieter Eendebak and Peter Vassiliou, who gave us numerous detailed comments, and to Robert Bryant and Michael Eastwood for their help developing the new material. We are also grateful to the AMS editorial staff, in particular Ed Dunne, Sergei Gelfand, and Christine Thivierge, for their help and patience.

One feature of this edition is that we have attempted to make more connections with the larger subject of differential geometry, mentioning major theorems and open questions with references to the literature. There are also three chapters of essentially new material:

Chapter 3 vastly expands the few pages on Riemannian geometry in the second chapter of the first edition. Notable here is the emphasis on a representation-theoretic perspective for the Riemann curvature tensor and its covariant derivative. There is also a proof of Killing's theorem describing the space of Killing vector fields on a Riemannian manifold and a discussion of homogeneous Riemannian manifolds, the latter following unpublished notes of R. Bryant.

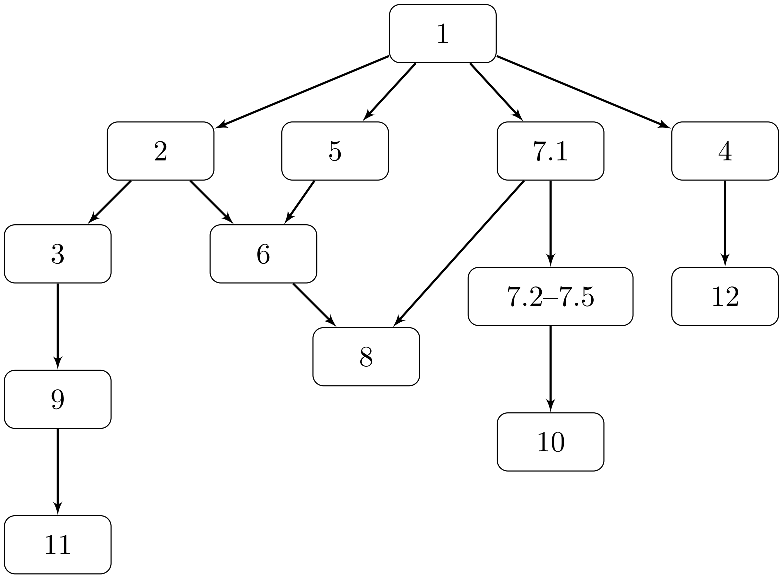
Chapter 10 is devoted to the latest development in the study of Darboux-integrable exterior differential systems, namely the work of Anderson, Fels and Vassiliou [**6**] on superposition formulas for such systems. These structures, which enable one to write down explicit solutions, are based on the action of the system's Vessiot group, which is generated by a set of vector fields which define its Vessiot algebra. After discussing the generalized definition of Darboux integrability formulated by these authors, we explain the construction of the Vessiot algebra by a sequence of delicate coframe adaptations. These adaptations are illustrated using a running example, and other recent applications of Darboux integrability in diverse settings, including Toda lattice systems and wave maps, are detailed in the exercises.

In Chapter 11 we discuss conformal differential geometry. A central goal of this chapter is to take steps to bring the EDS and parabolic geometry perspectives together by discussing a particular geometry. It should enable readers familiar with the parabolic geometry perspective to place this book in better context and enable readers from an EDS perspective to start reading the parabolic geometry literature (see [**33**] for an excellent introduction). Conformal geometry is a beautiful subject and we only touch on several topics: conformal Killing fields, the conformal Laplacian, and Gover's work on Einstein metrics in a given conformal class [**78**]. This chapter was heavily influenced by the expositions [**29**] and [**56**], as well as conversations with M. Eastwood, who we thank for his help.

In addition, we have re-arranged some of the material from the first edition. Grassmannians are introduced earlier in section 1.9. The first edition's Chapter 3, on projective geometry, has been split into two chapters: Chapter 4, which contains material that every differential geometer should learn and material needed later in the book, and Chapter 12, which contains more specialized and advanced topics. The chapter on G-structures (now Chapter 9) has also been substantially re-written for clarity, and a new section on G-Killing vector fields, based on conversations with R. Bryant, has been added.

With the re-arrangement of material, the interdependence of chapters in the second edition is described by the following diagram:

Dependence of Chapters



Suggested uses of this book:

- a year-long graduate course covering moving frames and exterior differential systems (chapters 1–9);
- a one-semester course on exterior differential systems and applications to partial differential equations (chapters 1 and 7–8);
- a one-semester course on the use of moving frames in algebraic geometry (chapters 4 and 12, preceded by part of chapter 1);
- a one-semester beginning graduate course on differential geometry (chapters 1, 2, 3 and 9)
- a year-long differential geometry course based on chapters 1, 2, 3 and 11, interspersed with an introduction to differentiable manifolds from e.g., [**174**, Vol. I].

Preface to the First Edition

In this book, we use moving frames and exterior differential systems to study geometry and partial differential equations. These ideas originated about a century ago in the works of several mathematicians, including Gaston Darboux, Edouard Goursat and, most importantly, Elie Cartan. Over the years these techniques have been refined and extended; major contributors to the subject are mentioned below, under "Further Reading".

The book has the following features: It concisely covers the classical geometry of surfaces and basic Riemannian geometry in the language of moving frames. It includes results from projective differential geometry that update and expand the classic paper [**85**] of Griffiths and Harris. It provides an elementary introduction to the machinery of exterior differential systems (EDS), and an introduction to the basics of G-structures and the general theory of connections. Classical and recent geometric applications of these techniques are discussed throughout the text.

This book is intended to be used as a textbook for a graduate-level course; there are numerous exercises throughout. It is suitable for a one-year course, although it has more material than can be covered in a year, and parts of it are suitable for a one-semester course (see the end of this preface for some suggestions). The intended audience is both graduate students who have some familiarity with classical differential geometry and differentiable manifolds, and experts in areas such as PDE and algebraic geometry who want to learn how moving frame and EDS techniques apply to their fields.

In addition to the geometric applications presented here, EDS techniques are also applied in CR geometry (see, e.g., [**120**]), robotics, and control theory (see [**68, 69, 159**]). This book prepares the reader for such areas, as well as for more advanced texts on exterior differential systems, such as [**27**], and papers on recent advances in the theory, such as [**71, 144**].

Overview. Each section begins with geometric examples and problems. Techniques and definitions are introduced when they become useful to help solve the geometric questions under discussion. We generally keep the presentation elementary, although advanced topics are interspersed throughout the text.

In Chapter 1 we introduce moving frames via the geometry of curves in the Euclidean plane $\mathbb{E}^2$. We define the Maurer-Cartan form of a Lie group G and explain its use in the study of submanifolds of G-homogeneous spaces. We give additional examples, including the equivalence of holomorphic mappings up to fractional linear transformation, where the machinery leads one naturally to the Schwarzian derivative.

We define exterior differential systems and jet spaces, and explain how to rephrase any system of partial differential equations as an EDS using jets. We state and prove the Frobenius system, leading up to it via an elementary example of an overdetermined system of PDE.

In Chapter 2 we cover traditional material—the geometry of surfaces in three-dimensional Euclidean space, submanifolds of higher-dimensional Euclidean space, and the rudiments of Riemannian geometry—all using moving frames. Our emphasis is on local geometry, although we include standard global theorems such as the rigidity of the sphere and the Gauss-Bonnet Theorem. Our presentation emphasizes finding and interpreting differential invariants to enable the reader to use the same techniques in other settings.

We begin Chapter 3[1] with a discussion of Grassmannians and the Plücker embedding.[2] We present some well-known material (e.g., Fubini's theorem on the rigidity of the quadric) which is not readily available in other textbooks. We present several recent results, including the Zak and Landman theorems on the dual defect, and results of the second author on complete intersections, osculating hypersurfaces, uniruled varieties and varieties covered by lines. We keep the use of terminology and results from algebraic geometry to a minimum, but we believe we have included enough so that algebraic geometers will find this chapter useful.

Chapter 4[3] begins our multi-chapter discussion of the Cartan algorithm and Cartan-Kähler Theorem. In this chapter we study constant coefficient homogeneous systems of PDE and the linear algebra associated to the corresponding exterior differential systems. We define tableaux and involutivity of tableaux. One way to understand the Cartan-Kähler Theorem is as follows: given a system of PDE, if the linear algebra at the infinitesimal level "works out right" (in a way explained precisely in the chapter), then existence of solutions follows.

In Chapter 5[4] we present the Cartan algorithm for linear Pfaffian systems, a very large class of exterior differential systems that includes systems of PDE rephrased as exterior differential systems. We give numerous examples, including many from Cartan's classic treatise [**40**], as well as the isometric immersion problem, problems related to calibrated submanifolds, and an example motivated by the variation of the Hodge structure.

[1]Now Chapters 4 and 12
[2]Now in Chapter 1
[3]Now Chapter 5
[4]Now Chapter 6

In Chapter 7[5] we take a detour to discuss the classical theory of characteristics, Darboux's method for solving PDE, and Monge-Ampère equations in modern language. By studying the exterior differential systems associated to such equations, we recover the sine-Gordon representation of pseudospherical surfaces, the Weierstrass representation of minimal surfaces, and the one-parameter family of noncongruent isometric deformations of a surface of constant mean curvature. We also discuss integrable extensions and Bäcklund transformations of exterior differential systems, and the relationship between such transformations and Darboux integrability.

In Chapter 6[6] we present the general version of the Cartan-Kähler Theorem. Doing so involves a detailed study of the integral elements of an EDS. In particular, we arrive at the notion of a Kähler-regular flag of integral elements, which may be understood as the analogue of a sequence of well-posed Cauchy problems. After proving both the Cartan-Kähler Theorem and Cartan's test for regularity, we apply them to several examples of non-Pfaffian systems arising in submanifold geometry.

Finally, in Chapter 8[7] we give an introduction to geometric structures (G-structures) and connections. We arrive at these notions at a leisurely pace, in order to develop the intuition as to why one needs them. Rather than attempt to describe the theory in complete generality, we present one extended example, path geometry in the plane, to give the reader an idea of the general theory. We conclude with a discussion of some recent generalizations of G-structures and their applications.

There are four appendices, covering background material for the main part of the book: linear algebra and rudiments of representation theory, differential forms and vector fields, complex and almost complex manifolds, and a brief discussion of initial value problems and the Cauchy-Kowalevski Theorem, of which the Cartan-Kähler Theorem is a generalization.

Layout. All theorems, propositions, remarks, examples, etc., are numbered together within each section; for example, Theorem 1.3.1 is the second numbered item in section 1.3. Equations are numbered sequentially within each chapter. We have included hints for selected exercises, those marked with the symbol ⊚ at the end, which is meant to be suggestive of a life preserver.

[5] Now Chapter 8
[6] Now Chapter 7
[7] Now Chapter 9

Further reading on EDS. To our knowledge, there are only a small number of textbooks on exterior differential systems. The first is Cartan's classic text [**40**], which has an extraordinarily beautiful collection of examples, some of which are reproduced here. We learned the subject from our teacher Bryant and the book by Bryant, Chern, Griffiths, Gardner and Goldschmidt [**27**], which is an elaboration of an earlier monograph [**26**], and is at a more advanced level than this book. One text at a comparable level to this book, but more formal in approach, is [**190**]. The monograph [**86**], which is centered around the isometric embedding problem, is similar in spirit but covers less material. The memoir [**189**] is dedicated to extending the Cartan-Kähler Theorem to the C^∞ setting for hyperbolic systems, but contains an exposition of the general theory. There is also a monograph by Kähler [**109**] and lectures by Kuranishi [**119**], as well the survey articles [**82, 110**]. Some discussion of the theory may be found in the differential geometry texts [**174**] and [**177**].

We give references for other topics discussed in the book in the text.

History and Acknowledgements. This book started out about a decade ago. We thought we would write up notes from Robert Bryant's Tuesday night seminar, held in 1988–89 while we were graduate students, as well as some notes on exterior differential systems which would be more introductory than [**27**]. The seminar material is contained in §9.8 and parts of Chapter 7. Chapter 2 is influenced by the many standard texts on the subject, especially [**55**] and [**174**], while Chapter 4 is influenced by the paper [**85**]. Several examples in Chapter 6 and Chapter 8 are from [**40**], and the examples of Darboux's method in Chapter 7 are from [**76**]. In each case, specific attributions are given in the text. Chapter 8 follows Chapter III of [**27**] with some variations. In particular, to our knowledge, Lemmas 8.1.10 and 8.1.13 are original. The presentation in §9.7 is influenced by [**15**], [**114**] and unpublished lectures of Bryant.

The first author has given graduate courses based on the material in Chapters 6 and 7 at the University of California, San Diego and at Case Western Reserve University. The second author has given year-long graduate courses using Chapters 1, 2, 4, 5, and 8 at the University of Pennsylvania and Université de Toulouse III, and a one-semester course based on Chapters 1, 2, 4 and 5 at Columbia University. He has also taught one-semester undergraduate courses using Chapters 1 and 2 and the discussion of connections in Chapter 8 (supplemented by [**173**] and [**174**] for background material) at Toulouse and at Georgia Institute of Technology, as well as one-semester graduate courses on projective geometry from Chapters 1 and 3 (supplemented by some material from algebraic geometry), at Toulouse, Georgia Tech. and the University of Trieste. He also gave more advanced

lectures based on Chapter 3 at Seoul National University, which were published as [**129**] and became a precursor to Chapter 3. Preliminary versions of Chapters 5 and 8 respectively appeared in [**126, 125**].

We would like to thank the students in the above classes for their feedback. We also thank Megan Dillon, Phillipe Eyssidieux, Daniel Fox, Sung-Eun Koh, Emilia Mezzetti, Joseph Montgomery, Giorgio Ottaviani, Jens Piontkowski, Margaret Symington, Magdalena Toda, Sung-Ho Wang and Peter Vassiliou for comments on the earlier drafts of this book, and Annette Rohrs for help with the figures. The staff of the publications division of the AMS—in particular, Ralph Sizer, Tom Kacvinsky, and our editor, Ed Dunne—were of tremendous help in pulling the book together. We are grateful to our teacher Robert Bryant for introducing us to the subject. Lastly, this project would not have been possible without the support and patience of our families.

Hints and Answers to Selected Exercises

Chapter 1

1.2.4 Use Stokes' Theorem applied to integrating about the boundary of a rectangle.

1.4.3(f) Let $A(t)$ be a family of circles with radius $r(t)$ and centers $c(t)$, and show that if $\frac{dr}{dt} > |\frac{dc}{dt}|$, then the circles are nested.

1.7.3.1(d) Consider the area of the region bounded by the curve and a line parallel to the curve, and take the limit as the line approaches the tangent line (see [**83**]).

1.8.4.3 For (b): Write $\overline{c}(t) = c(t)+r(t)e_2(t)$, and show that r is constant.

1.8.4.5 Consider the point $c(t) + \rho N + \sigma B$.

1.9.1.2 Use the skew-symmetry of the wedge product, defined in Appendix A.

1.10.7 For part (c), $n + m\sum_{j=0}^{k} \binom{n+j-1}{j}$.

Chapter 2

2.1.2.2 Compare the speed of a curve to its curvature.

2.1.2.3 What is the characteristic polynomial of h?

2.2.2.5 Take a line in the $x - z$ plane and rotate it about the z-axis.

2.2.2.6 Let $r(v) = C\cos(v)$, $t(v) = \int_0^v \sqrt{1 - C^2 \sin^2 v}\, dv$. The only complete ones are the spheres.

2.3.1.3 Differentiate $\omega_1^3 = k_1\omega^1$ and $\omega_2^3 = k_2\omega^2$.

2.3.1.4 Use (B.2).

2.3.1.5 Differentiate your answer from Exercise 2.3.1.3.

2.3.3 M compact implies there exists a point p at which k_1 has a local maximum. Then K constant implies that k_2 must have a local minimum at p as well. Deduce that p must be an umbilic point. Let $q \in M$ be any other point, and compare $k_j(p)$ with $k_j(q)$.

2.4.1 Answer: $H \equiv 0, K = -\frac{1}{a^2\cosh^4(v)}$.

2.4.3 Locally we may write $\omega_1^2 = du$ for some function $u : \mathcal{F} \to \mathbb{R}$. Since ω_1^2 is not semi-basic for the projection to M, locally there exists a vertical vector field X such that $\omega_1^2(X) > 0$, or $du(X) > 0$, which implies u is not constant in vertical directions. Thus we may choose a local section $F : M \to \mathcal{F}$ such that $F^*(u)$ is constant and therefore $F^*(\omega_1^2) = 0$.

2.4.5 Instead of calculating the whole frame, just show that $d\omega_1^2 = 0$.

2.5.4.2 It is sufficient to show that if we adapt frames such that e_1 is tangent to the fiber of the Gauss map, then the integral curves to the vector field e_1 are lines; see Chapter 4 for more details.

2.5.9 Use the identification in Exercise 3.7.2.4.

2.6.5 Extend c' to a neighborhood and show independence of extension.

2.7.2 Use Proposition 2.5.3.

2.7.3 Use (2.18) and apply Stokes' Theorem to $d\omega_1^2$.

2.8.1 Show that if one refines a triangulation by adding a new vertex and corresponding edges, then χ_Δ doesn't change.

3.2.2 If $\tilde{f} = Af$ for $A \in GL(n)$, then $(g_{ij}(\tilde{f})) = {}^tA^{-1}(g_{ij}(f))A^{-1}$.

2.8.5 First observe that this is true for orthonormal frames.

2.8.6 Normalize $e_1 \times e_2$ to obtain N.

Chapter 3

3.1.1 For (b) differentiate the result of part (a). For (d) use the Cartan Lemma A.1.9.

3.1.3 The fibers are isomorphic to $SO(V)$ without the identity labeled. If we consider the point we are taking the differential at as the identity, then that fiber becomes $SO(V)$ and thus its tangent space at that point becomes $\mathfrak{so}(V)$.

3.1.8 Show that $\tilde{\Theta}$ is unchanged under the action of $O(n)$ on the fiber of $\mathcal{F}_{\mathsf{on}}(M)$.

3.1.16 $d\beta$ is minus the skew-symmetrization of $\nabla\beta$.

3.1.19 For arbitrary $\alpha = a_i\eta^i$, we have $\nabla(\alpha(X)) = d(a_ix^i) = a_idx^i + x^ida_i$ and $X \lrcorner \nabla\alpha = x^i(da_i - a_j\eta^j_i)$ from (3.3).

3.1.21.3 Use $d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha(X,Y)$ for $\alpha = \eta^i$, and combine with (3.4).

3.1.21.4 Differentiating (3.4), with Z in place of X, gives a relationship between Θ and the second derivatives of Z.

3.2.3 To do this exercise, you need to use the higher-dimensional version of the Poincaré-Hopf Theorem; see p. 450 in Volume 1 of [**173**] for further details.

3.2.7 On one hand $\nabla h = \omega^i\omega^j \otimes (dh_{ij} - h_{ik}\omega^k_j - h_{jk}\omega^k_i)$ by the definition of the covariant derivative. On the other hand taking $0 = d(\omega^{n+1}_j - h_{jk}\omega^k$ shows $(dh_{ij} - h_{ik}\omega^k_j - h_{jk}\omega^k_i) \wedge \omega^j = 0$. Now apply the Cartan Lemma to conclude $(dh_{ij} - h_{ik}\omega^k_j - h_{jk}\omega^k_i) = h_{ijk}$ with $h_{ijk} = h_{ikj}$, but we also have $h_{ijk} = h_{jik}$.

3.2.10 The differentials of $\eta^i - \omega^i$ are clearly zero modulo I, those of $\eta^i_j - \omega^i_j$ are zero thanks to the Gauss equation, and that of $\omega^{n+1}, \omega^{n+1}_j - h_{jk}\omega^k$ is zero thanks to the Codazzi equation.

3.3.7.2 Show that $S^2(\Lambda^2 V)$ is spanned by elements of the form $(v \wedge w)^2$.

3.3.10 Differentiate $0 = d\eta^i_j + \eta^i_k \wedge \eta^k_j - \Theta^i_j$.

3.4.2.5 Consider the orbit of a one-parameter subgroup passing through a point x with the same tangent direction as a geodesic.

3.4.4.1 Use Proposition 11.2.10.

3.6.3.2 It is sufficient, by functoriality, to prove this is true on vector fields $Y \in \Gamma(TM)$.

3.7.2.1 Use Gram-Schmidt.

3.7.7 Use Schur's Lemma to show that for any group G and any irreducible G-module W, there exists at most one G-invariant quadratic form on W.

Chapter 4

4.2.2 Use the fundamental theorem of algebra.

4.2.16.1 When we project, we lose a conormal direction and the tangent space remains the same. Thus $II_{X',x'}$ is the same as $II_{X,x}$, except one quadric is lost. If we let $p = [e_{n+a}]$, and $II_{X,x} = q^{\mu}_{\alpha\beta}\omega^{\alpha}\omega^{\beta}\underline{e}_{\mu}$, then $II_{X',x'} = q^{\phi}_{\alpha\beta}\omega^{\alpha}\omega^{\beta}\underline{e}_{\phi}$, where $n+1 \leq \phi \leq n+a-1$.

4.2.16.2 Taking a hyperplane section, we lose a tangent direction; say this direction is $\underline{e}_n$. If $II_{X,x} = q^{\mu}_{\alpha\beta}\omega^{\alpha}\omega^{\beta}\underline{e}_{\mu}$, then $II_{X\cap H,x} = q^{\mu}_{st}\omega^{s}\omega^{t}\underline{e}_{\mu}$, where $1 \leq s,t \leq n-1$.

4.3.4.1 Calculate $d(e_0 + te_1)$.

4.3.4.2 Use Terracini's Lemma.

4.3.4.3 Use Exercise 4.3.4.2.

Chapter 5

5.1.11.1 $A = \{p^1_{11}w_1 \otimes v^1 \circ v^1 + p^1_{12}w_1 \otimes v^1 \circ v^2 + p^1_{22}w_1 \otimes v^2 \circ v^2 + p^2_{11}w_2 \otimes v^1 \circ v^1 + p^2_{12}w_2 \otimes v^1 \circ v^2 + p^2_{22}w_2 \otimes v^2 \circ v^2 \mid p^1_{11} + p^2_{22} = 0\}$ has dimension $5 = 6 - 1$.

Chapter 7

7.1.2.1 Use the fact that $\mathcal{L}_{[\mathsf{v},\mathsf{w}]} = \mathcal{L}_{\mathsf{v}} \circ \mathcal{L}_{\mathsf{w}} - \mathcal{L}_{\mathsf{w}} \circ \mathcal{L}_{\mathsf{v}}$. This identity is easily verified on 0-forms and exact 1-forms, and then follows for all forms from the Leibniz rule (B.3) for Lie derivatives.

7.1.2.2 Suppose $\mathcal{L}_{\mathsf{v}}\psi \in \mathcal{I}$ for $\psi \in \mathcal{I}$. Then use the Leibniz rule to show that $\mathcal{L}_{\mathsf{v}}(\alpha \wedge \psi) \in \mathcal{I}$ for any α, and use (B.4) to get $\mathcal{L}_{\mathsf{v}}(d\psi) \in \mathcal{I}$.

7.1.2.3 The most general v is given by (7.2) for $f = -h_z$ and $g = h + zh_z$, where h is an arbitrary function of x, y, z.

7.1.3 Note that if $\psi \in \mathcal{I}$, then $\phi_t^*(\psi) \in \mathcal{I}$ also.

7.1.6.1 Use (B.4).

7.1.6.2 Use the identity $[\mathsf{v},\mathsf{w}] \lrcorner\, \psi = \mathcal{L}_{\mathsf{v}}(\mathsf{w} \lrcorner\, \psi) - \mathsf{w} \lrcorner\, (\mathcal{L}_{\mathsf{v}}\psi)$, which may be derived in the same way as the formula for $\mathcal{L}_{[\mathsf{v},\mathsf{w}]}$ above.

7.1.6.3 Use the formula $\mathsf{v} \lrcorner\, (\alpha \wedge \psi) = (\mathsf{v} \lrcorner\, \alpha) \wedge \psi + (-1)^{\deg \alpha}\alpha \wedge (\mathsf{v} \lrcorner\, \psi)$, and let ψ be one of the algebraic generators of $\mathcal{I}$.

7.1.15.1 $u(x,y) = \frac{2}{5}\left(x \pm \frac{1}{2}y\right)^2$.

7.1.15.2 $u(x,y) = \frac{1}{2}(1 - x^2) \pm y$.

7.1.15.3 $u(x, y) = 2x^{3/2}/\sqrt{1 - 27y}$. In general, the solution of $u(s, 0) = f(s)$ is $u = f(s) - 2yf'(s)^3$, where s is implicitly defined as a function of x, y by $s = x + 3yf'(s)^2$. If $f(s)$ isn't linear, then there exists an s such that $f''(s) \neq 0$, and hence there exists a y such that $\partial s/\partial x$ is undefined. Then $\partial^2 u/\partial x^2$ is undefined.

7.1.19.1 Use the Pfaff Theorem 1.10.16.

7.1.19.2 Using x, y, z, p, q, s as coordinates, the Cauchy characteristic is given by

$$\mathsf{v} = \frac{\partial}{\partial x} + s^2 \frac{\partial}{\partial y} + (p + qs^2) \frac{\partial}{\partial z} + \frac{2}{3} s^3 \frac{\partial}{\partial p} + 2s \frac{\partial}{\partial q}.$$

7.1.22.2 See Proposition B.3.3.

7.2.8 The hyperbolic system of class zero is generated by $(du - f(v)dx) \wedge dy$ and $(dv - g(u)dy) \wedge dx$.

7.3.3 $I^{(2)}$ is spanned by θ_1.

7.3.13 $q = sx - y \ln s + \int \phi(y)dy + \int \psi(s)/s\, ds$ and $z = xp + qy - xys - \frac{1}{2}(x^2 r - y^2 \ln s) + \int y\phi(y)dy - \frac{1}{2} \int \psi(s)/s^2\, ds$.

7.3.15.2 The class-zero system is integrable if f, g are constants. The class-two system is integrable if f, g are exponential functions.

7.4.4.1 Compute $d\Psi$, noting that $dp, dq \equiv 0$ modulo θ, dx, dy. Then mod out by $d\theta$.

7.4.8 First apply the Pfaff Theorem 1.10.16 to obtain local coordinates in which θ is a multiple of $dz - p\, dx - q\, dy$. Then show that, in these coordinates, the generator 2-form can be taken to have the form (7.19).

7.4.10 $\mathcal{M}_{1,2} = \{\theta, dx \pm dq, dy \pm dp\}$.

7.4.11(b) Determine how $\beta_0, \beta_1, \beta_2$ can be modified, by adding linear combinations of the coframe 1-forms, and still satisfy the structure equations. Then, note that the derivative of the first equation implies $\beta_1 \equiv \beta_0 \bmod \theta, \omega_1, \pi_1$ and $\beta_2 \equiv \beta_0 \bmod \theta, \omega_2, \pi_2$.

7.4.11(c) The first structure equation now implies $d\beta_0 \equiv a(\omega_1 \wedge \pi_1 - \omega_2 \wedge \pi_2)$ mod θ. Then differentiate $\beta_1 = \beta_0 - a\theta$ modulo θ and compare with the derivative of the second structure equation.

7.4.13 If $f_z = 0$, we have a first-order PDE for $q = z_y$. If $f_z \neq 0$, then the equation is semi-integrable if and only if $f(x, y, z, q) = A(x, y, z)q + B(x, y, z)$ and $A_y = B_z$.

7.4.15 For part (a), note that ω^3 is the result of taking the dot product of dx with e_3. For part (b), note that F_r is covered by the transformation on $\mathcal{F}$ given by $(x; e_1, e_2, e_3) \mapsto (x+re_3; e_1, e_2, e_3)$. Then use (1.26) and (1.27) to compute pullbacks of the ω's.

7.4.16.1 Hint: when is $\Theta + \lambda\Psi$ decomposable, modulo ω^3?

7.4.16.2 In a space form of constant curvature ϵ, the Weingarten equation $AK+2BH+C=0$ is hyperbolic if and only if $B^2-A(C+\epsilon A)<0$. For example, the system for flat surfaces in S^3 is equivalent to the wave equation.

7.4.19 Hint: It's easier to use the ansatz $u = 2\arctan(f(x+y))$.

7.4.23 The minimal surfaces produced in question 1 are a plane, Enneper's surface, the catenoid, and the helicoid, respectively. The catenoid is also the mystery surface in question 3.

7.5.8 Let $V \subset TB$ be the bundle of vertical vectors for π. Then $J \subset T^*B$ defines a splitting T^*B as the direct sum $J \oplus V^\perp$ of vector bundles, with dual splitting $TB = V \oplus H$.
Let $\theta^i \in \Gamma(J)$ be a (local) basis for sections of J, with dual basis $e_i \in \Gamma(V)$. Then by definition

$$d\theta^i \equiv \Theta^i_j \wedge \theta^j \mod \pi^*\mathcal{I}$$

for some forms Θ^i_j on B. Thinking of H as a kind of connection, we define the *torsion* of J as

$$\Theta = \Theta^i_j \otimes (\theta^i \otimes e_j) \in \Gamma((T^*B/(J \oplus \pi^*\mathcal{I}^1)) \otimes \operatorname{End}(V)).$$

Then $(B, \mathcal{J})$ splits locally if there exists a subbundle $W \subset V$ preserved by Θ.

7.5.13.2 Condition (7.37a) implies that decomposable 2-forms in one system are congruent to decomposables for the other, modulo the contact forms $\omega^3, \bar{\omega}^3$. The asymptotic lines are dual to the factors of these decomposables.

7.5.13.5 Although the Cole-Hopf transformation can be described in terms of a double fibration, with the heat equation on one side and Burger's equation on the other, the pullback of the heat equation system doesn't include the 2-forms for Burger's equation. Equivalently, given a solution of the heat equation, there is no Frobenius system available for us to produce multiple solutions of Burger's equation.

7.5.13.6 The point is to show that Bäcklund-equivalence is transitive: given $(M_1, \mathcal{I}_1)$ linked to $(M_2, \mathcal{I}_2)$ by a double fibration carrying a Bäcklund transformation, and $(M_2, \mathcal{I}_2)$ similarly linked to $(M_3, \mathcal{I}_3)$, use this data to construct a Bäcklund transformation linking $(M_1, \mathcal{I}_1)$ and $(M_3, \mathcal{I}_3)$.

Chapter 8

8.1.5 Consider the bundle map $i : T\Sigma \to T^*\Sigma$ given by $\mathsf{v} \mapsto \mathsf{v} \lrcorner (\omega^1 \wedge \omega^2)$, which is well-defined up to multiple. Then the singular points of $\mathcal{V}_2(\mathcal{I})$ are those 2-planes to which this map restricts to be zero.

8.1.7.3 If $\psi \in \mathcal{I}^i$ for $k \leq n$, then $v \lrcorner \psi|_E \neq 0$ implies that there exists $\psi \wedge \alpha \in \mathcal{I}^{n+1}$ such that $v \lrcorner \psi|_E \neq 0$.

8.2.1.1 Express the right-hand side of (8.6) in terms of the Ω^i and solve for the Ω^i in terms of the $d\omega^i \wedge \omega^i$'s.

8.2.1.3 Consider the intersection of a hyperboloid and an ellipsoid in a system of confocal quadrics.

8.5.2 Let $q = \kappa e^{i\theta}$ and differentiate $N = e_2 \sin\theta + e_3 \cos\theta$.

8.5.9.1 In fact, $d(x + (1/k_0)e_3) \wedge \omega^1 = 0$, so each line of curvature in the e_2-direction lies on a sphere of radius $1/k_0$. The surface is the envelope of these spheres, and the two functions may be taken to be the curvature and torsion of the curve traced out by the centers of the spheres.

8.5.9.2 These surfaces may also be characterized as those for which both of the focal surfaces are degenerate. One such surface is the torus swept out by revolving a circle about an axis. The EDS for these surfaces becomes Frobenius after two prolongations, and the solutions are the *cyclides of Dupin*. These are the surfaces that are obtained by acting on circular tori of revolution by the group $O(4,1)$ of conformal transformations of $\mathbb{E}^3$. See [**104**] for more information, including an interesting application of conformal moving frames.

8.5.9.4 Add the 3-form $\omega^4_1 \wedge \omega^2 \wedge \omega^3 + \omega^4_2 \wedge \omega^3 \wedge \omega^1 + \omega^4_3 \wedge \omega^1 \wedge \omega^3$ to the ideal.

Chapter 9

9.1.3 All are equivalent.

9.1.12.5 $\dfrac{1}{2}\dfrac{\partial^2}{\partial x \partial y}(\log \dfrac{H_x}{H_y}) dx \wedge dy.$

9.1.12.6 $\dfrac{FF_{xy} - F_x F_y}{F^2} dx \wedge dy.$

9.7.1 Show that the components of θ span a codimension-one Pfaffian system on $\alpha^* \mathcal{F}_G$.

9.7.2 Use the equivariance of θ.

9.8.10 Consider H^2 as half of a two-sheeted hyperboloid in Lorentz space.

Chapter 10

10.2.6 Given $\alpha \in \Gamma(\hat{V})$, let $\alpha = \alpha_1 + \alpha_0$ where $\alpha_1 \in \Gamma(\check{V})$ and $\alpha_0 \in \Gamma(\hat{V}^{(\infty)})$. Conclude that α_1 is a section of $\hat{V} \cap \check{V} = I$. Thus, $\hat{V} = \{\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \boldsymbol{\theta}, \check{\boldsymbol{\eta}}\}$, and then use $T^*M = \hat{V} + \check{V}^{(\infty)}$.

10.2.9 To show, e.g., that $\hat{\mathcal{V}}$ is a differential ideal, note that modulo the 1-forms in $\hat{V}$, the $\check{\boldsymbol{\Omega}}$ are spanned by the derivatives of $\boldsymbol{\theta}$ and $\check{\boldsymbol{\eta}}$.

10.2.11 If $\alpha \in \hat{\mathcal{V}} \cap\!\!\!\!+ \check{\mathcal{V}}$ and β is any k-form, consider evaluating $\alpha \wedge \beta$ on the direct sum of an integral element of $\hat{\mathcal{V}}$ and an integral element of $\check{\mathcal{V}}$. (Take a basis for each.)

10.3.2 Show that $\hat{\eta}^m([\hat{S}_a, \hat{S}_b]) = -\hat{F}^m_{ab}$. More generally, derive a recursive formula for $\hat{\eta}^m([\hat{S}_A, \hat{S}_b])$ and conclude (using the last statement of Proposition 10.2.7) that $\hat{\eta}^m(\hat{S}_A)$ is always a first integral of $\hat{V}^{(\infty)}$.

10.3.7 It is straightforward algebra to show that $\hat{\omega}^i = \Lambda^i_j \check{\omega}^j$. By evaluating $d\theta^i_X(X_j, Y_k)$ two different ways, show that $X_j(Q^i_k) = C^i_{j\ell} Q^\ell_k$. Use this and the 4-adapted structure equations to compute dQ^i_j. Then, letting $\hat{\psi}^j = \hat{S}^j_a \hat{\pi}^a$ and $\check{\psi}^j = \check{S}^j_\alpha \check{\pi}^\alpha$, compare the 4-adapted and 5-adapted structure equations to compute $d\hat{\boldsymbol{R}}$, $d\check{\boldsymbol{R}}$, $d\hat{\psi}^j$ and $d\check{\psi}^j$. Finally, using all of this, compute $d\hat{\omega}^i$ and $d\check{\omega}^i$.

10.5.7 By Proposition 10.2.10, the span of the $\check{\boldsymbol{\Omega}}_1$ coincides with the pullbacks of $d\check{\boldsymbol{\eta}}$ and $\boldsymbol{B}\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}$ to M_1, and these in turn are derivatives of 1-forms in $\mathcal{W}_1$.

10.5.11.8 When the system is encoded in the usual way on manifold $\Sigma \subset J^2(\mathbb{R}^3, \mathbb{R})$, there is a decomposition $T^*\Sigma = I \oplus J_1 \oplus J_2 \oplus J_3$ such that any generator 2-form belongs to $\Lambda^2 J_i$ for some i. Then one can take $\hat{V}$ to be the sum of I and any two of the J_i, while $\check{V}$ is the sum of I and the remaining term in the decomposition.

Chapter 11

11.1.6 Under a change of metric

$$\hat{\Theta}^i_j = [\epsilon + \sigma^{-2}(\sum_k \sigma_k^2)]\eta^i \wedge \eta^j + \sigma^{-1}(s_{jm}\eta^i \wedge \eta^m - s_{im}\eta^j \wedge \eta^m).$$

In order that this is zero we see $s_{ij} = 0$ if $i \neq j$ and $s_{jj} = \frac{1}{2}s$ is independent of j, where

$$s = -\sigma\epsilon - \sigma^{-1}(\sum_k \sigma_k^2). \tag{4.5}$$

Set up an EDS on $\mathcal{F}^{(1)}_{CO(V)} \times \mathbb{R} \times \mathbb{R}^n$ where the last two factors have coordinates (σ, σ_i) given by

$$I = \{\theta = d\sigma - \sigma_i\omega^i,\ \theta_i = d\sigma_i - \sigma_j\theta^j_i - \mu\omega^i\}.$$

Show the system is Frobenius, keeping in mind that the expression for ds is obtained by differentiating (4.5).

11.1.8 Since there is only one relation, the characters of the tableau (ignoring the $d\theta$ and $d\lambda$ terms) are $s_1 = \ldots = s_{n-1} = n$, $s_n = n-1$. Any tableau with these characters cannot have torsion and is involutive.

11.2.1 Since the map $\delta : \mathfrak{so}(V)\otimes V^* \to V\otimes\Lambda^2 V^*$ is an isomorphism, the map $\mathfrak{co}(V)\otimes V^* \to V\otimes\Lambda^2 V^*$, whose kernel is $\mathfrak{co}(V)^{(1)}$, must have an n-dimensional kernel. Write an element of V^* as $t = t_k v^k$. Map this to $(\delta^i_j t_k - \delta^j_k t_i + \delta^i_k t_j)(v_i\otimes v^j)\otimes v^k$, and observe this is a $\mathfrak{co}(V)$-module map. Note that this lies in $\mathfrak{co}(V)\otimes V^*$ and in $V\otimes S^2 V^*$ and thus in $\mathfrak{co}(V)^{(1)}$. Then to show $\mathfrak{co}(V)^{(2)} = 0$, show that this space tensored with each v_m maps injectively under $\delta : \mathfrak{co}(V)^{(1)}\otimes V^* \to V\otimes V^*\otimes\Lambda^2 V^*$.
We remark that the first part can be solved without calculation: This map is a $\mathfrak{co}(V)$-module map, in particular an $\mathfrak{so}(V)$-module map. As an $\mathfrak{so}(V)$-module map, its kernel is isomorphic to $V \simeq V^*$. As a $\mathfrak{co}(V)$-module map, its kernel is therefore of the form $V\otimes\mathbb{R}_\mu$, where the first factor is the $\mathfrak{so}(V)$-module V, and the second tells how $\lambda\,\mathrm{Id}$ acts, namely by $\lambda\,\mathrm{Id} \mapsto \lambda^\mu\,\mathrm{Id}$. The case $\mu = 1$ corresponds to the $\mathfrak{co}(V)$-module V and the case $\mu = -1$ corresponds to the $\mathfrak{co}(V)$-module V^*. But on the whole space $V\otimes V^*\otimes V^*$, the action of $\lambda\,\mathrm{Id}$ is with $\mu = -1$, and so we conclude.

11.2.2 See Equation (11.6).

11.4.2 Sections of $\mathcal{R}_+$ correspond to sections of $\mathbb{R}[2]$.

11.4.4 Since it is semi-basic, it is sufficient to prove its exterior derivative is semi-basic.

11.5.5 Use (11.27) and (11.3), keeping in mind that $s = \frac{1}{2(n-1)}\mathrm{P}$.

Chapter 12

12.1.8.1 Use equation (12.5).

12.1.8.3 Consider a cubic form in two variables and differentiate.

12.2.1 Use the prolongation property.

12.2.2.3 $T_E G_\omega(k, W) \simeq E^* \otimes (E^\perp/E) \oplus S^2E^*$. Here $E^\perp = \{w \in W \mid \omega(v, w) = 0\ \forall v \in E\}$.

12.2.5 Note that M_r is a join.

12.4.3 The cases (a), (b), (e) are always hypersurfaces, (c), (d) are hypersurfaces respectively when $a = b$ and m is even. The smooth quadric surface in $\mathbb{P}^3$, $v_2(\mathbb{P}^1)$, $G(2, 5)$ and $\mathbb{P}^1 \times \mathbb{P}^n$ are all self-dual. The self-duality can be proven directly; or, once one knows the dimension is correct, since the dual variety of a homogeneous variety inherits a group action and they are the dimension of the minimal orbit, they must be the minimal orbit.

12.4.14.1 Given $A \in M_{p\times q}$, consider

$$\begin{pmatrix} 0 & A \\ {}^tA & 0 \end{pmatrix}.$$

12.4.14.2 Any nonzero 3×3 skew-symmetric matrix has rank two.

12.4.14.3 Answer: $|II_{\mathrm{Seg}(\mathbb{P}^1\times\mathbb{P}^n)}|$.

12.5.5 Since we are working with $e_\mu \operatorname{mod}\{\hat{T}, \widehat{II_v(T)}\}$, adapt frames further such that $\{e_\xi\} \simeq II_v(T)$ and $\{e_\phi\}$ gives a complement. You will need to show that if $w \in \mathrm{Singloc}(\mathrm{Ann}(v))$, then $II_w(T) \subseteq II_v(T)$.

12.5.6.1 Note that III can be recovered from $III(v, v, v)$ for all $v \in T$.

12.6.12.1 Note that $II(w_1 \wedge w_2, u_1 \wedge u_2) = w_1 \wedge w_2 \wedge u_1 \wedge u_2$.

12.8.9 Adapt frames so that $II = \omega^i\omega^j \otimes e_{ij}$ and calculate F_4, F_5. The calculation is a little involved; see [**127**] for the details.

12.9.5 Use the calculation (12.14).

12.11.4 Use Corollary 12.11.2. What is $\ker II$?

12.13.2 If $x \in X$ is a general point and $III_{X,x} = 0$, then a hyperplane osculating to order two at x contains X.

Appendix A

A.1.6.3 Use the preceding exercise.

A.1.8.2 $\Lambda^k V$ is the span of the wedge products (A.3).

A.1.8.4 $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $1 \leq i_1 < i_2 < \cdots < i_k < n$ is a basis of $\Lambda^k V$.

A.1.11 When $\dim V > n$, there are vectors $r_{jk} = r_{kj}$ in V such that $R_j = r_{jk} \wedge v_k$.

A.2.2.1 Consider the kernel of f.

A.2.2.2 Consider the map $\psi : V \otimes S^2 V \to \Lambda^2 V \otimes V$ given by $w \otimes v \otimes v \mapsto v \mathsf{w} w \otimes v$, and show that the image of ψ is $K_\Lambda(V)$.

A.3.1.1 Consider the characteristic polynomial of J.

A.5.2 If you have trouble doing this invariantly, use the answer to Exercise A.5.3.4.

A.5.3.4 $\phi = dx^{123} + dx^{345} + dx^{156} + dx^{246} - dx^{147} - dx^{367} + dx^{257}$, where we have written dx^{ijk} for $dx^i \wedge dx^j \wedge dx^k$.

A.6.3 Take a basis of $\mathbb{C}^{d+1}$ consisting of points that project to be on C_d.

A.6.4 First consider the case where all the eigenvalues are distinct and let v be an eigenvector of A_1. Compute $A_1 A_2 v = A_2 A_1 v$ to see that $A_2 v$ is also an eigenvector of A_1 with the same eigenvalue as v.

A.6.5 Schur's Lemma A.2.1 holds for Lie algebras, and the span of an eigenvector gives a subrepresentation.

A.7.9.3 Calculate

A.7.10.4 Use Exercise A.7.10.2; the algebra of $m \times m$ matrices cannot act nontrivially on a vector space of dimension less than m.

Appendix B

B.1.3.2 $[X, Y] = a^i \dfrac{\partial b^j}{\partial x^i} \dfrac{\partial}{\partial x^j} - b^i \dfrac{\partial a^j}{\partial x^i} \dfrac{\partial}{\partial x^j}$.

B.2.1 Let y^j be a second set of coordinates and similarly let $\alpha = (1/k!) \sum_J b_J dy^{j_1} \wedge \ldots \wedge dy^{j_k}$. Using the chain rule, show that $d\alpha$ is the same computed in either coordinate system.

Appendix C

C.1.10(b) Recall from Chapter 4 that the forms ω_i^k for $3 \leq k \leq n$ and $i = 1, 2$ are semi-basic for the projection from $GL(V)$ to G. Show that the system spanned by the forms $\omega_1^k - i\omega_2^k$ is Frobenius, and drops to G and forms a basis for $T^{*(1,0)}$.

Bibliography

1. L. Ahlfors, *Complex Analysis*, McGraw-Hill, 1966.
2. D.N. Akhiezer, *Lie group actions in complex analysis*, Vieweg, 1995.
3. M. Akivis, Webs and almost Grassmann structures, *Soviet Math. Dokl.* 21 (1980), 707–709.
4. M. Akivis, V. Goldberg, On the structure of submanifolds with degenerate Gauss maps, *Geom. Dedicata* 86 (2001), 205–226.
5. I. Anderson, M. Fels, Bäcklund Transformations for Darboux Integrable Differential Systems, *Selecta Mathematica* 21 (2015), 379–448.
6. I. Anderson, M. Fels, P. Vassiliou, Superposition formulas for exterior differential systems, *Advances in Mathematics* 221 (2009), 1910–1963.
7. I. Anderson, N. Kamran, The variational bicomplex for hyperbolic second-order scalar partial differential equations in the plane, *Duke Math. J.* 87 (1997), 265–319.
8. I. Anderson, N. Kamran, P. Olver, Internal, External, and Generalized Symmetries, *Adv. Math* 100 (1993), 53–100.
9. M. Audin, J. Lafontaine (eds.), *Holomorphic curves in symplectic geometry*, Birkhäuser, 1994.
10. T. N. Bailey, M. G. Eastwood, and A. R. Gover, Thomas's structure bundle for conformal, projective and related structures, *Rocky Mountain J. Math.* 24 (1994), no. 4, 1191–1217.
11. W. Barth, M. Larsen, On the homotopy groups of complex projective algebraic manifolds, *Math. Scand.* 30 (1972), 88–94.
12. R. Baston, M. Eastwood, *The Penrose transform. Its interaction with representation theory*, Oxford University Press, 1989.
13. E. Berger, R. Bryant, P. Griffiths, The Gauss equations and rigidity of isometric embeddings, *Duke Math. J.* 50 (1983) 803–892.
14. Jonas Bergman, S. Brian Edgar, and Magnus Herberthson, The Bach tensor and other divergence-free tensors, *Int. J. Geom. Methods Mod. Phys.* 2 (2005), no. 1, 13–21.
15. A.L. Besse, *Einstein Manifolds*, Springer, 1987.

16. R. Bishop, There is more than one way to frame a curve, *Am. Math. Monthly* 82 (1975), 246–251.
17. A. Bobenko, Exploring surfaces through methods from the theory of integrable systems: Lectures on the Bonnet Problem, preprint (1999), `arXiv:math.DG/9909003`
18. R. Bott, L. Tu, *Differential forms in algebraic topology*, Springer, 1982.
19. N. Bourbaki, *Groupes et algèbres de Lie*, Chap. 4–6, Hermann, 1968.
20. Simon Brendle, Embedded minimal tori in S^3 and the Lawson conjecture, *Acta Math.* 211 (2013), no. 2, 177–190.
21. Simon Brendle and Fernando C. Marques, Recent progress on the Yamabe problem, pp. 29–47 in *Surveys in geometric analysis and relativity*, International Press, 2011.
22. R. Bryant, Conformal and Minimal Immersions of Compact Surfaces into the 4-sphere, *J. Diff. Geom.* 17 (1982), 455–473.
23. —, Metrics with exceptional holonomy, *Ann. of Math.* 126 (1987), 525–576.
24. —, An introduction to Lie groups and symplectic geometry, pp. 5–181 in *Geometry and quantum field theory*, Amer. Math. Soc., Providence, 1995.
25. —, *Rigidity and quasi-rigidity of extremal cycles in Hermitian symmetric spaces*, Annals of Mathematics Studies vol. 153, Princeton, 2002.
26. R. Bryant, S.-S. Chern, P. Griffiths, Exterior Differential Systems, pp. 219–338 in *Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations*, Science Press, Beijing, 1982.
27. R. Bryant, S.-S. Chern, R.B. Gardner, H. Goldschmidt, P. Griffiths, *Exterior Differential Systems*, MSRI Publications, Springer, 1990.
28. R. Bryant, P. Griffiths, Characteristic Cohomology of Differential Systems (II): Conservation Laws for a Class of Parabolic Equations, *Duke Math. J.* 78 (1995), 531–676.
29. R. Bryant, P. Griffiths, D. Grossman, *Exterior differential systems and Euler-Lagrange partial differential equations*, University of Chicago Press, 2003.
30. R. Bryant, P. Griffiths, L. Hsu, Hyperbolic exterior differential systems and their conservation laws (I), *Selecta Mathematica (N.S.)* 1 (1995), 21–112.
31. —, Toward a geometry of differential equations, pp. 1–76 in *Geometry, topology, and physics*, International Press, 1995.
32. R. Bryant, S. Salamon, On the construction of some complete metrics with exceptional holonomy, *Duke Math. J.* 58 (1989), 829–850.
33. Andreas Čap and Jan Slovák, *Parabolic geometries I: Background and general theory*, American Mathematical Society, 2009.
34. J. Carlson, D. Toledo, Generic Integral Manifolds for weight two period domains, *Trans. AMS* 356 (2004), 2241-2249.
35. E. Cartan, Sur la structure des groupes infinis de transformations, *Ann. Sci. Ecole Norm. Sup.* 26 (1909), 93–161.
36. —, Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre, *Ann. Ecole Norm. Sup.* 27 (1910), 109–192.
37. —, Sur les variétés de courbure constante d'un espace euclidien ou non euclidien, *Bull. Soc. Math France* 47 (1919) 125–160 and 48 (1920), 132–208; see also pp. 321–432 in *Oeuvres Complètes* Part 3, Gauthier-Villars, 1955.
38. —, Sur les variétés à connexion projective, *Bull. Soc. Math. Fr.* 52 (1924), 205–241.
39. —, Sur la théorie des systèmes en involution et ses applications à la relativité, *Bull. Soc. Math. Fr.* 59 (1931), 88–118; see also pp. 1199–1230 in *Oeuvres Complètes*, Part 2.

40. —, *Les Systèmes Extérieurs et leurs Applications Géométriques*, Hermann, 1945.
41. P. Chaput, Severi varieties, *Math. Z.* 240 (2002), 451–459.
42. S.-S. Chern, *Selected Papers* Vols. 1–4, Springer, 1985/1989.
43. —, A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds, *Ann. Math* 45 (1944), 747–752.
44. —, Pseudo-groupes continus infinis, pp. 119–136 in *Colloques Internationaux du Centre National de la Recherche Scientifique: Géométrie différentielle*, C.N.R.S, 1953.
45. —, *Complex Manifolds without Potential Theory*, Springer, 1979.
46. —, Deformations of surfaces preserving principal curvatures, pp. 155–163 in *Differential Geometry and Complex Analysis*, Springer, 1984. (See also *Selected Papers*, vol. IV.)
47. S.-S. Chern, R. Osserman, Remarks on the Riemannian metric of a minimal submanifold, pp. 49–90 in *Geometry Symposium, Utrecht 1980*, Lecture Notes in Math. 894, Springer, 1981.
48. S.-S. Chern, C.-L. Terng, An analogue of Bäcklund's theorem in affine geometry, *Rocky Mountain Math J.* 10 (1980), 105–124.
49. S.-S. Chern, J.G. Wolfson, A simple proof of Frobenius theorem, pp. 67–69 in *Manifolds and Lie groups (Notre Dame, Ind., 1980)*, Birkhäuser, 1981.
50. J. Clelland, T. Ivey, Parametric Bäcklund Transformations I: Phenomenology, *Trans. Amer. Math. Soc.* 357 (2005), 1061–1093.
51. —, Bäcklund Transformations and Darboux Integrability for Nonlinear Wave Equations (with J. Clelland), *Asian J. Math.* 13 (2009), 13–64.
52. J. Clelland, P. Vassiliou, A solvable string on a Lorentzian surface, *Diff. Geom. Appl.* 33 (2014), 177–198.
53. D. Cox, S. Katz, *Mirror symmetry and algebraic geometry*, AMS, 1999.
54. G. Darboux, *Leçons sur la Théorie Générale des Surfaces* (3rd ed.), Chelsea, 1972.
55. M. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.
56. M.G. Eastwood, A.R. Gover, and K. Neusser, *Conformal differential geometry*, in preparation.
57. L. Ein, Varieties with small dual varieties, I, *Inventiones Math.* 86 (1986), 63–74.
58. —, Varieties with small dual varieties, II, *Duke Math. J.* 52 (1985), 895–907.
59. Ph. Ellia, D. Franco, On codimension two subvarieties of $\mathbf{P}^5$ and $\mathbf{P}^6$, *J. Algebraic Geom.* 11 (2002), 513–533.
60. Y. Eliashberg, L. Traynor (eds.) *Symplectic Geometry and Topology*, AMS, 1999.
61. L. Evans, *Partial Differential Equations*, AMS, 1998.
62. A.R. Forsyth, *Theory of Differential Equations (Part IV): Partial Differential Equations*, Cambridge University Press, 1906; also, Dover Publications, 1959.
63. G. Fubini, E. Cech, *Géométrie projective différentielle des surfaces*, Gauthier-Villars, 1931.
64. W. Fulton, J. Hansen, A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings, *Ann. of Math.* (2) 110 (1979), 159–166.
65. W. Fulton, J. Harris, *Representation theory – a first course*, Springer, 1991.
66. R.B. Gardner, Invariants of Pfaffian Systems, *Trans. A.M.S.* 126 (1967), 514–533.
67. —, *The Method of Equivalence and Its Applications*, SIAM, 1989.
68. —, Differential geometric methods interfacing control theory, pp. 117–180 in *Differential Geometric Control Theory*, Birkhäuser, 1983.

69. R.B. Gardner, W.F. Shadwick, The GS algorithm for exact linearization, *IEEE Trans. Autom. Control* 37 (1992), 224–230.
70. J. Gasqui, Sur la résolubilité locale des équations d'Einstein, *Compositio Math.* 47 (1982), 43–69.
71. —, Formal integrability of systems of partial differential equations, pp. 21–36 in *Nonlinear equations in classical and quantum field theory*, Lecture Notes in Phys. 226, Springer, 1985.
72. I.M. Gel'fand, S. Fomin, *Calculus of Variations*, Prentice-Hall, 1963.
73. I.M. Gel'fand, M. Kapranov, A. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, 1994.
74. H. Goldschmidt, Existence theorems for analytic linear partial differential equations, *Ann. of Math.* 86 (1967), 246–270.
75. A.B. Goncharov, Generalized Conformal Structures on Manifolds, *Selecta Math. Sov.* 6 (1987), 307–340.
76. E. Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, Gauthier-Villars, 1890.
77. —, Recherches sur quelques équations aux dérivées partielles du second ordre, *Annales de la Faculté de Toulouse, deuxième serie* 1 (1899), 31–78.
78. A. Rod Gover, Almost conformally Einstein manifolds and obstructions, pp. 247–260 in *Differential geometry and its applications*, Matfyzpress, Prague, 2005.
79. —, Almost Einstein, Poincaré-Einstein manifolds in Riemannian signature, *J. Geom. Phys.* 60 (2010), no. 2, 182–204.
80. C. Robin Graham and Kengo Hirachi, The ambient obstruction tensor and Q-curvature, pp. 59–71 in *AdS/CFT correspondence: Einstein metrics and their conformal boundaries*, Eur. Math. Soc., Zürich, 2005.
81. Mark L. Green, Generic initial ideals, pp. 119–186 in *Six lectures on commutative algebra* (J. Elias *et al*, eds.), *Progr. Math.* 166, Birkhäuser, 1998.
82. P. Griffiths, Some aspects of exterior differential systems, pp. 151–173 in *Complex geometry and Lie theory*, Proc. Sympos. Pure Math. 53 (1991), AMS.
83. —, *Exterior Differential Systems and the Calculus of Variations*, Birkhäuser, 1983.
84. P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley, 1978.
85. —, Algebraic geometry and local differential geometry, *Ann. Sci. Ecole Norm. Sup.* 12 (1979), 355–432.
86. P. Griffiths, G. Jensen, *Differential systems and isometric embeddings*, Princeton University Press, 1987.
87. M. Guest, *Harmonic Maps, Loop Groups and Integrable Systems*, Cambridge University Press, 1997.
88. V. Guillemin, The integral geometry of line complexes and a theorem of Gel'fand-Graev, pp. 135–149 in *The mathematical heritage of Elie Cartan*, *Astérisque*, Numero Hors Serie (1985).
89. J. Harris, *Algebraic geometry, a first course*, Springer, 1995.
90. Robin Hartshorne, Varieties of small codimension in projective space, *Bull. Amer. Math. Soc.* 80 (1974), 1017–1032.
91. F.R. Harvey, *Spinors and Calibrations*, Academic Press, 1990.
92. F.R. Harvey, H.B. Lawson, Calibrated geometries, *Acta Math.* 148 (1982), 47–157.

93. T. Hawkins, *Emergence of the theory of Lie groups. An essay in the history of mathematics 1869–1926*, Springer, 2000.
94. S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, 1978.
95. N. Hitchin, Kählerian twistor spaces, *Proc. London Math. Soc. (3)* 43 (1981), 133–150.
96. —, Complex manifolds and Einstein's equations, pp. 73–99 in *Twistor geometry and nonlinear systems*, Lecture Notes in Math. 970, Springer, 1982.
97. W.V.D. Hodge, D. Pedoe, *Methods of Algebraic Geometry* Vol. 2, Cambridge University Press, 1954.
98. J. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, 1972.
99. J.M. Hwang, N. Mok, Uniruled projective manifolds with irreducible reductive G-structures, *J. reine angew. Math.* 490 (1997), 55–64.
100. J.M. Hwang, N. Mok, Rigidity of irreduicible Hermitian symmetric spaces of the compact type under Kahler deformation, *Invent. Math.* 131 (1998), 393–418.
101. J.M. Hwang, K. Yamaguchi, Characterization of Hermitian symmetric spaces by fundamental forms, *Duke Math. J.* 120 (2003), 621–634.
102. J. Jost, *Riemannian geometry and geometric analysis*, Springer, 2011.
103. Bo Ilic, J. M. Landsberg, On symmetric degeneracy loci, spaces of symmetric matrices of constant rank and dual varieties, *Math. Ann.* 314 (1999), no. 1, 159–174.
104. T. Ivey, Surfaces with orthogonal families of circles, *Proc. AMS* **123** (1995), 865–872.
105. F. John, *Partial Differential Equations* (4th ed.), Springer, 1982.
106. D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2, I, II, *J. Diff. Geom.* 43 (1996), 291–375.
107. D. Joyce, Compact 8-manifolds with holonomy $\mathrm{Spin}(7)$, *Invent. Math.* 123 (1996), 507–552.
108. M. Juráš, I. Anderson, Generalized Laplace invariants and the method of Darboux, *Duke Math. J.* 89 (1997), 351–375.
109. E. Kähler, *Einfürhung in die Theorie der Systeme von Differentialgleichungen*, Teubner, 1934.
110. N. Kamran, An elementary introduction to exterior differential systems, pp. 151–173 in *Geometric approaches to differential equations*, Cambridge University Press, 2000.
111. S. Kleiman, Tangency and duality, pp. 163–225 in *Proceedings of the 1984 Vancouver conference in algebraic geometry*, CMS Conf. Proc., vol. 6, Amer. Math. Soc., 1986.
112. M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972.
113. A. Knapp, *Lie groups beyond an introduction*, Birkhäuser, 1996.
114. S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Vols. 1,2, Wiley, 1963/1969.
115. S. Kobayashi, T. Ochiai, Holomorphic structures modeled after compact Hermitian symmetric spaces, pp. 207–222 in *Manifolds and Lie groups*, Birkhäuser, 1981.
116. Bertram Kostant, Holonomy and the Lie algebra of infinitesimal motions of a Riemannian manifold, *Trans. Amer. Math. Soc.* 80 (1955), 528–542.
117. J.Krasil'shchick, A.M. Vinogradov (eds.), *Symmetries and conservation laws for differential equations of mathematical physics*, American Math. Society, 1999.
118. Wolfgang Kühnel and Hans-Bert Rademacher, Einstein spaces with a conformal group, *Results Math.* 56 (2009), no. 1-4, 421–444.

119. M. Kuranishi, Lectures on exterior differential systems, Tata Institute, Bombay, 1962.
120. —, CR geometry and Cartan geometry, Forum Math. 7 (1995), 147–205.
121. J.M. Landsberg, Minimal submanifolds defined by first-order systems of PDE, *J. Diff. Geom.* 36 (1992), 369–415.
122. —, On second fundamental forms of projective varieties, *Inventiones Math.* 117 (1994), 303–315.
123. —, On degenerate secant and tangential varieties and local differential geometry, *Duke Mathematical Journal* 85 (1996), 605–634.
124. —, Differential-geometric characterizations of complete intersections, *J. Diff. Geom.* 44 (1996), 32–73.
125. —, Introduction to G-structures via three examples, pp. 133–149 in *Web theory and related topics*, World Scientific, 2001,
126. —, Exterior differential systems: a geometric approach to pde, pp. 77–101 in *Topology and Geometry*, Proc. Workshop Pure Math. vol. 17, Korean Academic Council, 1998.
127. —, On the infinitesimal rigidity of homogeneous varieties, *Compositio Math.* **118** (1999), 189–201.
128. —, Is a linear space contained in a submanifold? - On the number of derivatives needed to tell, *J. reine angew. Math.* 508 (1999), 53–60.
129. —, *Algebraic geometry and projective differential geometry*, Lecture Notes Series, vol. 45, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1999.
130. —, Lines on projective varieties, *J. Reine Angew. Math.* 562 (2003), 1–3.
131. —, Griffiths-Harris rigidity of compact Hermitian symmetric spaces, *J. Diff. Geom.* 74 (2006), 395–405.
132. —, The border rank of the multiplication of 2×2 matrices is seven, *J. Amer. Math. Soc.* 19 (2006), no. 2, 447–459.
133. —, *Tensors: geometry and applications*, American Mathematical Society, Providence, RI, 2012.
134. J.M. Landsberg, L. Manivel, The projective geometry of Freudenthal's magic square, *J. Algebra* 239 (2001), 477–512.
135. —, Construction and classification of complex simple Lie algebras via projective geometry, *Selecta Math. (N.S.)* 8 (2002), 137–159.
136. —, On the projective geometry of homogeneous varieties, *Commentari Math. Helv.* 78 (2003), 65–100.
137. J.M. Landsberg, C. Robles, Fubini-Griffiths-Harris rigidity and Lie algebra cohomology, *Asian J. Math.* 16 (2012), 561–586.
138. H.B. Lawson, M. Michelsohn, *Spin geometry*, Princeton University Press, 1989.
139. R. Lazarsfeld, *Positivity in algebraic geometry*, Volume I, Springer, 2004.
140. Claude R. LeBrun, Ambi-twistors and Einstein's equations, *Classical and Quantum Gravity* 2 (1985), no. 4, 555–563.
141. Claude LeBrun and Simon Salamon, Strong rigidity of positive quaternion-Kähler manifolds, *Invent. Math.* 118 (1994), no. 1, 109–132.
142. M. van Leeuwen, LiE, a computer algebra package, available at `http://young.sp2mi.univ-poitiers.fr/~marc/LiE/`.
143. S. L'vovsky, On Landsberg's criterion for complete intersections, *Manuscripta Math.* 88 (1995), 185–189.

144. B. Malgrange, L'involutivité générique des systèmes différentiels analytiques, *C. R. Acad. Sci. Paris, Ser. I* 326 (1998), 863–866.
145. R. Mayer, Coupled contact systems and rigidity of maximal dimensional variations of Hodge structure, *Trans. AMS* 352 (2000), 2121–2144.
146. R. McLachlan, A Gallery of Constant-Negative-Curvature Surfaces, *Math. Intelligencer* 16 (1994), 31–37.
147. M. Melko, I. Sterling, Integrable systems, harmonic maps and the classical theory of surfaces, pp. 129–144 in *Harmonic Maps and Integrable Systems*, Vieweg, 1994.
148. J. Milnor, *Topology from the differentiable viewpoint*, U. Virginia Press, 1965.
149. —, *Morse theory*, Princeton University Press, 1963.
150. J. Milnor, J.D. Stasheff, *Characteristic classes*, Princeton University Press, 1974.
151. D. Mumford, *Algebraic Geometry I: Complex projective varieties*, Springer, 1976.
152. R. Muñoz, Varieties with degenerate dual variety, *Forum Math.* 13 (2001), 757–779.
153. I. Nakai, Web geometry and the equivalence problem of the first order partial differential equations, pp. 150–204 in *Web theory and related topics*, World Scientific, 2001.
154. Z. Nie, On characteristic integrals of Toda field theories, *J. Nonlinear Math. Phys.* 21 (2014), 120–131.
155. T. Ochiai, Geometry associated with semisimple flat homogeneous spaces, *Trans. AMS* 152 (1970), 159–193.
156. C. Okonek, M. Schneider, H. Spindler, *Vector bundles on complex projective spaces*, Progress in Mathematics 3, Birkhauser, 1980.
157. P. Olver, *Applications of Lie Groups to Differential Equations* (2nd ed.), Springer, 1993.
158. —, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, 1995.
159. G. Pappas, J. Lygeros, D. Tilbury, S. Sastry, Exterior differential systems in control and robotics, pp. 271–372 in *Essays on mathematical robotics*, Springer, 1998.
160. J. Piontkowski, Developable varieties of Gauss rank 2, *Internat. J. Math.* 13 (2002), 93–110.
161. Y. S. Poon, S. M. Salamon, Quaternionic Kähler 8-manifolds with positive scalar curvature, *Journal of Differential Geometry* 33 (1991), 363–378.
162. Z. Ran, On projective varieties of codimension 2 *Invent. Math.* 73 (1983), 333–336.
163. —, The (dimension $+2$)-secant lemma, *Invent. Math.* 106 (1991), no. 1, 65–71.
164. C. Rogers, Bäcklund transformations in soliton theory, pp. 97–130 in *Soliton theory: a survey of results* (A. P. Fordy, ed.), St. Martin's Press, 1990.
165. C. Rogers, W. Schief, *Bäcklund and Darboux Transformations*, Cambridge University Press, 2002.
166. C. Rogers, W. Shadwick, *Bäcklund Transformations and Their Applications*, Academic Press, 1982.
167. E. Sato, Uniform vector bundles on a projective space, *J. Math. Soc. Japan* 28 (1976), 123–132.
168. B. Segre, Bertini forms and Hessian matrices, *J. London Math. Soc.* 26 (1951), 164–176.
169. C. Segre, Preliminari di una teoria delle varietà luoghi di spazi, *Rend. Circ. Mat. Palermo* XXX (1910), 87–121.
170. W. Shadwick, The KdV Prolongation Algebra, *J. Math. Phys.* 21 (1980), 454–461.

171. R. Sharpe, *Differential geometry: Cartan's generalization of Klein's Erlangen program*, Springer, 1997.
172. G.F. Simmons, *Differential Equations, with Applications and Historical Notes*, McGraw-Hill, 1972.
173. M. Spivak, *Calculus on Manifolds*, Benjamin, 1965.
174. —, *A Comprehensive Introduction to Differential Geometry* (3rd ed.), Publish or Perish, 1999.
175. O. Stormark, *Lie's Structural Approach to PDE Systems*, Encyclopedia of Mathematics and its Applications, v. 80, Cambridge University Press, 2000.
176. V. Strassen, Relative bilinear complexity and matrix multiplication, J. Reine Angew. Math. 413 (1991), 127-180.
177. S. Sternberg, *Lectures on differential geometry*, Chelsea, 1983.
178. D.J. Struik, *Lectures on Classical Differential Geometry* (2nd ed.), Addison-Wesley, 1961; Dover, 1988.
179. A. Terracini, Alcune questioni sugli spazi tangenti e osculatori ad una varieta, I, II, III, *Atti della Societa dei Naturalisti e Matematici Torino* 49 (1914), 214–247.
180. E. Tevelev, Projectively dual varieties of homogeneous spaces, pp. 183–225 in *Surveys in geometry and number thoery: reports on contemporary Russian mathematics*, London Math. Soc. Lecture Notes vol. 338, Cambridge, 2007.
181. T.Y. Thomas, Announcement of a projective theory of affinely connected manifolds, *Proc. Nat. Acad. Sci.* 11 (1925), no. 11, 588–589.
182. P.J. Vassiliou, Darboux Integrability and Symmetry, *Trans. AMS* 353 (2001), 1705–1739.
183. M. Wadati, H. Sanuki, and K. Konno, Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws, *Progr. Theoret. Phys.* 53 (1975), 419–436.
184. E. Vessiot, Sur les équations aux dérivées partielles du second ordre intégrables par la methode de Darboux, *J. Math Pures Appl.* 18 (1939), 1–61 and 21 (1942), 1–66.
185. H. Wahlquist, F. Estabrook, Prolongation structures of nonlinear evolution equations, *J. Math. Phys.* 16 (1975), 1–7.
186. —, Prolongation structures, connection theory and Bäcklund transformations, pp. 64–83 in *Nonlinear evolution equations solvable by the spectral transform*, Res. Notes in Math., vol. 26, Pitman, 1978.
187. F. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer, 1983.
188. K. Yamaguchi, Differential Systems Associated with Simple Graded Lie Algebras, pp. 413–494 in *Progress in Differential Geometry*, Adv. Stud. Pure Math., vol 22., Math. Soc. Japan, 1993.
189. D. Yang, Involutive hyperbolic differential systems, *A.M.S. Memoirs* # 370 (1987).
190. K. Yang, *Exterior differential systems and equivalence problems*, Kluwer, 1992.
191. F. Zak, *Tangents and Secants of Algebraic Varieties*, Translations of mathematical monographs vol. 127, AMS, 1993.
192. M.Y. Zvyagin, Second order equations reducible to $z_{xy} = 0$ by a Bäcklund transformation, *Soviet Math. Dokl.* 43 (1991) 30–34.

Index

SELECTED PUBLISHED TITLES IN THIS SERIES

175 **Thomas A. Ivey and Joseph M. Landsberg,** Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems, Second Edition, 2016
174 **Alexander Kirillov Jr.,** Quiver Representations and Quiver Varieties, 2016
173 **Lan Wen,** Differentiable Dynamical Systems, 2016
172 **Jinho Baik, Percy Deift, and Toufic Suidan,** Combinatorics and Random Matrix Theory, 2016
171 **Qing Han,** Nonlinear Elliptic Equations of the Second Order, 2016
170 **Donald Yau,** Colored Operads, 2016
169 **András Vasy,** Partial Differential Equations, 2015
168 **Michael Aizenman and Simone Warzel,** Random Operators, 2015
167 **John C. Neu,** Singular Perturbation in the Physical Sciences, 2015
166 **Alberto Torchinsky,** Problems in Real and Functional Analysis, 2015
165 **Joseph J. Rotman,** Advanced Modern Algebra: Third Edition, Part 1, 2015
164 **Terence Tao,** Expansion in Finite Simple Groups of Lie Type, 2015
163 **Gérald Tenenbaum,** Introduction to Analytic and Probabilistic Number Theory, Third Edition, 2015
162 **Firas Rassoul-Agha and Timo Seppäläinen,** A Course on Large Deviations with an Introduction to Gibbs Measures, 2015
161 **Diane Maclagan and Bernd Sturmfels,** Introduction to Tropical Geometry, 2015
160 **Marius Overholt,** A Course in Analytic Number Theory, 2014
159 **John R. Faulkner,** The Role of Nonassociative Algebra in Projective Geometry, 2014
158 **Fritz Colonius and Wolfgang Kliemann,** Dynamical Systems and Linear Algebra, 2014
157 **Gerald Teschl,** Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators, Second Edition, 2014
156 **Markus Haase,** Functional Analysis, 2014
155 **Emmanuel Kowalski,** An Introduction to the Representation Theory of Groups, 2014
154 **Wilhelm Schlag,** A Course in Complex Analysis and Riemann Surfaces, 2014
153 **Terence Tao,** Hilbert's Fifth Problem and Related Topics, 2014
152 **Gábor Székelyhidi,** An Introduction to Extremal Kähler Metrics, 2014
151 **Jennifer Schultens,** Introduction to 3-Manifolds, 2014
150 **Joe Diestel and Angela Spalsbury,** The Joys of Haar Measure, 2013
149 **Daniel W. Stroock,** Mathematics of Probability, 2013
148 **Luis Barreira and Yakov Pesin,** Introduction to Smooth Ergodic Theory, 2013
147 **Xingzhi Zhan,** Matrix Theory, 2013
146 **Aaron N. Siegel,** Combinatorial Game Theory, 2013
145 **Charles A. Weibel,** The K-book, 2013
144 **Shun-Jen Cheng and Weiqiang Wang,** Dualities and Representations of Lie Superalgebras, 2012
143 **Alberto Bressan,** Lecture Notes on Functional Analysis, 2013
142 **Terence Tao,** Higher Order Fourier Analysis, 2012
141 **John B. Conway,** A Course in Abstract Analysis, 2012
140 **Gerald Teschl,** Ordinary Differential Equations and Dynamical Systems, 2012
139 **John B. Walsh,** Knowing the Odds, 2012
138 **Maciej Zworski,** Semiclassical Analysis, 2012
137 **Luis Barreira and Claudia Valls,** Ordinary Differential Equations, 2012

Two central aspects of Cartan's approach to differential geometry are the theory of exterior differential systems (EDS) and the method of moving frames. This book presents thorough and modern treatments of both subjects, including their applications to both classic and contemporary problems in geometry. It begins with the classical differential geometry of surfaces and basic Riemannian geometry in the language of moving frames, along with an elementary introduction to exterior differential systems. Key concepts are developed incrementally, with motivating examples leading to definitions, theorems, and proofs.

Once the basics of the methods are established, the authors develop applications and advanced topics. One notable application is to complex algebraic geometry, where they expand and update important results from projective differential geometry. As well, the book features an introduction to G-structures and a treatment of the theory of connections. The techniques of EDS are also applied to obtain explicit solutions of PDEs via Darboux's method, the method of characteristics, and Cartan's method of equivalence.

This text is suitable for a one-year graduate course in differential geometry, and parts of it can be used for a one-semester course. It has numerous exercises and examples throughout. It will also be useful to experts in areas such as geometry of PDE systems and complex algebraic geometry who want to learn how moving frames and exterior differential systems apply to their fields.

The second edition features three new chapters: on Riemannian geometry, emphasizing the use of representation theory; on the latest developments in the study of Darboux-integrable systems; and on conformal geometry, written in a manner to introduce readers to the related parabolic geometry perspective.

ISBN 978-1-4704-0986-9

9 781470 409869

GSM/175

For additional information and updates on this book, visit

www.ams.org/bookpages/gsm-175

www.ams.org