Differential Galois Theory through Riemann-Hilbert Correspondence
An Elementary Introduction
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Jacques Sauloy
This book is dedicated to the Department of Mathematics of Wuda, in particular to its director Chen Hua, and to Wenyi Chen for giving me the occasion to teach this course and Jean-Pierre Ramis for giving me the ability to do so.

The math buildings at Wuda (2012) and at Toulouse University (2013)
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Nowadays differential Galois theory is a topic appearing more and more in graduate courses. There are several reasons. It mixes fundamental objects from very different areas of mathematics and uses several interesting theories. Moreover, during the last decades, mathematicians have discovered a lot of important and powerful applications of differential Galois theory. Some are quite surprising, such as some criteria for the classical problem of integrability of dynamical systems in mechanics and physics.

If differential Galois theory is a beautiful subject for a learning mathematician, it is not an easy one. There are several reference books on this topic but they are too difficult for a beginner. Jacques Sauloy’s book is a wonderful success because even if it is accessible to a graduate (or a motivated undergraduate) student, it nevertheless introduces sophisticated tools in a remarkably accessible way and gives an excellent and self-contained introduction to the modern developments of the subject. Until now, I thought that such a pedagogical presentation was impossible and I envy the young beginners for having the possibility to learn the subject in such a fantastic book.

When the French mathematician Émile Picard created differential Galois theory at the end of the nineteenth century he started from some analogies between the roots of an algebraic equation and the solutions of an algebraic linear differential equation in the complex domain, which, in general, are many-valued analytic functions. Picard built on preceding works, in particular on the work of B. Riemann on the famous hypergeometric functions of Euler and Gauss, introducing the notion of monodromy representation. After Picard and his follower Vessiot, differential Galois theory was successfully algebraized by Ellis Kolchin. Today the reference books use the purely
algebraic approach of Kolchin. Jacques Sauloy returns to the origin, working only in the complex domain and making essential use of transcendental tools as the monodromy representation and the Riemann-Hilbert correspondence. He does not at all avoid the use of algebra and of modern tools such as categories, functors, sheaves, algebraic groups, etc., but, sewing a transcendental flesh on the algebraic bones, he throws a very interesting light on the algebra. Moreover his point of view allows for a rather elementary presentation of the subject with concrete approaches and a lot of interesting examples.

The last chapter of the book (more difficult for a beginner) gives several directions where one could extend the topics presented before and gives a flavor of the richness of the theory.

Jean-Pierre Ramis, member of the French Academy of Sciences
Preface

In 2012, the University of Wuhan (nicknamed Wuda by its friends) asked me to give a course on differential Galois theory. Books have been published on the subject and courses have been given, all based on differential algebra and Picard-Vessiot theory and always with a very algebraic flavor (or sometimes even computer-algebra oriented). I did not feel competent to offer anything new in this direction. However, as a student of Jean-Pierre Ramis, I have been deeply influenced by his point of view that tagging algebraic objects with transcendental information enriches our understanding and brings not only new points of view but also new solutions. This certainly was illustrated by Ramis’ solution to the inverse problem in differential Galois theory! Note that although there has been an overwhelming trend to algebraization during the twentieth century, the opposite tendency never died, whether in algebraic geometry or in number theory. There still are function theoretic methods in some active areas of Langlands’ program! And pedagogically, I found the functional approach a nice shortcut to Tannaka theory.

On a more personal level, I remember my first advisor, Jean Giraud, an algebraist if there ever was one (he worked on the resolution of singularities in positive characteristics), sending me the beautiful 1976 article of Griffith “Variations on a theorem of Abel”, which, if my memory doesn’t fail me, conveyed a similar message. I explain better in the introduction why it could be a sensible choice to organize my course around such ideas.

The important point is that Chen Hua (the director of the math department at Wuda) and Wenyi Chen (who organized my stay there) accepted my proposition, and that the course took existence not only on paper. To Chen
Hua and Wenyi Chen my greatest thanks for that, and also to the whole math department of Wuda, in particular to the students who attended the course. There is no greater thrill than to teach to Chinese students. When I came back to France, I got the opportunity to teach the same course to master students. The background of French students is somewhat different from that of Chinese students, so this required some adaptation: I insisted less on complex function theory, that these students knew well (while my very young Chinese students had not yet studied it), and I used the time thus saved by giving more abstract algebraic formalism (for which French students have a strong taste).

In Wuda, the task of leading the exercise sessions with students was separated from the course proper. Luo Zhuangchu, a colleague and friend, took it in charge as well as the organization of the final exam. Xiexie Luo!

When my 2012 course in Wuhan ended, there was a two-week conference on differential equations, one of whose organizers was Changgui Zhang, who first had introduced me to the colleagues at Wuda. As a friend, as a coauthor and for having brought me first to China, I thank Changgui very much.

That conference was attended (as a speaker) by Michael Singer, also a friend and a colleague. Michael had an opportunity to see my course and found it interesting enough to suggest publication by the AMS. He put me in touch with Ina Mette of the AMS, who also encouraged me. So thanks to them both for giving me the motivation to transform my set of notes into a book. Ina Mette, along with another AMS staff member, Marcia C. Almeida, helped me all along the production process, for which I am grateful to them both. My thanks extend to production editor Mike Saitas, who handled the final publication steps. The drawings were first done by hand by my son Louis, then redrawn to professional standards by Stephen Gossman, one of the wizards of the AMS production team led by Mary Letourneau. Thanks to all!

Last, my dearest thanks go to Jean-Pierre Ramis who, to a great extent, forged my understanding of mathematics.
Introduction

The course will involve only complex analytic linear differential equations. Thus, it will not be based on the general algebraic formalism of differential algebra, but rather on complex function theory.

There are two main approaches to differential Galois theory. The first one, usually called Picard-Vessiot theory, and mainly developed by Kolchin, is in some sense a transposition of the Galois theory of algebraic equations in the form it was given by the German algebraists: to a (linear) differential equation, one attaches an extension of differential fields and one defines the Galois group of the equation as the group of automorphisms compatible with the differential structure. This group is automatically endowed with a structure of an algebraic group, and one must take in account that structure to get information on the differential equation. This approach has been extensively developed, it has given rise to computational tools (efficient algorithms and software) and it is well documented in a huge literature.

A more recent approach is based on so-called “tannakian duality”. It is very powerful and can be extended to situations where the Picard-Vessiot approach is not easily extended (like $q$-difference Galois theory). There is less literature and it has a reputation of being very abstract. However, in some sense, the tannakian approach can be understood as an algebraic transposition of the Riemann-Hilbert correspondence. In this way, it is rooted in very concrete and down-to-earth processes: the analytic continuation of power series solutions obtained by the Cauchy theorem and the ambiguity introduced by the multivaluedness of the solutions. This is expressed by the monodromy group, a precursor of the differential Galois group, and by
the monodromy representation. The Riemann-Hilbert correspondence is the other big galoisian theory of the nineteenth century, and it is likely that Picard had it in mind when he started to create differential Galois theory. The moral of this seems to be that, as understood and/or emphasized by such masters as Galois, Riemann, Birkhoff, etc., ambiguities give rise to the action of groups and the objects subject to these ambiguities are governed and can be classified by representations of groups.

Therefore, I intend to devote the first two parts\footnote{The first audience of this course consisted of young enthusiastic and gifted Chinese students who had been especially prepared, except for complex function theory. Therefore I added a “crash course” on analytic functions at the beginning, which the reader may skip, although it also serves as an introduction to the more global aspects of the theory which are seldom taught at an elementary level.} of the course to the study of the monodromy theory of complex analytic linear differential equations and of the Riemann-Hilbert correspondence, which is, anyhow, a must for anyone who wants to work with complex differential equations. In the third part of the course, I introduce (almost from scratch) the basic tools required for using algebraic groups in differential Galois theory, whatever the approach (Picard-Vessiot or tannakian). Last, I shall show how to attach algebraic groups and their representations to complex analytic linear differential equations. Some algebraic and functorial formalism is explained when needed, without any attempt at a systematic presentation.

The course is centered on the local analytic setting and restricted to the case of regular singular (or fuchsian) equations. One can consider it as a first-semester course. To define a second-semester course following this one would depend even more on one’s personal tastes; I indicate some possibilities in Chapter\footnote{The first audience of this course consisted of young enthusiastic and gifted Chinese students who had been especially prepared, except for complex function theory. Therefore I added a “crash course” on analytic functions at the beginning, which the reader may skip, although it also serves as an introduction to the more global aspects of the theory which are seldom taught at an elementary level.} which gives some hints of what lies beyond, with sufficient bibliographical references. In the appendices that follow, I give some algebraic complements to facilitate the reader’s work and to avoid having him or her delve into the literature without necessity. Also, the course having been taught twice before a class has given rise to two written examinations and some oral complementary tests. These are reproduced in the last appendix.

Prerequisites. The main prerequisites are: linear algebra (mostly reduction of matrices); elementary knowledge of groups and of polynomials in many variables; elementary calculus in \( n \) variables, including topology of the real euclidean spaces \( \mathbb{R}^n \). Each time a more advanced result will be needed, it will be precisely stated and explained and an easily accessible reference will be given.
Exercises. Two kinds of exercises are presented: some are inserted in the main text and serve as an illustration (e.g., examples, counterexamples, explicit calculations, etc). Some come at the end of the chapters and may be either application exercises or deepening or extensions of the main text. Some solutions or hints will be posted on [www.ams.org/bookpages/gsm-177](http://www.ams.org/bookpages/gsm-177).

Errata. I cannot hope to have corrected all the typographical and more substantial errors that appeared in the long process of making this book. A list of errata will be maintained on the same webpage mentioned above.

Notational conventions. Notation $A \colonequals B$ means that the term $A$ is defined by formula $B$. New terminology is written in emphatic style when first defined. Note that a definition can appear in the course of a theorem, an example, an exercise, etc.

Example 0.1. The monodromy group of $\mathcal{F}$ at the base point $a$ is the image of the monodromy representation:

$$\text{Mon}(\mathcal{F}, a) := \text{Im} \rho_{\mathcal{F},a} \subset \text{GL}(\mathcal{F}_a).$$

We mark the end of a proof, or its absence, by the symbol $\Box$.

We use commutative diagrams. For instance, to say that the diagram

$$\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{F}'(U) \\
\rho'_V & \downarrow & \rho'_V \\
\mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{F}'(V)
\end{array}$$

is commutative means that $\phi_V \circ \rho'_V = \rho'_V \circ \phi_U$.

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General notation.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mat$<em>n$(C), Mat$</em>{p,n}$(C)</td>
<td>Spaces of matrices</td>
</tr>
<tr>
<td>GL$_n$(C)</td>
<td>Group of invertible matrices</td>
</tr>
<tr>
<td>$\overline{D}(a,R)$, $\partial \overline{D}(a,R)$</td>
<td>Closed and open disk</td>
</tr>
<tr>
<td>$\partial D(a,R)$, $C(r,R)$</td>
<td>Circle, open annulus $r &lt;</td>
</tr>
<tr>
<td>Sp(A)</td>
<td>Spectrum$^2$ of a matrix</td>
</tr>
<tr>
<td>$n \gg -\infty$</td>
<td>“For $n$ big enough”</td>
</tr>
</tbody>
</table>

$^2$Sometimes, the spectrum is seen as a plain set and, for instance, writing $0_n$ as the null $n \times n$ matrix, $\text{Sp} 0_n = \{0\}$. Sometimes, it is seen as a multiset (elements have multiplicities) and then $\text{Sp} 0_n = \{0, \ldots, 0\}$ (counted $n$ times).
\[ \ln, e^x \] \hspace{1em} \text{Usual logarithm and exponential}
\[ \text{Diag}(a_1, \ldots, a_n) \] \hspace{1em} \text{Diagonal matrix}
\[ \mu_n \subset \mathbb{C}^* \] \hspace{1em} \text{Group of } n\text{th roots of unity}
\[ \dim_{\mathbb{C}} \] \hspace{1em} \text{Dimension of a complex space}
\[ \overline{D} \] \hspace{1em} \text{Closure of a set}
\[ N_m \] \hspace{1em} \text{Elementary nilpotent Jordan block}
\[ \langle x_1, \ldots, x_r \rangle \] \hspace{1em} \text{Group generated by } x_1, \ldots, x_r
\[ [A, B] \] \hspace{1em} \text{Commutator } AB - BA \text{ of matrices}

**Specific notation.** They are more competely defined in the text.

**Appearing in Chapter 1**
\[ e^z, \exp(z) \] \hspace{1em} \text{Complex exponential function}
\[ JF(x, y) \] \hspace{1em} \text{Jacobian matrix}
\[ I(a, \gamma) \] \hspace{1em} \text{Index of a loop relative to a point}
\[ e^A, \exp(A) \] \hspace{1em} \text{Exponential of a square matrix}

**Appearing in Chapter 2**
\[ \mathbb{C}[[z]] \] \hspace{1em} \text{Ring of formal power series}
\[ v_0(f) \] \hspace{1em} \text{Valuation, or order, of a power series}
\[ \mathbb{C}((z)) \] \hspace{1em} \text{Field of formal Laurent series}
\[ \mathcal{O}_0 = \mathbb{C}\{z\} \] \hspace{1em} \text{Ring of convergent power series, holomorphic germs at } 0

**Appearing in Chapter 3**
\[ \mathcal{O}_a = \mathbb{C}\{z - a\} \] \hspace{1em} \text{Ring of convergent power series in } z - a, \text{ holomorphic germs at } a
\[ \mathcal{O}(\Omega) \] \hspace{1em} \text{Ring of holomorphic functions over } \Omega
\[ B_n \] \hspace{1em} \text{Bernoulli numbers}
\[ v_{z_0}(f) \] \hspace{1em} \text{Valuation, or order, at } z_0
\[ \mathcal{M}(\Omega) \] \hspace{1em} \text{Field of meromorphic functions over a domain } \Omega
\[ I(z_0, \gamma) \] \hspace{1em} \text{Index of a loop around a point}

**Appearing in Chapter 4**
\[ \log \] \hspace{1em} \text{Complex logarithm (principal determination)}
\[ M_n \] \hspace{1em} \text{Nilpotent component of a matrix}
\[ M_s \] \hspace{1em} \text{Semi-simple component of a matrix}
\[ M_u \] \hspace{1em} \text{Unipotent component of an invertible matrix}
Appearing in Chapter 5

\( f^\gamma \) .......................... Result of analytic continuation of \( f \) along \( \gamma \)

\( \mathcal{O}_a \) ................................. Ring of germs of continuable functions

\( \gamma_1 \cdot \gamma_2 \) .......................... Composition of two paths

\( \gamma_1 \sim \gamma_2 \) .......................... Homotopy relation among paths (or loops)

\([\gamma]\) ................................. Homotopy class of a path (or loop)

\( \Pi_1(\Omega; a, b) \) .......................... Set of homotopy classes

\( \text{Iso}_{C-algdiff} \) .......................... Set of differential algebra isomorphisms

\([\gamma_1],[\gamma_2]\) .......................... Composition of two homotopy classes

\( \text{Aut}_{C-algdiff} \) .......................... Set of differential algebra automorphisms

\( \pi_1(\Omega; a) \) .......................... Fundamental group

\( \text{Sol}(z^{-1}A, \Omega) \) .......................... Space of solutions

\( z^A \) ........................................ \( \exp(A \log z) \)

Appearing in Chapter 6

\( z^\alpha \) ........................................ \( \exp(\alpha \log z) \)

\( H_0 \) ........................................... Right half-plane

\( H_{\theta_0} \) .................................. Half-plane with mediatrix at angle \( \theta_0 \)

\( IC \) ........................................... Initial condition map

\( f|_V \) ........................................ Restriction of function \( f \) to subset \( V \)

Appearing in Chapter 7

\( S \) ............................................. Riemann sphere

\( \mathcal{O}(\Omega), \mathcal{M}(\Omega), \mathcal{O}(S), \mathcal{M}(S) \) As in Chapter 3 but on the Riemann sphere

\( E_a, F_a \) . Scalar differential equation with coefficients \( a_i \) and its sheaf of solutions

\( S_A, F_A \) . Vectorial differential equation with matrix \( A \) and its sheaf of solutions

\( A_a \) .......................... Vectorialization matrix for \( E_a \) (companion matrix)

\( X_f \) .................................. Vectorialization of unknown function \( f \)

\( W_n, w_n \) .......................... Wronskian matrix and determinant

\( \rho^U, F_a \) .......................... Restriction map and stalk (or fiber) for a sheaf

\( \text{Iso}(F_a, F_b) \) .......................... Set of linear isomorphisms

\( G^\circ \) ...................................... Opposite group

\( \rho_{F,a}, \text{Mon}(F, a) \) ........ Monodromy representation and group for the local system \( F \)
\(\rho_{a,z_0}, \text{Mon}(E_a,z_0)\) Monodromy representation and group for the scalar equation \(E_a\)

\(\rho_{A,z_0}, \text{Mon}(S_A,z_0)\) ....... Monodromy representation and group for the vectorial equation \(S_A\)

\(F[A]\) Transform of \(A\) by the gauge matrix \(F\)

\(\chi_\lambda, M_\lambda\) Analytic continuation of a fundamental matricial solution along the loop \(\lambda\) and corresponding monodromy matrix

\(\sim_h, \sim_m\) Holomorphic and meromorphic equivalence  

**Appearing in Chapter** \([8]\)

\(\mathcal{D}\mathcal{S}\) Category of differential systems

\(\mathcal{L}\mathcal{S}\) Category of local systems

\(\mathcal{E}\mathcal{H}\) Category of sheaves

\(\text{Ob}(\mathcal{C}), \text{Mor}_\mathcal{C}(X,Y), \text{Id}_X\) Class of objects, sets of morphisms, identity morphisms in a category

\(\mathcal{F}\) Constant sheaf

\(\rightarrow\) Funny arrow reserved for functors

\(\mathcal{R}\text{Rep}_\mathcal{C}(G), \mathcal{R}\text{Rep}_\mathcal{C}^f(G)\) Category of complex representations, of finite-dimensional complex representations, of \(G\)

**Appearing in Chapter** \([9]\)

\(D, \delta\) Usual and Euler differential operators

\((\alpha)_n\) Pochhammer symbols

\(F(\alpha, \beta, \gamma; z)\) Hypergeometric series

**Appearing in Chapter** \([10]\)

\(\mathcal{E}^{(0)}, \mathcal{E}_f^{(0)}, \mathcal{E}_f^{(0)}_{R}, \mathcal{E}_f^{(0)}_{\infty}\) Various categories of differential systems

\(\mathcal{M}_0 = \mathcal{C}(\{z\})\) Meromorphic germs at 0

**Appearing in Chapter** \([11]\)

\(\mathcal{B}^{\lambda}\) Analytic continuation of a basis of solutions

\(HG_{\alpha,\beta,\gamma}, HG'_{\alpha,\beta,\gamma}\) Hypergeometric equation using \(\delta\) or \(D\)

**Appearing in Chapter** \([12]\)

RS Regular singular (abbreviation used only in this chapter)

**Appearing in Chapter** \([13]\)

\(K\) Differential field, usually \(\mathcal{M}_0\)

\(\mathcal{A}(A,z_0) = K[\mathcal{X}]\) Differential, algebra generated by solutions

\(\text{Gal}(A,z_0)\) Differential Galois group computed at \(z_0\)
Index of notation

Appearing in Chapter 14

\[
\text{Mon}(A) \subset \text{Gal}(A) \subset \text{GL}_n(C) . \quad \text{Matricial realizations of the monodromy and Galois group}
\]

\[V(E) \quad \text{Algebraic subset defined by a set of equations}\]

Appearing in Chapter 15

\[\hat{\rho}_A, \hat{\pi}_1 \quad \text{Universal representation and group}\]

\[\mathcal{X}(H) \quad \text{Proalgebraic group of characters}\]

\[\iota : \pi_1 \mapsto \hat{\pi}_1 \quad \text{Canonical inclusion}\]

Appearing in Chapter 16

\[\hat{\pi}_{1,s}, \hat{\pi}_{1,u} \quad \text{Semi-simple and unipotent components of the universal group}\]

\[\mathcal{A}(F) \quad \text{Affine algebra of } F\]

\[\Gamma_\Sigma \quad \text{Group generated by } \Sigma \subset C^*\]

\[\text{lim} \quad \text{Inverse limit}\]

\[\sqrt{I_0} \quad \text{Radical of an ideal}\]

\[[\Gamma, \Gamma] \quad \text{Derived subgroup}\]

\[\Gamma^{ab} \quad \text{Abelianization of a group}\]

Appearing in Appendix A

\[M^x \quad \text{Complex powers of matrix } M\]

Appearing in Appendix B

\[\log A \quad \text{Logarithm of a matrix}\]

\[R(z, A) \quad \text{Resolvent}\]

Appearing in Appendix C

\[\phi_s, \phi_n \quad \text{Semi-simple and nilpotent components of an endomorphism}\]

\[M_s, M_u \quad \text{Semi-simple and unipotent components of an invertible matrix}\]

Appearing in Appendix D

\[G^{alg}, Z^{alg} \quad \text{Proalgebraic hull of } G, \text{ of } Z\]

\[\text{Gal}(\mathcal{C}, \omega) := \text{Aut}^{\otimes}(\omega) \quad \text{Galois group of a tannakian category}\]

\[\iota : G \to G^{alg} \quad \text{Canonical inclusion}\]

Appearing in Appendix E

\[\mathbf{X}(G) \quad \text{Group of rational characters of an algebraic group}\]
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Differential Galois theory is an important, fast developing area which appears more and more in graduate courses since it mixes fundamental objects from many different areas of mathematics in a stimulating context. For a long time, the dominant approach, usually called Picard-Vessiot Theory, was purely algebraic. This approach has been extensively developed and is well covered in the literature. An alternative approach consists in tagging algebraic objects with transcendental information which enriches the understanding and brings not only new points of view but also new solutions. It is very powerful and can be applied in situations where the Picard-Vessiot approach is not easily extended. This book offers a hands-on transcendental approach to differential Galois theory, based on the Riemann-Hilbert correspondence. Along the way, it provides a smooth, down-to-earth introduction to algebraic geometry, category theory and tannakian duality.

Since the book studies only complex analytic linear differential equations, the main prerequisites are complex function theory, linear algebra, and an elementary knowledge of groups and of polynomials in many variables. A large variety of examples, exercises, and theoretical constructions, often via explicit computations, offers first-year graduate students an accessible entry into this exciting area.