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Introduction to Algebraic Geometry

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To Hema, Ashok, and Maya

Contents

Preface

This book is an introductory course in algebraic geometry, proving most of the fundamental classical results of algebraic geometry.

Algebraic geometry combines the intuition of geometry with the precision of algebra. Starting with geometric concepts, we introduce machinery as necessary to model important ideas from algebraic geometry and to prove fundamental results. Emphasis is put on developing facility with connecting geometric and algebraic concepts. Examples are constructed or cited illustrating the scope of these results. The theory in this book is developed in increasing sophistication, giving (and refining) definitions as required to accommodate new geometric ideas.

We work as much as possible with quasi-projective varieties over an algebraically closed field of arbitrary characteristic. This allows us to interpret varieties through their function fields. This approach and the use of methods of algebraic number theory in algebraic geometry have been central to algebraic geometry at least since the time of Dedekind and Weber (*Theorie der algebraischen Functionen einer Veränderlichen* [**46**], translated in [**47**]). By interpreting the geometric concept of varieties through their regular functions, we are able to use the techniques of commutative algebra.

Differences between the theory in characteristic 0 and positive characteristic are emphasized in this book. We extend our view to schemes, allowing rings with nilpotents, to study fibers of regular maps and to develop intersection theory. We discuss the cases of nonclosed ground fields and nonseparated schemes and some of the extra considerations which appear in these situations. A list of exercises is given at the end of many sections and chapters.

The classic textbooks *Basic Algebraic Geometry* [**136**] by Shafarevich, *Introduction to Algebraic Geometry* [**116**] by Mumford, and *Algebraic Geometry* [**73**] by Hartshorne, as well as the works of Zariski, Abhyankar, Serre, and Grothendieck, have been major influences on this book.

The necessary commutative algebra is introduced and reviewed, beginning with Chapter 1, "A Crash Course in Commutative Algebra". We state definitions and theorems, explain concepts, and give examples from commutative algebra for everything that we will need, proving some results and giving a few examples, but mostly giving references to books on commutative algebra for proofs. As such, this book is intended to be self-contained, although a reader may be curious about the proofs for some cited results in commutative algebra and will want to either derive them or look up the references. We give references to several books, mostly depending on which book has the exact statement we require.

A reader should be familiar with the material through Section 1.6 on primary decomposition before beginning Chapter 2 on affine varieties. Depending on the background of students, the material in Chapter 1 can be skipped, quickly reviewed at the beginning of a course, or be used as an outline of a semester-long course in commutative algebra before beginning the study of geometry in Chapter 2.

Chapters 2–10 give a one-semester introduction to algebraic geometry, through affine and projective varieties. The sections on integral extensions, dimension, depth, and normal and regular local rings in Chapter 1 can be referred to as necessary as these concepts are encountered within a geometric context. Chapters 11–20 provide a second-semester course, which includes sheaves, schemes, cohomology, divisors, intersection theory, and the application of these concepts to curves and surfaces.

Chapter 21 (on ramification and étale maps) and Chapter 22 (on Bertini's theorems and general fibers of maps) could be the subject of a topics course for a third-semester course. The distinctions between characteristic 0 and positive characteristic are especially explored in these last chapters. These two chapters could be read any time after the completion of Chapter 14.

I thank the students of my classes, especially Razieh Ahmadian, Navaneeth Chenicheri Chathath, Suprajo Das, Arpan Dutta, Melissa Emory, Kyle Maddox, Smita Praharaj, Thomas Polstra, Soumya Sanyal, Pham An Vinh, and particularly Roberto Núñez, for their feedback and helpful comments on preliminary versions of this book. I thank Maya Cutkosky for help with the figures.

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Index

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