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Chapter 7

Semigroups of Operators

Strongly continuous semigroups play an important role in the study of many linear partial differential equations such as the heat equation, the wave equation, and the Schrödinger equation. The finite-dimensional model of a strongly continuous semigroup is the exponential matrix associated to a first order linear ordinary differential equation. The concept of the exponential operator carries over naturally to infinite-dimensional Banach spaces X and can be used to find a solution of the **Cauchy problem**

$$\dot{x} = Ax, \qquad x(0) = x_0$$

for every bounded linear operator $A \in \mathcal{L}(X)$ and every initial value $x_0 \in X$. The unique solution $x : \mathbb{R} \to X$ of this equation is given by

$$x(t) = e^{tA}x_0 = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x_0 \qquad \text{for } t \in \mathbb{R}.$$

(See Exercise 5.2.13.) The aforementioned partial differential equations can be expressed in the same form, however, with the caveat that the operator A is unbounded with a dense domain and that the solutions may only exist in forward time. In such situations it is convenient to use the solutions, rather than the equation, as the starting point. This leads to the notion of a strongly continuous semigroup, introduced in Section 7.1 along with several examples. That section also derives some of their basic properties and discusses the infinitesimal generator. The main result is the Hille–Yosida–Phillips Theorem in Section 7.2 which characterizes infinitesimal generators

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of strongly continuous semigroups. The dual semigroup is the subject of Section 7.3 and analytic semigroups are discussed in Section 7.4. A preparatory Section 7.5 is devoted to Banach space valued measurable functions, and inhomogeneous equations are examined in Section 7.6.

7.1. Strongly Continuous Semigroups

7.1.1. Definition and Examples. The existence and uniqueness theorem for solutions of a time-independent ordinary differential equation implies that the solutions define a flow. This means that the value of the solution with initial condition x_0 at time $s+t$ agrees with the value at time s of the solution whose initial condition is taken to be the value of the original solution at time t. For linear differential equations on Banach spaces this translates into a semigroup condition on the family of linear operators, parametrized by a nonnegative real variable t, that assign to a given initial condition the solution of the respective linear differential equation at time t. Continuous dependence on time translates into strong continuity of the semigroup of operators and continuous dependence on the initial condition translates into boundedness of the operators.

Definition 7.1.1 (Strongly Continuous Semigroup). Let X be a real Banach space. A **one-parameter semigroup (of operators on X)** is a map $S : [0,\infty) \to \mathcal{L}(X)$ that satisfies

$$S(0) = \mathbb{1}, \qquad S(s+t) = S(s)S(t) \tag{7.1.1}$$

for all $s, t \geq 0$. A **one-parameter group (of operators on X)** is a map $S : \mathbb{R} \to \mathcal{L}(X)$ that satisfies (7.1.1) for all $s, t \in \mathbb{R}$. A **strongly continuous semigroup (of operators on X)** is a map $S : [0,\infty) \to \mathcal{L}(X)$ that satisfies (7.1.1) for all $s, t \geq 0$ and satisfies

$$\lim_{t\to 0} \|S(t)x - x\| = 0 \tag{7.1.2}$$

for all $x \in X$. A **strongly continuous group (of operators on X)** is a map $S : \mathbb{R} \to \mathcal{L}(X)$ that satisfies (7.1.1) for all $s, t \in \mathbb{R}$ and satisfies (7.1.2) for all $x \in X$.

Example 7.1.2 (Groups Generated by Bounded Operators). Let X be a real Banach space and let $A : X \to X$ be a bounded linear operator. Then the operators

$$S(t) := e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \tag{7.1.3}$$

for $t \in \mathbb{R}$ form a strongly continuous group of operators on X. In this example the map $\mathbb{R} \to \mathcal{L}(X) : t \mapsto S(t)$ is continuous with respect to the norm topology on $\mathcal{L}(X)$ (see Exercise 5.2.13).

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Example 7.1.3 (Semigroups and Orthonormal Bases). Let H be a separable complex Hilbert space, let $(e_i)_{i\in\mathbb{N}}$ be a complex orthonormal basis, and let $(\lambda_i)_{i\in\mathbb{N}}$ be a sequence of complex numbers such that

$$\sup_{i\in\mathbb{N}} \mathrm{Re}\lambda_i < \infty.$$

Define the map $S : [0,\infty) \to \mathcal{L}^c(H)$ by

$$S(t)x := \sum_{i=1}^{\infty} e^{\lambda_i t}\langle e_i, x\rangle e_i \tag{7.1.4}$$

for $x \in H$ and $t \geq 0$. **Exercise:** Show that this is a strongly continuous semigroup of operators on H. Show that it extends to a strongly continuous group $S : \mathbb{R} \to \mathcal{L}^c(H)$ if and only if

$$\sup_{i\in\mathbb{N}} |\mathrm{Re}\lambda_i| < \infty.$$

Example 7.1.4 (Shift Semigroups). Fix a constant $1 \leq p < \infty$ and let $X = L^p([0,\infty))$ be the Banach space of real valued L^p-functions on $[0,\infty)$ with respect to the Lebesgue measure.

(i) Define the map $L : [0,\infty) \to \mathcal{L}(X)$ by

$$(L(t)f)(s) := f(s+t) \tag{7.1.5}$$

for $f \in L^p([0,\infty))$ and $s,t \geq 0$. **Exercise:** Show that this is a strongly continuous semigroup of operators. Show that this example extends to the space of continuous functions on $[0,\infty)$ that converge to zero at infinity. Show that strong continuity fails when $L^p([0,\infty))$ is replaced by $L^\infty([0,\infty))$ or by the space of bounded continuous real valued functions on $[0,\infty)$. Show that the formula (7.1.5) defines a group on $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

(ii) Define the map $R : [0,\infty) \to \mathcal{L}(X)$ by

$$(R(t)f)(s) := \begin{cases} 0, & \text{if } s < t, \\ f(s-t), & \text{if } s \geq t, \end{cases} \tag{7.1.6}$$

for $f \in L^p([0,\infty))$ and $s,t \geq 0$. **Exercise:** Show that this is a strongly continuous semigroup of isometric embeddings. Show that this example extends to the space of continuous functions $f : [0,\infty) \to \mathbb{R}$ that vanish at the origin and converge to zero at infinity.

(iii) Define the map $S : [0,\infty) \to \mathcal{L}(L^p([0,1]))$ by

$$(S(t)f)(s) := \begin{cases} f(s+t), & \text{if } s+t \leq 1, \\ 0, & \text{if } s+t > 1, \end{cases} \tag{7.1.7}$$

for $f \in L^p([0,1])$, $s \in [0,1]$, and $t \geq 0$. **Exercise:** Show that this is a strongly continuous semigroup of operators such that $S(t) = 0$ for $t \geq 1$.

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Example 7.1.5 (Flows). Let (M, d) be a compact metric space and suppose that the map

$$\mathbb{R} \times M \to M : (t, p) \mapsto \phi_t(p)$$

is a **continuous flow**, i.e. it is continuous and satisfies

$$\phi_0 = \mathrm{id}, \qquad \phi_{s+t} = \phi_s \circ \phi_t$$

for all $s, t \in \mathbb{R}$. Let $X := C(M)$ be the Banach space of continuous real valued functions on M equipped with the supremum norm. Define

$$S(t)f := f \circ \phi_t \qquad \text{for } t \in \mathbb{R} \text{ and } f \in C(M). \tag{7.1.8}$$

Then $S : \mathbb{R} \to \mathcal{L}(C(M))$ is a strongly continuous group of operators.

Example 7.1.6 (Heat Equation). Fix a positive integer n and a real number $1 \le p < \infty$. Define the **heat kernel** $K_t : \mathbb{R}^n \to \mathbb{R}$ by

$$K_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \qquad \text{for } x \in \mathbb{R}^n \text{ and } t > 0. \tag{7.1.9}$$

Here $|x| := \sqrt{\sum_{i=1}^n x_i^2}$ denotes the Euclidean norm of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. These functions are nonnegative and Lebesgue integrable and satisfy

$$\int_{\mathbb{R}^n} K_t(\xi)\, d\xi = 1, \qquad K_{s+t} = K_s * K_t \tag{7.1.10}$$

for all $s, t > 0$, where $(f*g)(x) := \int_{\mathbb{R}^n} f(x-\xi)g(\xi)\, d\xi$ denotes the convolution of two Lebesgue integrable functions $f, g : \mathbb{R}^n \to \mathbb{R}$. Equation (7.1.10) implies that the operators $S(t) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, defined by

$$S(t)f := \begin{cases} K_t * f, & \text{for } t > 0, \\ f, & \text{for } t = 0, \end{cases} \tag{7.1.11}$$

define a semigroup of operators. Since $\lim_{t\to 0} \sup_{|x|\ge\delta} K_t(x) = 0$ for all $\delta > 0$ and $\int_{\mathbb{R}^n} K_t = 1$ for all $t > 0$, the functions $S(t)f = K_t * f$ converge uniformly to f for every continuous function $f : \mathbb{R}^n \to \mathbb{R}$ with compact support. The convergence is also in $L^p(\mathbb{R}^n)$. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ by [**75**, Thm. 4.15] and $\|S(t)\| \le 1$ for all $t \ge 0$ by Young's inequality, it follows from Theorem 2.1.5 that $\lim_{t\to 0} \|S(t)f - f\|_{L^p} = 0$ for all $f \in L^p(\mathbb{R}^n)$. Thus the semigroup (7.1.11) is strongly continuous. Moreover, for each $f \in L^p(\mathbb{R}^n)$, the function $u : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, defined by $u(t, x) := (K_t * f)(x)$ for $t > 0$ and $x \in \mathbb{R}^n$, is smooth and satisfies the **heat equation**

$$\frac{\partial u}{\partial t} = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}, \qquad \lim_{t\to 0} \int_{\mathbb{R}^n} |u(t, x) - f(x)|^p\, dx = 0. \tag{7.1.12}$$

Exercise: Fill in the details.

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Example 7.1.7 (Wave Equation). Let $L^2(\mathbb{R})$ be the space of square integrable real valued functions on $\mathbb{R}$ with respect to the Lebesgue measure, modulo equality almost everywhere, and let $W^{1,2}(\mathbb{R})$ denote the space of absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that f and f' are square integrable. Then $H := W^{1,2}(\mathbb{R}) \times L^2(\mathbb{R})$ is a Hilbert space with the norm

$$\|(f,g)\|_H := \sqrt{\int_{-\infty}^{\infty} \left(|f(x)|^2 + \left| \frac{df}{dx}(x) \right|^2 + |g(x)|^2 \right) dx}$$

for $f \in W^{1,2}(\mathbb{R})$ and $g \in L^2(\mathbb{R})$. Given a pair $(f,g) \in H$, define the function $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(t,x) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} g(s)\, ds \tag{7.1.13}$$

for $t, x \in \mathbb{R}$. Then $u(t,\cdot) \in W^{1,2}(\mathbb{R})$ and $\partial_t u(t,\cdot) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$, and the linear operators $S(t) : H \to H$ given by $S(t)(f,g) := (u(t,\cdot), \partial_t u(t,\cdot))$ for $(f,g) \in H$ and $t \in \mathbb{R}$ define a strongly continuous group of operators on H. If f and g are smooth, then the function (7.1.13) is the unique solution of the one-dimensional **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x). \tag{7.1.14}$$

The energy identity asserts that the function

$$E(t) := \frac{1}{2} \int_{-\infty}^{\infty} \left(\left| \frac{\partial u}{\partial x}(t,x) \right|^2 + \left| \frac{\partial u}{\partial t}(t,x) \right|^2 \right) dx$$

is constant for every solution of (7.1.14). Thus the operators $S(t) \in \mathcal{L}(H)$ extend to isometries of the completion $\mathcal{H}$ of H with respect to the norm

$$\|(f,g)\|_{\mathcal{H}} := \sqrt{\int_{-\infty}^{\infty} \left(\left| \frac{df}{dx}(x) \right|^2 + |g(x)|^2 \right) dx}.$$

The completion can be identified with the quotient of the space of all pairs (f,g), where $g \in L^2(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous with square integrable derivative, under the equivalence relation $(f_1, g_1) \sim (f_2, g_2)$ iff $g_1 = g_2$ and $f_1 - f_2$ is constant (Exercise 7.7.5). If one identifies $\mathcal{H}$ with $\mathscr{H} := L^2(\mathbb{R}, \mathbb{R}^2)$ via the isomorphism $\mathcal{H} \to \mathscr{H} : (f,g) \mapsto (f', g)$, one obtains the strongly continuous group $\mathscr{S} : \mathbb{R} \to \mathcal{L}(\mathscr{H})$ of isometries, given by $\mathscr{S}(t)(f,g) := (u(t,\cdot), v(t,\cdot))$ for $t \in \mathbb{R}$ and $f, g \in L^2(\mathbb{R})$, where

$$\begin{aligned} u(t,x) &:= \frac{f(x+t) + f(x-t)}{2} + \frac{g(x+t) - g(x-t)}{2}, \\ v(t,x) &:= \frac{f(x+t) - f(x-t)}{2} + \frac{g(x+t) + g(x-t)}{2}. \end{aligned} \tag{7.1.15}$$

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7.1.2. Basic Properties.

7.1.2. Basic Properties. The next two lemmas examine some of the elementary properties of strongly continuous semigroups.

Lemma 7.1.8. *Let X be a real Banach space and let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup. Then the following hold.*

(i) $\sup_{0 \le t \le T} \|S(t)\| < \infty$ *for all* $T > 0$.

(ii) *The function* $[0, \infty) \to X : t \mapsto S(t)x$ *is continuous for all* $x \in X$.

(iii) *The function* $t^{-1} \log \|S(t)\|$ *converges in* $\mathbb{R} \cup \{-\infty\}$ *as* t *tends to infinity and*

$$\lim_{t\to\infty} t^{-1} \log\|S(t)\| = \inf_{t>0} t^{-1} \log\|S(t)\| =: \omega_0. \tag{7.1.16}$$

(iv) *Let* ω_0 *be as in (iii) and fix a real number* $\omega > \omega_0$. *Then there exists a constant* $M \ge 1$ *such that*

$$\|S(t)\| \le Me^{\omega t} \qquad \text{for all } t \ge 0. \tag{7.1.17}$$

Proof. To prove (i) we show first that there exist constants $\delta > 0$ and $M \ge 1$ such that, for all $t \in \mathbb{R}$,

$$0 \le t \le \delta \qquad \Longrightarrow \qquad \|S(t)\| \le M. \tag{7.1.18}$$

Suppose by contradiction that there do not exist such constants. Then

$$\sup_{0\le t\le\delta} \|S(t)\| = \infty$$

for all $\delta > 0$. Hence there exists a sequence of real numbers $t_n > 0$ such that $\lim_{n\to\infty} t_n = 0$ and the sequence $\|S(t_n)\|$ is unbounded. By the Uniform Boundedness Theorem 2.1.1 this implies that there exists an element $x \in X$ such that the sequence $\|S(t_n)x\|$ is unbounded. This contradicts the fact that $\lim_{n\to\infty} \|S(t_n)x - x\| = 0$. Thus we have proved (7.1.18).

Now fix a number $T > 0$ and choose $N \in \mathbb{N}$ such that $N\delta > T$. Fix an element $t \in [0, T]$. Then there exists a unique integer $k \in \{0, 1, \dots, N-1\}$ such that $k\delta \le t < (k+1)\delta$ and hence, by (7.1.18),

$$\|S(t)\| = \|S(\delta)^k S(t - k\delta)\| \le \|S(\delta)\|^k \|S(t - k\delta)\| \le M^{k+1} \le M^N.$$

This proves part (i).

Part (ii) follows from part (i) and the inequalities

$$\|S(t+h)x - S(t)x\| \le \|S(t)\| \, \|S(h)x - x\|$$

and

$$\|S(t-h)x - S(t)x\| \le \|S(t-h)\| \, \|x - S(h)x\|$$

for $0 \le h \le t$.

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We prove part (iii). Equation (7.1.16) holds obviously with $\omega_0 = -\infty$ whenever $S(t) = 0$ for some $t > 0$. Hence assume $S(t) \neq 0$ for all $t > 0$. Then for every $t > 0$ there is a constant $c \geq 1$ such that $c^{-1} \leq \|S(s)\| \leq c$ for $0 \leq s \leq t$. Define the function $g : [0, \infty) \to \mathbb{R}$ by

$$g(t) := \log \|S(t)\| \qquad \text{for } t \geq 0.$$

Then it follows from the semigroup property and part (i) that

$$g(0) = 0, \qquad g(s+t) \leq g(s) + g(t), \qquad M(t) := \sup_{0 \leq s \leq t} |g(s)| < \infty$$

for all $s, t \geq 0$. Fix a real number $t_0 > 0$ and let $t > 0$ be any positive real number. Then there exists an integer $k \geq 0$ and a real number s such that

$$t = kt_0 + s, \qquad 0 \leq s < t_0.$$

Hence

$$\frac{g(t)}{t} \leq \frac{kg(t_0) + g(s)}{t} = \frac{g(t_0)}{t_0} - \frac{sg(t_0)}{t_0 t} + \frac{g(s)}{t} \leq \frac{g(t_0)}{t_0} + \frac{2M(t_0)}{t}.$$

This implies

$$\limsup_{t \to \infty} \frac{g(t)}{t} \leq \frac{g(t_0)}{t_0}.$$

Since this holds for all $t_0 > 0$, we have $\limsup_{t\to\infty} t^{-1}g(t) \leq \inf_{t>0} t^{-1}g(t)$ and this proves part (iii).

We prove part (iv). Fix a real number $\omega > \omega_0$. By part (iii) there exists a constant $T > 0$ such that

$$\frac{\log \|S(t)\|}{t} \leq \omega \qquad \text{for all } t \geq T.$$

Thus $\log \|S(t)\| \leq \omega t$ and so $\|S(t)\| \leq e^{\omega t}$ for all $t \geq T$. Define

$$M := \sup_{0 \leq t \leq T} \|S(t)\| \, e^{-\omega t}.$$

Then $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and this proves Lemma 7.1.8. □

Lemma 7.1.9. *Let X be a real Banach space and let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup. Then the following hold.*

(i) *The operator $S(t)$ is injective for some $t > 0$ if and only if it is injective for all $t > 0$.*

(ii) *The operator $S(t)$ is surjective for some $t > 0$ if and only if it is surjective for all $t > 0$.*

(iii) *The operator $S(t)$ has a dense image for some $t > 0$ if and only if it has a dense image for all $t > 0$.*

(iv) *Assume $S(t)$ is injective for all $t > 0$. Then $S(t)$ has a closed image for some $t > 0$ if and only if it has a closed image for all $t > 0$.*

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Proof. We prove part (i). Assume that there exists a real number $t_0 > 0$ such that $S(t_0)$ is injective. Let $t > 0$ and choose an integer $k > 0$ such that $kt_0 \geq t$. If $x \in X$ satisfies $S(t)x = 0$, then $S(t_0)^k x = S(kt_0 - t)S(t)x = 0$ and hence $x = 0$. Thus $S(t)$ is injective for all $t > 0$.

We prove part (ii). Assume that there exists a real number $t_0 > 0$ such that $S(t_0)$ is surjective. Let $t > 0$ and choose an integer $k > 0$ such that $kt_0 \geq t$. Then $S(kt_0) = S(t_0)^k$ is surjective and this implies that $\mathrm{im}(S(t)) \supset \mathrm{im}(S(t)S(kt_0 - t)) = \mathrm{im}(S(kt_0)) = X$. Thus $S(t)$ is surjective for all $t > 0$.

We prove part (iii). Assume that there exists a real number $t_0 > 0$ such that $S(t_0)$ has a dense image. Let $t > 0$ and choose an integer $k > 0$ such that $kt_0 \geq t$. Then the operator $S(kt_0) = S(t_0)^k$ has a dense image. Since $\mathrm{im}(S(t)) \supset \mathrm{im}(S(t)S(kt_0 - t)) = \mathrm{im}(S(kt_0))$ this implies that $S(t)$ has a dense image.

We prove part (iv). Thus assume $S(t)$ is injective for all $t > 0$ and that there exists a real number $t_0 > 0$ such that $S(t_0)$ has a closed image. Then it follows from part (ii) of Corollary 4.1.17 that there exists a constant $\delta > 0$ such that $\delta \|x\| \leq \|S(t_0)x\|$ for all $x \in X$. By induction this implies $\delta^k \|x\| \leq \|S(kt_0)x\|$ for all $x \in X$ and all $k \in \mathbb{N}$. Let $t > 0$ and choose an integer $k > 0$ such that $kt_0 \geq t$. Then

$$\|S(kt_0 - t)\| \, \|S(t)x\| \geq \|S(kt_0)x\| \geq \delta^k \|x\|$$

and so $\|S(t)x\| \geq \|S(kt_0 - t)\|^{-1} \delta^k \|x\|$ for all $x \in X$. Hence $S(t)$ has a closed image by Theorem 4.1.16 and this proves Lemma 7.1.9. □

Example 7.1.10. This example shows that the hypothesis that $S(t)$ is injective for all $t > 0$ cannot be removed in part (iv) of Lemma 7.1.9. Consider the real Banach space

$$X := \left\{ f \in \mathcal{L}^2([0,1]) \,\middle|\, f \text{ is continuous on } [0, \tfrac{1}{2}] \text{ and } f(0) = 0 \right\} \Big/ \sim .$$

Here the equivalence relation is defined by $f \sim g$ if and only if $f - g$ vanishes almost everywhere on the interval $[\frac{1}{2}, 1]$, and the norm is defined by

$$\|f\|_X := \sup_{0 \leq s \leq \frac{1}{2}} |f(s)| + \sqrt{\int_{\frac{1}{2}}^1 f(s)^2 \, ds}$$

for $f \in X$. Then the formula

$$(S(t)f)(s) := \begin{cases} f(s-t), & \text{if } s \geq t, \\ 0, & \text{if } s < t, \end{cases}$$

for $f \in X$, $t \geq 0$, and $0 \leq s \leq 1$ defines a strongly continuous semigroup on X. The operator $S(t)$ has a nontrivial kernel for all $t > 0$, does not have a closed image for $0 < t < 1$, and vanishes for all $t \geq 1$.

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7.1.3. The Infinitesimal Generator. The starting point of the present section was to introduce strongly continuous semigroups of operators as a generalization of the space of solutions of a linear differential equation. Given such a space of *"solutions"* it is then a natural question to ask whether there is actually a linear differential equation that a given strongly continuous semigroup provides the solutions of. The quest for such an equation leads to the following definition.

Definition 7.1.11 (**Infinitesimal Generator).** Let X be a real Banach space and let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup. The **infinitesimal generator of** S is the linear operator $A : \mathrm{dom}(A) \to X$, whose domain is the linear subspace $\mathrm{dom}(A) \subset X$ defined by

$$\mathrm{dom}(A) := \left\{ x \in X \,\middle|\, \text{the limit } \lim_{h \searrow 0} \frac{S(h)x - x}{h} \text{ exists} \right\}, \tag{7.1.19}$$

and which is given by

$$Ax := \lim_{h \searrow 0} \frac{S(h)x - x}{h} \qquad \text{for } x \in \mathrm{dom}(A). \tag{7.1.20}$$

Example 7.1.12. Let H be a separable complex Hilbert space, let $(e_i)_{i\in\mathbb{N}}$ be a complex orthonormal basis, and let $(\lambda_i)_{i\in\mathbb{N}}$ be a sequence of complex numbers such that

$$\sup_{i\in\mathbb{N}} \mathrm{Re}\lambda_i < \infty.$$

Let $S : [0, \infty) \to \mathcal{L}^c(H)$ be the strongly continuous semigroup in Example 7.1.3, i.e.

$$S(t)x = \sum_{i=1}^{\infty} e^{\lambda_i t} \langle e_i, x \rangle e_i$$

for $x \in H$ and $t \geq 0$. Then the infinitesimal generator of S is the linear operator

$$A : \mathrm{dom}(A) \to H$$

in Example 6.1.3, given by

$$\mathrm{dom}(A) = \left\{ x \in H \,\middle|\, \sum_{i=1}^{\infty} |\lambda_i \langle e_i, x \rangle|^2 < \infty \right\} \tag{7.1.21}$$

and

$$Ax = \sum_{i=1}^{\infty} \lambda_i \langle e_i, x \rangle e_i \qquad \text{for } x \in \mathrm{dom}(A). \tag{7.1.22}$$

Exercise: Prove this.

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Lemma 7.1.13. *Let X be a real Banach space and let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup with infinitesimal generator*

$$A : \mathrm{dom}(A) \to X.$$

Let $x \in X$. Then the following are equivalent.

(i) $x \in \mathrm{dom}(A)$.

(ii) *The function $[0, \infty) \to X : t \mapsto S(t)x$ is continuously differentiable, takes values in the domain of A, and satisfies the differential equation*

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax \qquad \text{for all } t \geq 0. \tag{7.1.23}$$

Proof. That (ii) implies (i) follows directly from the definitions. To prove the converse, fix an element $x \in \mathrm{dom}(A)$. Then, for $t \geq 0$, we have

$$S(t)Ax = \lim_{h \searrow 0} S(t)\frac{S(h)x - x}{h} = \lim_{h \searrow 0} \frac{S(t+h)x - S(t)x}{h}$$

and, for $t > 0$,

$$S(t)Ax = \lim_{h \searrow 0} S(t-h)\frac{S(h)x - x}{h} = \lim_{h \searrow 0} \frac{S(t-h)x - S(t)x}{-h}.$$

This shows that the function $[0, \infty) \to X : t \mapsto S(t)x$ is continuously differentiable and that its derivative at $t \geq 0$ is $S(t)Ax$. Moreover,

$$\lim_{h \searrow 0} \frac{S(h)S(t)x - S(t)x}{h} = \lim_{h \searrow 0} S(t)\frac{S(h)x - x}{h} = S(t)Ax.$$

Thus $S(t)x \in \mathrm{dom}(A)$ and

$$AS(t)x = S(t)Ax.$$

This proves Lemma 7.1.13. □

Lemma 7.1.14 (Variation of Constants). *Let X be a real Banach space and let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup with infinitesimal generator $A : \mathrm{dom}(A) \to X$. Let $f : [0, \infty) \to X$ be a continuously differentiable function and define the function $x : [0, \infty) \to X$ by*

$$x(t) := \int_0^t S(t-s)f(s)\,ds \qquad \text{for } t \geq 0. \tag{7.1.24}$$

Then x is continuously differentiable, $x(t) \in \mathrm{dom}(A)$ for all $t \geq 0$, and

$$\dot{x}(t) = Ax(t) + f(t) = S(t)f(0) + \int_0^t S(t-s)\dot{f}(s)\,ds \tag{7.1.25}$$

for all $t \geq 0$.

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Proof. Fix a constant $t \geq 0$ and let $h > 0$. Then

$$\begin{aligned}
\frac{S(h)x(t) - x(t)}{h} &= \frac{S(h) - 1\!\!1}{h} \int_0^t S(s)f(t-s)\,ds \\
&= \frac{1}{h}\int_0^t S(s+h)f(t-s)\,ds - \frac{1}{h}\int_0^t S(s)f(t-s)\,ds \\
&= \frac{1}{h}\int_h^{t+h} S(s)f(t+h-s)\,ds - \frac{1}{h}\int_0^t S(s)f(t-s)\,ds \\
&= \frac{1}{h}\int_t^{t+h} S(s)f(t+h-s)\,ds - \frac{1}{h}\int_0^h S(s)f(t+h-s)\,ds \\
&\quad + \int_0^t S(s)\frac{f(t+h-s) - f(t-s)}{h}\,ds.
\end{aligned}$$

Take the limit $h \to 0$ to obtain $x(t) \in \mathrm{dom}(A)$ and

$$Ax(t) = S(t)f(0) - f(t) + \int_0^t S(t-s)\dot{f}(s)\,ds. \tag{7.1.26}$$

This proves the second equation in (7.1.25) and shows that Ax is continuous. Next observe that

$$\begin{aligned}
\frac{x(t+h) - x(t)}{h} &= \frac{1}{h}\int_0^{t+h} S(t+h-s)f(s)\,ds - \frac{1}{h}\int_0^t S(t-s)f(s)\,ds \\
&= \frac{S(h)x(t) - x(t)}{h} + \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds
\end{aligned}$$

for all $h > 0$. Take the limit $h \to 0$ to obtain that x is right differentiable and $\frac{d}{dt^+}x(t) = Ax(t) + f(t)$. Third, observe that

$$\begin{aligned}
\frac{x(t) - x(t-h)}{h} &= \frac{1}{h}\int_0^t S(t-s)f(s)\,ds - \frac{1}{h}\int_0^{t-h} S(t-h-s)f(s)\,ds \\
&= \frac{1}{h}\int_0^t S(t-s)f(s)\,ds - \frac{1}{h}\int_h^t S(t-s)f(s-h)\,ds \\
&= \frac{1}{h}\int_0^h S(t-s)f(s)\,ds + \int_h^t S(t-s)\frac{f(s) - f(s-h)}{h}\,ds
\end{aligned}$$

for $0 < h < t$. Take the limit $h \to 0$ to obtain that x is left differentiable and $\frac{d}{dt^-}x(t) = S(t)f(0) + \int_0^t S(t-s)\dot{f}(s)\,ds = Ax(t) + f(t)$. Here the last equation follows from (7.1.26). This proves Lemma 7.1.14. □

Example 7.1.15. Let $x \in X$ and take $f(t) = x$ in Lemma 7.1.14. Then

$$\int_0^t S(s)x\,ds \in \mathrm{dom}(A), \qquad A\int_0^t S(s)x\,ds = S(t)x - x$$

for all $t > 0$.

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Lemma 7.1.16. *Let X be a real Banach space and let $S : [0,\infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup with infinitesimal generator*

$$A : \mathrm{dom}(A) \to X.$$

For $n \in \mathbb{N}$ define the linear subspaces $\mathrm{dom}(A^n) \subset X$ recursively by

$$\mathrm{dom}(A^1) := \mathrm{dom}(A), \qquad \mathrm{dom}(A^n) := \left\{x \in \mathrm{dom}(A) \,|\, Ax \in \mathrm{dom}(A^{n-1})\right\}$$

for $n \geq 2$. Then the linear subspace $\mathrm{dom}(A^\infty) := \bigcap_{n\in\mathbb{N}} \mathrm{dom}(A^n)$ is dense in X and A has a closed graph.

Proof. The proof has three steps.

Step 1. *Let $x \in X$ and let $\phi : \mathbb{R} \to X$ be a smooth function with compact support contained in the interval $[\delta, \delta^{-1}]$ for some constant $0 < \delta < 1$. Then, for every $n \in \mathbb{N}$, we have $\int_0^\infty \phi(t)S(t)x\,dt \in \mathrm{dom}(A^n)$ and*

$$A^n \int_0^\infty \phi(t)S(t)x\,dt = (-1)^n \int_0^\infty \phi^{(n)}(t)S(t)x\,dt.$$

For $n = 1$ this follows from Lemma 7.1.14 with $t > \delta^{-1}$ and $f(s) := \phi(t-s)x$ for $s \geq 0$. For $n \geq 2$ the assertion follows by induction.

Step 2. $\mathrm{dom}(A^\infty)$ *is dense in X.*

Let $x \in X$ and choose a smooth function $\phi : \mathbb{R} \to [0,\infty)$, with compact support in the interval $[1/2, 1]$, such that $\int_0^1 \phi(t)\,dt = 1$. Define

$$x_n := n\int_0^\infty \phi(nt)S(t)x\,dt \qquad \text{for } n \in \mathbb{N}.$$

Then $x_n \in \mathrm{dom}(A^\infty)$ by Step 1 and

$$\|x_n - x\| = \left\| n\int_0^{1/n} \phi(nt)(S(t)x - x)\,dt \right\| \leq \sup_{0\leq t\leq 1/n} \|S(t)x - x\|.$$

Hence $\lim_{n\to\infty}\|x_n - x\| = 0$ and this proves Step 2.

Step 3. *A has a closed graph.*

Choose a sequence $x_n \in \mathrm{dom}(A)$ and $x, y \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0, \qquad \lim_{n\to\infty} \|Ax_n - y\| = 0.$$

Then, by Lemma 7.1.13,

$$S(t)x - x = \lim_{n\to\infty}\big(S(t)x_n - x_n\big) = \lim_{n\to\infty}\int_0^t S(s)Ax_n\,ds = \int_0^t S(s)y\,ds$$

for all $t > 0$. Hence $y = \lim_{t\searrow 0} t^{-1}(S(t)x - x)$ and this implies $x \in \mathrm{dom}(A)$ and $Ax = y$. This proves Step 3 and Lemma 7.1.16. □

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Recall from Exercise 2.2.12 that the domain of a closed densely defined operator $A : \mathrm{dom}(A) \to X$ is a Banach space with the **graph norm**

$$\|x\|_A := \|x\|_X + \|Ax\|_X \qquad \text{for } x \in \mathrm{dom}(A).$$

Moreover, the operator A can be viewed as a bounded operator from $\mathrm{dom}(A)$ to X rather than as an unbounded densely defined operator from X to itself.

Lemma 7.1.17. *Let X be a real Banach space and let $S : [0,\infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup. Let $A : \mathrm{dom}(A) \to X$ be a linear operator with a dense domain $\mathrm{dom}(A) \subset X$ and a closed graph. Then the following are equivalent.*

(i) *The operator A is the infinitesimal generator of the semigroup S.*

(ii) *If $x \in \mathrm{dom}(A)$ and $t > 0$, then $S(t)x \in \mathrm{dom}(A)$, $AS(t)x = S(t)Ax$, and $S(t)x - x = \int_0^t S(s)Ax\,ds$.*

(iii) *If $x_0 \in \mathrm{dom}(A)$, then the function $[0,\infty) \to X : t \mapsto x(t) := S(t)x_0$ is continuously differentiable, takes values in $\mathrm{dom}(A)$, and satisfies the differential equation $\dot{x}(t) = Ax(t)$ for all $t \geq 0$.*

Proof. That (i) implies (ii) follows directly from Lemma 7.1.13. That (ii) implies (iii) follows directly from part (vii) of Lemma 5.1.10. We prove in three steps that (iii) implies (i). Assume A satisfies (iii).

Step 1. *Let $x \in \mathrm{dom}(A)$ and $t > 0$. Then*

$$\int_0^t S(s)x\,ds \in \mathrm{dom}(A), \qquad A\int_0^t S(s)x\,ds = S(t)x - x. \tag{7.1.27}$$

By part (iii) the function $\xi : [0,t] \to X$ defined by $\xi(s) := S(s)x$ for $0 \leq s \leq t$ takes values in $\mathrm{dom}(A)$ and the function $A\xi = \dot{\xi} : [0,t] \to X$ is continuous. Hence the function $\xi : [0,t] \to \mathrm{dom}(A)$ is continuous with respect to the graph norm. Thus it follows from part (iii) of Lemma 5.1.10 that

$$\int_0^t \xi(s)\,ds \in \mathrm{dom}(A)$$

and

$$A\int_0^t \xi(s)\,ds = \int_0^t A\xi(s)\,ds = \xi(t) - \xi(0) = S(t)x - x.$$

This proves Step 1.

Step 2. *If $x \in X$ and $t > 0$, then* (7.1.27) *holds.*

Let $x \in X$ and $t > 0$. Choose a sequence $x_i \in \mathrm{dom}(A)$ that converges to x. Then $\xi_i := \int_0^t S(s)x_i\,ds \in \mathrm{dom}(A)$ and $A\xi_i = S(t)x_i - x_i$ by Step 1. Since A has a closed graph, ξ_i converges to $\int_0^t S(s)x\,ds$, and $A\xi_i$ converges to $S(t)x - x$, it follows that x and t satisfy (7.1.27). This proves Step 2.

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Step 3. *Let $x, y \in X$. Then*

$$\lim_{h\to 0} \frac{S(h)x - x}{h} = y \qquad \Longleftrightarrow \qquad x \in \mathrm{dom}(A), \quad Ax = y. \tag{7.1.28}$$

If $x \in \mathrm{dom}(A)$ and $y = Ax$, then $\lim_{h\to 0} h^{-1}(S(h)x - x) = y$ by part (iii). Conversely, suppose that $\lim_{h\to 0} h^{-1}(S(h)x - x) = y$. For each $h > 0$ define $x_h := h^{-1}\int_0^h S(s)x\,ds$. Then $\lim_{h\to 0} x_h = x$ and by Step 2 $x_h \in \mathrm{dom}(A)$ and $Ax_h = h^{-1}(S(h)x - x)$. Hence $\lim_{h\to 0} Ax_h = y$. Since A has a closed graph this implies $x \in \mathrm{dom}(A)$ and $Ax = y$. This proves Lemma 7.1.17. □

Lemma 7.1.18. *Let X be a real Banach space and let $S : [0,\infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup with infinitesimal generator A. Then the following are equivalent.*

(i) $\mathrm{dom}(A) = X$.

(ii) *A is bounded.*

(iii) *The semigroup S is continuous in the norm topology on $\mathcal{L}(X)$.*

Proof. The Closed Graph Theorem 2.2.13 asserts that (i) and (ii) are equivalent. That (ii) implies (iii) follows from Exercise 1.5.4 and Corollary 7.2.3 below. We prove that (iii) implies (i), following [**26**, p. 615]. Assume that $S : [0,\infty) \to \mathcal{L}(X)$ is continuous with respect to the norm topology on $\mathcal{L}(X)$. Then $\lim_{t\to 0} \|S(t) - 1\!\!1\| = 0$. Hence there exists a $\delta > 0$ such that $\sup_{0\le t\le\delta} \|S(t) - 1\!\!1\| < 1$. For $0 \le t \le \delta$ define

$$B(t) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \big(S(t) - 1\!\!1\big)^n.$$

Then the following hold.

(I) The function $B : [0,\delta] \to \mathcal{L}(X)$ is norm-continuous.

(II) $e^{B(t)} = S(t)$ for $0 \le t \le \delta$.

(III) If $k \in \mathbb{N}$ and $0 \le t \le \delta/k$, then $B(kt) = kB(t)$.

Part (II) uses the fact that the power series $f(z) := \sum_{n=1}^{\infty}(-1)^{n-1}(z-1)^n/n$ satisfies $\exp(f(z)) = z$ for all $z \in \mathbb{C}$ with $|z-1| < 1$. Part (III) follows from the fact that $f(z^k) = kf(z)$ whenever $|z^j - 1| < 1$ for $j = 1, 2, \ldots, k$.

By (III), $B(\delta) = \ell B(\delta/\ell)$ and so $B(k\delta/\ell) = kB(\delta/\ell) = (k/\ell)B(\delta)$ for all integers $0 \le k \le \ell$. Since B is continuous by (I), this implies

$$B(t) = t\delta^{-1}B(\delta) \qquad \text{for } 0 \le t \le \delta.$$

(Approximate $t\delta^{-1}$ by a sequence of rational numbers in $[0,1]$.) Now define the operator $A := \delta^{-1}B(\delta) \in \mathcal{L}(X)$. Then by (II) we have $S(t) = e^{B(t)} = e^{tA}$ for $0 \le t \le \delta$. So $S(t) = e^{tA}$ for all $t \ge 0$ and this proves Lemma 7.1.18. □

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7.2. The Hille–Yosida–Phillips Theorem

7.2.1. Well-Posed Cauchy Problems. Let us now change the point of view and suppose that $A : \mathrm{dom}(A) \to X$ is a linear operator on a Banach space X whose domain is a linear subspace $\mathrm{dom}(A) \subset X$. Consider the **Cauchy problem**

$$\dot{x} = Ax, \qquad x(0) = x_0. \tag{7.2.1}$$

Definition 7.2.1. (i) Let $I \subset [0, \infty)$ be a closed interval with $0 \in I$. A continuously differentiable function $x : I \to X$ is called a **solution** of (7.2.1) if it takes values in $\mathrm{dom}(A)$ and $x(0) = x_0$ and $\dot{x}(t) = Ax(t)$ for all $t \in I$.

(ii) The Cauchy problem (7.2.1) is called **well-posed** if it satisfies the following axioms.

(Existence) For each $x_0 \in \mathrm{dom}(A)$ there is a solution of (7.2.1) on $[0, \infty)$.

(Uniqueness) Let $x_0 \in \mathrm{dom}(A)$ and $T > 0$. If $x, y : [0, T] \to X$ are solutions of (7.2.1), then $x(t) = y(t)$ for all $t \in [0, T]$.

(Continuous Dependence) Define the map $\phi : [0, \infty) \times \mathrm{dom}(A) \to X$ by $\phi(t, x_0) := x(t)$ for $t \geq 0$ and $x_0 \in \mathrm{dom}(A)$, where $x : [0, \infty) \to X$ is the unique solution of (7.2.1). Then, for every $T > 0$, there is a constant $M \geq 1$ such that $\|\phi(t, x_0)\| \leq M\|x_0\|$ for all $t \in [0, T]$ and all $x_0 \in \mathrm{dom}(A)$.

The next theorem characterizes well-posed Cauchy problems and was proved by Ralph S. Phillips [**68**] in 1954.

Theorem 7.2.2 (**Phillips**). *Let $A : \mathrm{dom}(A) \to X$ be a linear operator with a dense domain $\mathrm{dom}(A) \subset X$ and a closed graph. The following are equivalent.*

(i) *A is the infinitesimal generator of a strongly continuous semigroup.*

(ii) *The Cauchy problem* (7.2.1) *is well-posed.*

Proof. We prove that (i) implies (ii). Thus assume that A is the infinitesimal generator of a strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}(X)$ and fix an element $x_0 \in \mathrm{dom}(A)$. Then the function $[0, \infty) \to X : t \mapsto S(t)x_0$ is a solution of equation (7.2.1) by Lemma 7.1.13. To prove uniqueness, assume that $x : [0, \infty) \to X$ is any solution of (7.2.1). Fix a constant $t > 0$. We will prove that the function $[0, t] \to X : s \mapsto S(t - s)x(s)$ is constant. To see this, note that $x(s) \in \mathrm{dom}(A)$ and so

$$\lim_{\substack{h \to 0 \\ h \leq t-s}} \frac{S(t-s-h)x(s) - S(t-s)x(s)}{-h} = S(t-s)Ax(s) \qquad \text{for } 0 \leq s \leq t.$$

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This implies

$$\begin{aligned}
&\lim_{h\to 0} \frac{S(t-s-h)x(s+h) - S(t-s)x(s)}{h} \\
&= \lim_{h\to 0} S(t-s-h)\left(\frac{x(s+h)-x(s)}{h} - Ax(s)\right) \\
&\quad + \lim_{h\to 0}\left(\frac{S(t-s-h)x(s) - S(t-s)x(s)}{h} + S(t-s)Ax(s)\right) \\
&\quad + \lim_{h\to 0}\big(S(t-s-h)Ax(s) - S(t-s)Ax(s)\big) \\
&= 0.
\end{aligned}$$

Hence the function $[0,t] \to X : s \mapsto S(t-s)x(s)$ is everywhere differentiable and its derivative vanishes. Thus it is constant and hence $x(t) = S(t)x_0$. Since $t > 0$ was chosen arbitrarily this proves uniqueness. Continuous dependence follows from the estimate $\|S(t)\| \le Me^{\omega t}$ in Lemma 7.1.8. This shows that (i) implies (ii).

We prove that (ii) implies (i). Assume the Cauchy problem (7.2.1) is well-posed and let

$$\phi : [0,\infty) \times \mathrm{dom}(A) \to \mathrm{dom}(A)$$

be the map that assigns to each element $x_0 \in \mathrm{dom}(A)$ the unique solution $[0,\infty) \to X : t \mapsto \phi(t,x_0)$ of (7.2.1). We claim that, for each $t \ge 0$, there is a unique bounded linear operator $S(t) : X \to X$ such that

$$S(t)x_0 = \phi(t,x_0) \qquad \text{for all } x_0 \in \mathrm{dom}(A). \tag{7.2.2}$$

To see this, note first that the space of solutions $x : [0,\infty) \to X$ of (7.2.1) is a linear subspace of the space of all functions from $[0,\infty)$ to X. Hence it follows from uniqueness that the map $\mathrm{dom}(A) \to X : x_0 \mapsto \phi(t,x_0)$ is linear. Second, it follows from continuous dependence that the linear operator $\mathrm{dom}(A) \to X : x_0 \mapsto \phi(t,x_0)$ is bounded. Since $\mathrm{dom}(A)$ is a dense linear subspace of X it follows that this operator extends uniquely to a bounded linear operator $S(t) \in \mathcal{L}(X)$. More precisely, fix an element $x \in X$. Then there exists a sequence $x_n \in \mathrm{dom}(A)$ that converges to x. Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and so is the sequence $(\phi(t,x_n))_{n\in\mathbb{N}}$ by continuous dependence. Hence it converges and the limit

$$S(t)x := \lim_{n\to\infty} \phi(t,x_n)$$

is independent of the choice of the sequence $x_n \in \mathrm{dom}(A)$ used to define it. This proves the existence of a bounded linear operator $S(t)$ that satisfies (7.2.2).

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We prove that these operators form a one-parameter semigroup. Fix a real number $t \geq 0$ and an element $x_0 \in \mathrm{dom}(A)$. Then

$$S(t)x_0 = \phi(t, x_0) \in \mathrm{dom}(A)$$

and the function $[0, \infty) \to X : s \mapsto S(s+t)x_0 = \phi(s+t, x_0)$ is a solution of the Cauchy problem (7.2.1) with x_0 replaced by $S(t)x_0 = \phi(t, x_0)$. Hence

$$S(s+t, x_0) = \phi(s, S(t)x_0) = S(s)S(t)x_0.$$

Since this holds for all $x_0 \in \mathrm{dom}(A)$, the set $\mathrm{dom}(A)$ is dense in X, and the operators $S(s+t)$ and $S(s)S(t)$ are both continuous maps, it follows that $S(s+t) = S(s)S(t)$ for all $s \geq 0$. This shows that $S : [0, \infty) \to \mathcal{L}(X)$ is a one-parameter semigroup.

We prove that S is strongly continuous. To see this, fix an element $x \in X$ and a constant $\varepsilon > 0$. By continuous dependence there exists an $M \geq 1$ such that $\sup_{0 \leq t \leq 1} \|\phi(t, x_0)\| \leq M \|x_0\|$ for all $x_0 \in \mathrm{dom}(A)$. This shows that $\sup_{0 \leq t \leq 1} \|S(t)\| \leq M$. Choose an element $y \in \mathrm{dom}(A)$ such that

$$\|x - y\| \leq \frac{\varepsilon}{2(M+1)}.$$

Next choose a constant $0 < \delta < 1$ such that, for all $t \in \mathbb{R}$,

$$0 \leq t < \delta \qquad \Longrightarrow \qquad \|\phi(t, y) - y\| < \frac{\varepsilon}{2}.$$

Fix a real number $0 \leq t < \delta$. Then

$$\begin{aligned} \|S(t)x - x\| &\leq \|S(t)x - S(t)y\| + \|S(t)y - y\| + \|y - x\| \\ &\leq (M+1)\|x - y\| + \|\phi(t, y) - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that S is strongly continuous.

We prove that A is the infinitesimal generator of S. Let $x_0 \in \mathrm{dom}(A)$ and define the function $x : [0, \infty) \to X$ by $x(t) := S(t)x_0 = \phi(t, x_0)$. It is continuously differentiable, takes values in $\mathrm{dom}(A)$, and satisfies the equation $\dot{x}(t) = Ax(t)$ for all $t \geq 0$. Thus A and S satisfy condition (iii) in Lemma 7.1.17, so A is the infinitesimal generator of S. This proves Theorem 7.2.2. □

Corollary 7.2.3 (Uniqueness). *A linear operator on a Banach space is the infinitesimal generator of at most one strongly continuous semigroup.*

Proof. Let A be the infinitesimal generator of two strongly continuous semigroups $S, T : [0, \infty) \to \mathcal{L}(X)$. Let $x_0 \in \mathrm{dom}(A)$. Then the functions $x(t) := S(t)x_0$ and $y(t) := T(t)x_0$ both satisfy (7.2.1) and hence agree by Theorem 7.2.2. Since $\mathrm{dom}(A)$ is dense in X by Lemma 7.1.16, it follows that $S(t)x = T(t)x$ for all $x \in X$ and all $t \geq 0$. □

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Theorem 7.2.4 (Strongly Continuous Groups). *Let X be a real Banach space, let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup, and let $A : \mathrm{dom}(A) \to X$ be the infinitesimal generator of S. Then the following are equivalent.*

(i) *The semigroup S extends to a strongly continuous group $S : \mathbb{R} \to \mathcal{L}(X)$.*

(ii) *$-A$ is the infinitesimal generator of a strongly continuous semigroup.*

(iii) *The operator $S(t)$ is bijective for all $t > 0$.*

Proof. We prove that (i) implies (ii). Thus assume that S extends to a strongly continuous group $S : \mathbb{R} \to \mathcal{L}(X)$. Then

$$S(t)S(-t) = S(-t)S(t) = \mathbb{1}$$

for all $t > 0$ by definition of a one-parameter group of operators. This implies that $S(t)$ is bijective and

$$S(t)^{-1} = S(-t)$$

for all $t > 0$. Define the map $T : [0, \infty) \to \mathcal{L}(X)$ by

$$T(t) := S(-t) = S(t)^{-1} \qquad \text{for } t \geq 0.$$

Then T is a strongly continuous semigroup by definition. Denote its infinitesimal generator by $B : \mathrm{dom}(B) \to X$. We must prove that $B = -A$. To see this, choose a constant $M \geq 1$ such that

$$\|S(t)\| \leq M \quad \text{and} \quad \|T(t)\| \leq M \qquad \text{for } 0 \leq t \leq 1.$$

Now let $x \in \mathrm{dom}(A)$. Then

$$\begin{aligned} \left\| \frac{T(h)x - x}{h} + Ax \right\| &\leq \left\| T(h) \left(\frac{x - S(h)x}{h} + Ax \right) \right\| + \|Ax - T(h)Ax\| \\ &\leq M \left\| \frac{x - S(h)x}{h} + Ax \right\| + \|Ax - T(h)Ax\| \end{aligned}$$

for $0 < h < 1$. Since the right-hand side converges to zero it follows that

$$x \in \mathrm{dom}(B), \qquad Bx = -Ax.$$

Thus we have proved that

$$\mathrm{dom}(A) \subset \mathrm{dom}(B), \qquad B|_{\mathrm{dom}(A)} = -A.$$

Interchange the roles of S and T to obtain

$$\mathrm{dom}(B) = \mathrm{dom}(A), \qquad B = -A.$$

This shows that (i) implies (ii).

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We prove that (ii) implies (iii). Let $T : [0, \infty) \to \mathcal{L}(X)$ be the strongly continuous semigroup generated by $-A$. We prove that $S(t)$ is bijective and $T(t) = S(t)^{-1}$ for all $t > 0$. To see this, fix an element $x \in \mathrm{dom}(A)$ and a real number $t > 0$. Define the functions $y, z : [0, t] \to X$ by

$$y(s) := S(t-s)x, \qquad z(s) := T(t-s)x \qquad \text{for } 0 \le s \le t.$$

Then y and z are continuously differentiable, take values in the domain of A, and satisfy the Cauchy problems

$$\dot{y}(s) = -Ay(s) \quad \text{for } 0 \le s \le t, \qquad y(0) = S(t)x,$$

and

$$\dot{z}(s) = Az(s) \quad \text{for } 0 \le s \le t, \qquad z(0) = T(t)x.$$

By Theorem 7.2.2 this implies

$$y(s) = T(s)S(t)x, \qquad z(s) = S(s)T(t)x \qquad \text{for } 0 \le s \le t.$$

Take $s = t$ to obtain $T(t)S(t)x = y(t) = x$ and $S(t)T(t)x = z(t) = x$. Thus we have proved that $S(t)T(t)x = T(t)S(t)x = x$ for all $t > 0$ and all $x \in \mathrm{dom}(A)$. Since the domain of A is dense in X this implies

$$S(t)T(t) = T(t)S(t) = \mathbb{1} \qquad \text{for all } t > 0.$$

Hence $S(t)$ is bijective for all $t > 0$. This shows that (ii) implies (iii).

We prove that (iii) implies (i). Thus assume that $S(t)$ is bijective for all $t > 0$. Then $S(t)^{-1} : X \to X$ is a bounded linear operator for every $t > 0$ by the Open Mapping Theorem 2.2.1. Define

$$S(-t) := S(t)^{-1} \qquad \text{for } t > 0.$$

We prove that the extended function $S : \mathbb{R} \to \mathcal{L}(X)$ is a one-parameter group. The formula $S(t+s) = S(t)S(s)$ holds by definition whenever $s, t \ge 0$ or $s, t \le 0$. Moreover, if $0 \le s < t$, then $S(t-s)S(s) = S(t)$ and hence

$$S(t-s) = S(t)S(s)^{-1} = S(t)S(-s).$$

This implies that, for $0 \le t < s$, we have $S(s-t) = S(s)S(-t)$ and hence

$$S(t-s) = S(s-t)^{-1} = S(-t)^{-1}S(s)^{-1} = S(t)S(-s).$$

This shows that S is a one-parameter group. Strong continuity at $t = 0$ follows from the equation

$$S(-h)x - x = S(h)^{-1}(x - S(h)x)$$

for $h > 0$. Strong continuity at $-t < 0$ follows from the equation

$$S(-t+h)x - S(-t)x = S(t)^{-1}\big(S(h)x - x\big)$$

for $h \in \mathbb{R}$. This proves Theorem 7.2.4. □

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7.2.2. The Hille–Yosida–Phillips Theorem. The following theorem is the main result of this chapter. For the special case $M = 1$ it was discovered by Hille [**35**] and Yosida [**87**] independently in 1948. It was extended to the case $M > 1$ by Phillips [**67**] in 1952.

Theorem 7.2.5 (Hille–Yosida–Phillips). *Let X be a real Banach space and $A : \mathrm{dom}(A) \to X$ be a linear operator with a dense domain $\mathrm{dom}(A) \subset X$. Fix real numbers ω and $M \geq 1$. Then the following are equivalent.*

(i) *The operator A is the infinitesimal generator of a strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}(X)$ that satisfies*

$$\|S(t)\| \leq Me^{\omega t} \qquad \text{for all } t \geq 0. \tag{7.2.3}$$

(ii) *For every real number $\lambda > \omega$ the operator $\lambda 1\!\!1 - A : \mathrm{dom}(A) \to X$ is invertible and*

$$\|(\lambda 1\!\!1 - A)^{-k}\| \leq \frac{M}{(\lambda - \omega)^k} \qquad \text{for all } \lambda > \omega \text{ and all } k \in \mathbb{N}. \tag{7.2.4}$$

Proof. See page 371. □

The necessity of the condition (7.2.4) is a straightforward consequence of Lemma 7.2.6 below which expresses the resolvent operator $(\lambda 1\!\!1 - A)^{-1}$ in terms of the semigroup. At this point it is convenient to allow for λ to be a complex number and therefore to extend the discussion to complex Banach spaces. When X is a real Banach space we will tacitly assume that X has been complexified so as to make sense of the operator $\lambda 1\!\!1 - A : \mathrm{dom}(A) \to X$ for complex numbers λ (see Exercise 5.1.5).

Lemma 7.2.6 (Resolvent Identity for Semigroups). *Let X be a complex Banach space and let*

$$A : \mathrm{dom}(A) \to X$$

be the infinitesimal generator of a strongly continuous semigroup

$$S : [0, \infty) \to \mathcal{L}^c(X).$$

Let $\lambda \in \mathbb{C}$ such that

$$\mathrm{Re}\lambda > \omega_0 := \lim_{t \to \infty} \frac{\log\|S(t)\|}{t}. \tag{7.2.5}$$

Then $\lambda \in \rho(A)$ and

$$(\lambda 1\!\!1 - A)^{-k} x = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-\lambda t} S(t) x \, dt \tag{7.2.6}$$

for all $x \in X$ and all $k \in \mathbb{N}$.

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Proof. We first prove the assertion for $k = 1$. Fix a complex number λ such that $\mathrm{Re}\lambda > \omega_0$ and choose a real number ω such that $\omega_0 < \omega < \mathrm{Re}\lambda$. By Lemma 7.1.8, there exists a constant $M \geq 1$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Hence $\|e^{-\lambda t}S(t)x\| \leq Me^{(\omega-\mathrm{Re}\lambda)t}\|x\|$ for all $x \in X$ and all $t \geq 0$. This implies that the formula

$$R_\lambda x := \int_0^\infty e^{-\lambda t}S(t)x\,dt = \lim_{T\to\infty}\int_0^T e^{-\lambda t}S(t)x\,dt \quad \text{for } x \in X$$

defines a bounded linear operator $R_\lambda \in \mathcal{L}^c(X)$. We prove the following.

Claim 1. *If $x \in X$ and $T > 0$, then $\xi_T := \int_0^T e^{-\lambda t}S(t)x\,dt \in \mathrm{dom}(A)$ and*

$$A\xi_T = e^{-\lambda T}S(T)x - x + \lambda\int_0^T e^{-\lambda t}S(t)x\,dt =: \eta_T.$$

Claim 2. *If $x \in \mathrm{dom}(A)$ and $T > 0$, then $\int_0^T e^{-\lambda t}S(t)Ax\,ds = \eta_T$.*

Claim 1 follows from Lemma 7.1.14 with $t = T$ and $f(t) := e^{-\lambda(T-t)}x$. Claim 2 follows from integration by parts with $\frac{d}{dt}S(t)x = S(t)Ax$. Now

$$A\xi_T = \eta_T, \qquad \lim_{T\to\infty}\xi_T = R_\lambda x, \qquad \lim_{T\to\infty}\eta_T = \lambda R_\lambda x - x$$

by Claim 1. Since A has a closed graph this implies

$$R_\lambda x \in \mathrm{dom}(A), \qquad AR_\lambda x = \lambda R_\lambda x - x \qquad \text{for all } x \in X.$$

If $x \in \mathrm{dom}(A)$ it follows from Claim 2 that

$$R_\lambda Ax = \lim_{T\to\infty}\int_0^T e^{-\lambda t}S(t)Ax\,dt = \lambda R_\lambda x - x.$$

Thus $(\lambda\mathbb{1}-A)R_\lambda x = x$ for all $x \in X$ and $R_\lambda(\lambda\mathbb{1}-A)x = x$ for all $x \in \mathrm{dom}(A)$. Hence $\lambda\mathbb{1}-A$ is bijective and $(\lambda\mathbb{1}-A)^{-1} = R_\lambda$. This proves (7.2.6) for $k = 1$. To prove the equation for $k \geq 2$ observe that the function

$$\rho(A) \to X : \lambda \mapsto (\lambda\mathbb{1} - A)^{-1}x$$

is holomorphic by Lemma 6.1.10 and satisfies

$$\begin{aligned}
(\lambda\mathbb{1} - A)^{-k}x &= \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{d\lambda^{k-1}}(\lambda\mathbb{1} - A)^{-1}x \\
&= \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{d\lambda^{k-1}}\int_0^\infty e^{-\lambda t}S(t)x\,dt \\
&= \frac{1}{(k-1)!}\int_0^\infty t^{k-1}e^{-\lambda t}S(t)x\,dt
\end{aligned}$$

for all $x \in X$ and all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega_0$. This proves Lemma 7.2.6. □

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It follows from Lemma 7.2.6 that

$$\sup_{\lambda\in\sigma(A)} \mathrm{Re}\lambda \le \omega_0 = \lim_{t\to\infty} \frac{\log\|S(t)\|}{t} \tag{7.2.7}$$

for every strongly continuous semigroup S with infinitesimal generator A. The following example by Einar Hille shows that the inequality in (7.2.7) can be strict.

Example 7.2.7. Fix a real number $\omega > 0$ and consider the Banach space

$$X := \left\{ f : [0,\infty) \to \mathbb{C} \,\middle|\, \begin{array}{l} f \text{ is continuous and bounded} \\ \text{and } \int_0^\infty e^{\omega s}|f(s)|\,ds < \infty \end{array} \right\},$$

equipped with the norm

$$\|f\| := \sup_{s\ge 0} |f(s)| + \int_0^\infty e^{\omega s}|f(s)|\,ds \qquad \text{for } f \in X.$$

The formula

$$(S(t)f)(s) := f(s+t) \qquad \text{for } f \in X \text{ and } s,t \ge 0$$

defines a strongly continuous semigroup on X and its infinitesimal generator is the operator $A : \mathrm{dom}(A) \to X$ given by

$$\mathrm{dom}(A) = \left\{ u : [0,\infty) \to \mathbb{C} \,\middle|\, \begin{array}{l} u \text{ is continuously differentiable} \\ \text{and } u, \dot{u} \in X \end{array} \right\},$$
$$Au = \dot{u}.$$

The operator $S(t)$ satisfies $\|S(t)\| = 1$ for all $t \ge 0$ and so

$$\omega_0 = \lim_{t\to\infty} \frac{\log\|S(t)\|}{t} = 0.$$

Now let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > -\omega$ and let $f \in X$. Then, for $u \in \mathrm{dom}(A)$,

$$\lambda u - Au = f \qquad \iff \qquad \dot{u} = \lambda u - f.$$

This equation has a unique solution $u \in \mathrm{dom}(A)$ given by

$$u(s) = \int_s^\infty e^{\lambda(s-t)} f(t)\,dt \qquad \text{for } s \ge 0.$$

Thus the operator $\lambda\mathbb{1} - A$ is bijective for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > -\omega$. It has a one-dimensional kernel for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda < -\omega$. Thus

$$\sup_{\lambda\in\sigma(A)} \mathrm{Re}\lambda = -\omega < 0 = \lim_{t\to\infty} \frac{\log\|S(t)\|}{t}.$$

Exercise: For $t > 0$ the spectrum of $S(t)$ is the closed unit disc and the point spectrum of $S(t)$ is the open disc of radius $e^{-\omega t}$ centered at the origin.

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Proof of Theorem 7.2.5. We prove that (i) implies (ii). Thus assume that $A : \mathrm{dom}(A) \to X$ is the infinitesimal generator of a strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}(X)$ that satisfies (7.2.3). Fix a real number

$$\lambda > \omega$$

and a positive integer k. Then

$$(\lambda 1\!\!1 - A)^{-k} x = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-\lambda t} S(t) x \, dt$$

for all $x \in X$ by Lemma 7.2.6 and hence

$$\begin{aligned} \|(\lambda 1\!\!1 - A)^{-k} x\| &\le \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-\lambda t} \|S(t)x\| \, dt \\ &\le \frac{M\|x\|}{(k-1)!} \int_0^\infty t^{k-1} e^{-(\lambda-\omega)t} \, dt \\ &= \frac{M\|x\|}{(\lambda-\omega)^k}. \end{aligned}$$

Hence the operator A satisfies (ii).

We prove that (ii) implies (i). Thus assume that $A : \mathrm{dom}(A) \to X$ is a linear operator with a dense domain such that

$$\lambda 1\!\!1 - A : \mathrm{dom}(A) \to X$$

is bijective and satisfies the estimate (7.2.4) for $\lambda > \omega$. We prove in five steps that A is the infinitesimal generator of a strongly continuous semigroup that satisfies the estimate (7.2.3).

Step 1. $x = \lim_{\lambda\to\infty} \lambda(\lambda 1\!\!1 - A)^{-1} x$ *for all* $x \in X$.

If $x \in \mathrm{dom}(A)$, then

$$\lambda(\lambda 1\!\!1 - A)^{-1} x - x = A(\lambda 1\!\!1 - A)^{-1} x = (\lambda 1\!\!1 - A)^{-1} A x$$

for all $\lambda > \omega$ and so it follows from (7.2.4) that

$$\left\|\lambda(\lambda 1\!\!1 - A)^{-1} x - x\right\| \le \frac{M}{\lambda - \omega} \|Ax\| .$$

Thus

$$x = \lim_{\lambda\to\infty} \lambda(\lambda 1\!\!1 - A)^{-1} x$$

for all $x \in \mathrm{dom}(A)$. Moreover

$$\left\|\lambda(\lambda 1\!\!1 - A)^{-1}\right\| \le \frac{M\lambda}{\lambda-\omega} \le 2M \qquad \text{for all } \lambda > 2\omega.$$

Hence Step 1 follows from Theorem 2.1.5.

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Step 2. *For $\lambda > \omega$ and $t \geq 0$ define*

$$A_\lambda := \lambda A(\lambda 1\!\!1 - A)^{-1}, \qquad S_\lambda(t) := e^{tA_\lambda} = \sum_{k=0}^\infty \frac{t^k A_\lambda^k}{k!}.$$

Then

$$\|S_\lambda(t)\| \leq M e^{\frac{\lambda\omega t}{\lambda-\omega}}$$

for all $\lambda > \omega$ and all $t \geq 0$.

The operator A_λ can be written as

$$A_\lambda = \lambda^2(\lambda 1\!\!1 - A)^{-1} - \lambda 1\!\!1.$$

Hence

$$\begin{aligned}
\|S_\lambda(t)\| &= e^{-\lambda t}\left\|e^{t\lambda^2(\lambda 1\!\!1 - A)^{-1}}\right\| \\
&\leq e^{-\lambda t}\sum_{k=0}^\infty \frac{t^k\lambda^{2k}}{k!}\left\|(\lambda 1\!\!1 - A)^{-k}\right\| \\
&\leq e^{-\lambda t}\sum_{k=0}^\infty \frac{t^k\lambda^{2k}}{k!}\frac{M}{(\lambda-\omega)^k} \\
&= Me^{-\lambda t}e^{\frac{\lambda^2 t}{\lambda-\omega}} = Me^{\frac{\lambda\omega t}{\lambda-\omega}}
\end{aligned}$$

for all $\lambda > \omega$ and all $t \geq 0$. This proves Step 2.

Step 3. *Fix real numbers $\lambda > \mu > \omega$. Then*

$$\|S_\lambda(t)x - S_\mu(t)x\| \leq M^2 e^{\frac{\mu\omega t}{\mu-\omega}} t \|A_\lambda x - A_\mu x\|$$

for all $x \in X$ and all $t \geq 0$.

Since $A_\lambda A_\mu = A_\mu A_\lambda$, we have $A_\lambda S_\mu(t) = S_\mu(t)A_\lambda$ and so

$$\begin{aligned}
S_\lambda(t)x - S_\mu(t)x &= \int_0^t \frac{d}{ds} S_\mu(t-s)S_\lambda(s)x\, ds \\
&= \int_0^t S_\mu(t-s)S_\lambda(s)(A_\lambda x - A_\mu x)\, ds
\end{aligned}$$

for all $x \in X$ and all $t \geq 0$. Hence

$$\begin{aligned}
\|S_\lambda(t)x - S_\mu(t)x\| &\leq \int_0^t \|S_\mu(t-s)\|\,\|S_\lambda(s)\|\, ds\, \|A_\lambda x - A_\mu x\| \\
&\leq M^2 e^{\frac{\mu\omega t}{\mu-\omega}} \int_0^t e^{-\frac{\mu\omega s}{\mu-\omega}} e^{\frac{\lambda\omega s}{\lambda-\omega}}\, ds\, \|A_\lambda x - A_\mu x\| \\
&\leq M^2 e^{\frac{\mu\omega t}{\mu-\omega}} t \|A_\lambda x - A_\mu x\|.
\end{aligned}$$

Here the last step uses the inequality $\frac{\lambda\omega}{\lambda-\omega} \leq \frac{\mu\omega}{\mu-\omega}$. This proves Step 3.

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Step 4. *The limit*

$$S(t)x := \lim_{\lambda\to\infty} S_\lambda(t)x \tag{7.2.8}$$

exists for all $x \in X$ and all $t \geq 0$. The resulting map $S : [0,\infty) \to \mathcal{L}(X)$ is a strongly continuous semigroup that satisfies (7.2.3)*.*

Assume first that $x \in \mathrm{dom}(A)$. Then $\lim_{\lambda\to\infty} A_\lambda x = Ax$ by Step 1. Hence the limit (7.2.8) exists for all $t \geq 0$ by Step 3 and the convergence is uniform on every compact interval $[0,T]$. Since the operator family $\{S_\lambda(t)\}_{\lambda\geq 2\omega}$ is bounded by Step 2 it follows from Theorem 2.1.5 that the limit (7.2.8) exists for all $x \in X$ and that $S(t) \in \mathcal{L}(X)$ for all $t \geq 0$. Apply Theorem 2.1.5 to the operator family $X \to C([0,T],X) : x \mapsto S_\lambda(\cdot)x$ to deduce that the map $[0,T] \to X : t \mapsto S(t)x$ is continuous for all $x \in X$ and all $T > 0$. Moreover,

$$S(s)S(t)x = \lim_{\lambda\to\infty} S_\lambda(s)S_\lambda(t)x = \lim_{\lambda\to\infty} S_\lambda(s+t)x = S(s+t)x$$

for all $s,t \geq 0$ and all $x \in X$ and $S(0)x = \lim_{\lambda\to\infty} S_\lambda(t)x = x$ for all $x \in X$. Thus S is a strongly continuous semigroup. By Step 2 it satisfies the estimate

$$\|S(t)x\| = \lim_{\lambda\to\infty} \|S_\lambda(t)x\| \leq \lim_{\lambda\to\infty} Me^{\frac{\lambda\omega t}{\lambda-\omega}} \|x\| = Me^{\omega t} \|x\|$$

and this proves Step 4.

Step 5. *The operator A is the infinitesimal generator of S.*

Let B be the infinitesimal generator of S and let $x \in \mathrm{dom}(A)$. Then

$$\|S_\lambda(t)A_\lambda x - S(t)Ax\| \leq \|S_\lambda(t)\| \|A_\lambda x - Ax\| + \|S_\lambda(t)Ax - S(t)Ax\|$$

for all $t \geq 0$. Hence it follows from Step 1 and Step 2 that the functions $S_\lambda(\cdot)A_\lambda x : [0,h] \to X$ converge uniformly to $S(\cdot)Ax$ as λ tends to infinity. This implies

$$\int_0^h S(t)Ax\,dt = \lim_{\lambda\to\infty} \int_0^h S_\lambda(t)A_\lambda x\,dt = \lim_{\lambda\to\infty} S_\lambda(h)x - x = S(h)x - x$$

for all $h > 0$ and so

$$\lim_{h\to 0} \frac{S(h)x - x}{h} = \lim_{h\to 0} \frac{1}{h} \int_0^h S(t)Ax\,dt = Ax.$$

This shows that $\mathrm{dom}(A) \subset \mathrm{dom}(B)$ and $B|_{\mathrm{dom}(A)} = A$. Now let $y \in \mathrm{dom}(B)$ and $\lambda > \omega$. Define $x := (\lambda\mathbb{1} - A)^{-1}(\lambda y - By)$. Then $x \in \mathrm{dom}(A) \subset \mathrm{dom}(B)$ and $\lambda x - Bx = \lambda x - Ax = \lambda y - By$. Since $\lambda\mathbb{1} - B : \mathrm{dom}(B) \to X$ is injective by Lemma 7.2.6, this implies $y = x \in \mathrm{dom}(A)$. Thus $\mathrm{dom}(B) \subset \mathrm{dom}(A)$ and so $\mathrm{dom}(B) = \mathrm{dom}(A)$. This proves Step 5 and Theorem 7.2.5. □

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Corollary 7.2.8. *Let X be a complex Banach space and let*

$$A : \mathrm{dom}(A) \to X$$

be a complex linear operator with a dense domain $\mathrm{dom}(A) \subset X$*. Fix two real numbers $M \geq 1$ and ω. Then the following are equivalent.*

(i) *The operator A is the infinitesimal generator of a strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}^c(X)$ that satisfies the estimate* (7.2.3)*.*

(ii) *For every real number $\lambda > \omega$ the operator $\lambda 1\!\!1 - A : \mathrm{dom}(A) \to X$ is bijective and satisfies the estimate* (7.2.4)*.*

(iii) *For every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega$ the operator $\lambda 1\!\!1 - A : \mathrm{dom}(A) \to X$ is bijective and satisfies the estimate*

$$\|(\lambda 1\!\!1 - A)^{-k}\| \leq \frac{M}{(\mathrm{Re}\lambda - \omega)^k} \qquad \text{for all } k \in \mathbb{N}. \tag{7.2.9}$$

Proof. That (i) implies (iii) follows from Lemma 7.2.6 by the same argument that was used in the proof of Theorem 7.2.5. That (iii) implies (ii) is obvious and that (ii) implies (i) follows from Theorem 7.2.5 and the fact that the operators $S_\lambda(t)$ in the proof of Theorem 7.2.5 are complex linear whenever A is complex linear. This proves Corollary 7.2.8. □

7.2.3. Contraction Semigroups. The archetypal example of a contraction semigroup is the heat flow in Example 7.1.6. Here is the general definition.

Definition 7.2.9 (Contraction Semigroup). Let X be a real Banach space. A **contraction semigroup on** X is a strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}(X)$ that satisfies the inequality

$$\|S(t)\| \leq 1 \tag{7.2.10}$$

for all $t \geq 0$.

Definition 7.2.10 (Dissipative Operator). Let X be a complex Banach space. A complex linear operator $A : \mathrm{dom}(A) \to X$ with a dense domain $\mathrm{dom}(A) \subset X$ is called **dissipative** if, for every $x \in \mathrm{dom}(A)$, there exists an element $x^* \in X^*$ such that

$$\|x^*\|^2 = \|x\|^2 = \langle x^*, x \rangle, \qquad \mathrm{Re}\langle x^*, Ax \rangle \leq 0. \tag{7.2.11}$$

When $X = H$ is a complex Hilbert space, a linear operator $A : \mathrm{dom}(A) \to H$ with a dense domain $\mathrm{dom}(A) \subset H$ is dissipative if and only if

$$\mathrm{Re}\langle x, Ax \rangle \leq 0 \tag{7.2.12}$$

for all $x \in \mathrm{dom}(A)$.

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The next theorem characterizes the infinitesimal generators of contraction semigroups. It was proved by Lumer–Phillips [**58**] in 1961. They also introduced the notion of a dissipative operator.

Theorem 7.2.11 (**Lumer–Phillips).** *Let X be a complex Banach space and let $A : \mathrm{dom}(A) \to X$ be a complex linear operator with a dense domain $\mathrm{dom}(A) \subset X$. Then the following are equivalent.*

(i) *The operator A is the infinitesimal generator of a contraction semigroup.*

(ii) *For every real number $\lambda > 0$ the operator $\lambda\mathbb{1} - A : \mathrm{dom}(A) \to X$ is bijective and satisfies the estimate*

$$\left\|(\lambda\mathbb{1} - A)^{-1}\right\| \le \frac{1}{\lambda}. \tag{7.2.13}$$

(iii) *For every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ the operator $\lambda\mathbb{1} - A : \mathrm{dom}(A) \to X$ is bijective and satisfies the estimate*

$$\left\|(\lambda\mathbb{1} - A)^{-1}\right\| \le \frac{1}{\mathrm{Re}\lambda}. \tag{7.2.14}$$

(iv) *The operator A is dissipative and there exists a $\lambda > 0$ such that the operator $\lambda\mathbb{1} - A : \mathrm{dom}(A) \to X$ has a dense image.*

Proof. The equivalence of (i), (ii), and (iii) follows from Corollary 7.2.8 with $M = 1$ and $\omega = 0$. We prove the remaining implications in three steps.

Step 1. *If A is dissipative, then*

$$\|\lambda x - Ax\| \ge \mathrm{Re}\lambda\, \|x\| \tag{7.2.15}$$

for all $x \in \mathrm{dom}(A)$ and all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$.

Let $x \in \mathrm{dom}(A)$ and $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > 0$. Since A is dissipative, there exists an element $x^* \in X^*$ such that (7.2.11) holds. This implies

$$\begin{aligned} \|x\|\,\|\lambda x - Ax\| &= \|x^*\|\,\|\lambda x - Ax\| \\ &\ge \mathrm{Re}\langle x^*, \lambda x - Ax\rangle \\ &= \mathrm{Re}\lambda\langle x^*, x\rangle - \mathrm{Re}\langle x^*, Ax\rangle \\ &\ge \mathrm{Re}\lambda\,\|x\|^2. \end{aligned}$$

Hence

$$\|\lambda x - Ax\| \ge \mathrm{Re}\lambda\,\|x\|$$

and this proves Step 1.

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Step 2. *We prove that (iv) implies (iii).*

Assume A satisfies (iv) and define the set

$$\Omega = \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda > 0 \text{ and } \lambda\mathbb{1} - A \text{ has a dense image}\}.$$

This set is nonempty by (iv). Moreover, it follows from Step 1 that the operator $\lambda\mathbb{1} - A : \mathrm{dom}(A) \to X$ is injective and has a closed image for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$. Hence $\Omega \subset \rho(A)$ and

$$\left\|(\lambda\mathbb{1} - A)^{-1}\right\| \le \frac{1}{\mathrm{Re}\lambda} \qquad \text{for all } \lambda \in \Omega \subset \rho(A). \tag{7.2.16}$$

If $\lambda \in \Omega$ and $|\mu - \lambda| < \mathrm{Re}\lambda$, then $\mathrm{Re}\mu > 0$ and $|\mu - \lambda|\|(\lambda\mathbb{1} - A)^{-1}\| < 1$, hence $\mu \in \rho(A)$ by Lemma 6.1.10, and hence $\mu \in \Omega$. Thus

$$\lambda \in \Omega \text{ and } |\mu - \lambda| < \mathrm{Re}\lambda \qquad \Longrightarrow \qquad \mu \in \Omega. \tag{7.2.17}$$

Fix an element $\lambda \in \Omega$. Then it follows from (7.2.17) that

$$\{\mu \in \mathbb{C} \,|\, \mathrm{Im}\mu = \mathrm{Im}\lambda,\, 0 < \mathrm{Re}\mu < 2\mathrm{Re}\lambda\} \subset \Omega.$$

Thus an induction argument shows that

$$\{\mu \in \mathbb{C} \,|\, \mathrm{Im}\mu = \mathrm{Im}\lambda,\, \mathrm{Re}\mu > 0\} \subset \Omega.$$

Hence it follows from (7.2.17) that $B_{\mathrm{Re}\mu}(\mu) \subset \Omega$ for every $\mu \in \mathbb{C}$ such that $\mathrm{Im}\mu = \mathrm{Im}\lambda$ and $\mathrm{Re}\mu > 0$. The union of these open discs is the entire positive half-plane in $\mathbb{C}$. Thus $\{z \in \mathbb{C} \,|\, \mathrm{Re}z > 0\} = \Omega \subset \rho(A)$ and hence it follows from (7.2.16) that A satisfies (iii). This proves Step 2.

Step 3. *We prove that (i) implies (iv).*

Assume that $A : \mathrm{dom}(A) \to X$ is the infinitesimal generator of a contraction semigroup $S : [0, \infty) \to \mathcal{L}^c(X)$. Let $x \in \mathrm{dom}(A)$. By the Hahn–Banach Theorem (Corollary 2.3.23) there exists an element $x^* \in X^*$ such that

$$\|x^*\|^2 = \|x\|^2 = \langle x^*, x\rangle.$$

Since S is a contraction semigroup this implies

$$\mathrm{Re}\langle x^*, S(h)x - x\rangle \le \|x^*\|\|S(h)x\| - \|x\|^2 \le 0$$

for all $h > 0$ and hence

$$\mathrm{Re}\langle x^*, Ax\rangle = \lim_{h\to 0} \frac{\mathrm{Re}\langle x^*, S(h)x - x\rangle}{h} \le 0.$$

This proves Step 3 and Theorem 7.2.11. □

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7.3. The Dual Semigroup

When $S : [0, \infty) \to \mathcal{L}(X)$ is a strongly continuous semigroup on a real Banach space X the dual operators define a semigroup

$$S^* : [0, \infty) \to \mathcal{L}(X^*),$$

called the **dual semigroup**. One might expect that the dual semigroup is again strongly continuous, however, an elementary example shows that this need not always be the case (see Example 7.3.3 below). The failure of strong continuity of the dual semigroup is related to the fact that the Banach space X in Example 7.3.3 is not reflexive. On a reflexive Banach space it turns out that the dual semigroup is always strongly continuous and this is the content of Corollary 7.3.2 below, which will be derived as a consequence of the main theorem about the dual semigroup. The other subsections deal with self-adjoint semigroups and with unitary groups.

7.3.1. The Dual Semigroup and its Infinitesimal Generator. The following theorem is the main result of the present section. It was proved in 1955 by R. S. Phillips [**69**].

Theorem 7.3.1 (Phillips). *Let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup on a real Banach space X and let $A : \mathrm{dom}(A) \to X$ be its infinitesimal generator. Denote by*

$$[0, \infty) \to \mathcal{L}(X^*) : t \mapsto S^*(t) := S(t)^*$$

the dual semigroup and by

$$E := \left\{ x^* \in X^* \,\middle|\, \begin{array}{l} \text{there exists a sequence } x_i^* \in \mathrm{dom}(A^*) \\ \text{such that } \lim_{i \to \infty} \|x_i^* - x^*\| = 0 \end{array} \right\} \tag{7.3.1}$$

the strong closure of the domain of the dual operator $A^* : \mathrm{dom}(A^*) \to X^*$. Then the following hold.

(i) *Let $x^* \in X^*$. Then $x^* \in E$ if and only if $\lim_{t \to 0} \|S^*(t)x^* - x^*\| = 0$.*

(ii) *The closed subspace $E \subset X^*$ is invariant under the operator $S^*(t)$ for every $t \geq 0$ and the map $T : [0, \infty) \to \mathcal{L}(E)$, defined by*

$$T(t) := S^*(t)|_E \qquad \text{for } t \geq 0,$$

is a strongly continuous semigroup.

(iii) *The infinitesimal generator of the strongly continuous semigroup T in part (ii) is the operator $B : \mathrm{dom}(B) \to E$ with*

$$\mathrm{dom}(B) = \{x^* \in \mathrm{dom}(A^*) \mid A^*x^* \in E\}$$

and $Bx^* = A^*x^*$ for $x^* \in \mathrm{dom}(B)$.

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Proof. It follows directly from Lemma 4.1.3 that S^* is a one-parameter semigroup. The remaining assertions are proved in eight steps.

Step 1. *Let $x^* \in X^*$ and $h > 0$ and define the element $x_h^* \in X^*$ by*

$$\langle x_h^*, x\rangle = \frac{1}{h}\int_0^h \langle x^*, S(t)x\rangle\, dt \tag{7.3.2}$$

for $x \in X$. Then $x_h^* \in \mathrm{dom}(A^*)$ and $A^*x_h^* = h^{-1}(S^*(h)x^* - x^*)$.

Let $M := \sup_{0\le t\le h}\|S(t)\|$. The functional $X \to \mathbb{R} : x \mapsto \frac{1}{h}\int_0^h\langle x^*, S(t)x\rangle dt$ is linear and satisfies the inequality

$$\begin{aligned}\left|\frac{1}{h}\int_0^h \langle x^*, S(t)x\rangle\, dt\right| &\le \frac{1}{h}\int_0^h |\langle x^*, S(t)x\rangle|\; dt \\ &\le \frac{1}{h}\int_0^h \|x^*\|\,\|S(t)x\|\; dt \\ &\le M\,\|x^*\|\,\|x\|\end{aligned}$$

for all $x \in X$. Hence (7.3.2) defines an element $x_h^* \in X^*$. For $x \in \mathrm{dom}(A)$ this element satisfies the equation

$$\begin{aligned}\langle x_h^*, Ax\rangle &= \left\langle x^*, \int_0^h \frac{S(t)Ax}{h}dt\right\rangle \\ &= \left\langle x^*, \frac{S(h)x - x}{h}\right\rangle \\ &= \left\langle \frac{S^*(h)x^* - x^*}{h}, x\right\rangle.\end{aligned}$$

Here the second step follows from Lemma 7.1.13. This implies $x_h^* \in \mathrm{dom}(A^*)$ and $A^*x_h^* = h^{-1}(S^*(h)x^* - x^*)$. This proves Step 1.

Step 2. *Let $x^* \in \mathrm{dom}(A^*)$ and $t > 0$. Then $S^*(t)x^* \in \mathrm{dom}(A^*)$ and*

$$A^*S^*(t)x^* = S^*(t)A^*x^*.$$

If $x\in\mathrm{dom}(A)$, then $S(t)x\in\mathrm{dom}(A)$ and $S(t)Ax = AS(t)x$ by Lemma 7.1.13, and hence $\langle S^*(t)A^*x^*, x\rangle = \langle A^*x^*, S(t)x\rangle = \langle x^*, AS(t)x\rangle = \langle S^*(t)x^*, Ax\rangle$. By definition of the dual operator, this implies that $S^*(t)x^* \in \mathrm{dom}(A^*)$ and $A^*S^*(t)x^* = S^*(t)A^*x^*$. This proves Step 2.

Step 3. *Let $x^* \in E$ and $t \ge 0$. Then $S^*(t)x^* \in E$.*

Choose a sequence $x_i^* \in \mathrm{dom}(A^*)$ such that $\lim_{i\to\infty}\|x_i^* - x^*\| = 0$. Then it follows from Step 2 that $S^*(t)x_i^* \in \mathrm{dom}(A^*)$. Since $S^*(t) : X^* \to X^*$ is a bounded linear operator, we also have $\lim_{i\to\infty}\|S^*(t)x_i^* - S^*(t)x^*\| = 0$, and hence $S^*(t)x^* \in E$. This proves Step 3.

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Step 4. *Let* $x^* \in \mathrm{dom}(A^*)$ *and* $x \in X$. *Then*

$$\langle S^*(t)x^* - x^*, x\rangle = \int_0^t \langle S^*(s)A^*x^*, x\rangle\, ds.$$

By Example 7.1.15 we have

$$\int_0^t S(s)x\, ds \in \mathrm{dom}(A), \qquad A\int_0^t S(s)x\, ds = S(t)x - x,$$

and hence

$$\begin{aligned}\langle S^*(t)x^* - x^*, x\rangle &= \langle x^*, S(t)x - x\rangle \\ &= \langle x^*, A\int_0^t S(s)x\, ds\rangle \\ &= \langle A^*x^*, \int_0^t S(s)x\, ds\rangle \\ &= \int_0^t \langle A^*x^*, S(s)x\rangle\, ds \\ &= \int_0^t \langle S^*(s)A^*x^*, x\rangle\, ds.\end{aligned}$$

Here the fourth equality follows from Lemma 5.1.8. This proves Step 4.

Step 5. *If* $x^* \in E$, *then* $\lim_{t\to 0} \|S^*(t)x^* - x^*\| = 0$.

Define $M := \sup_{0\le t\le 1} \|S(t)\|$ and let $x^* \in \mathrm{dom}(A^*)$. Then, by Step 4,

$$\begin{aligned}\langle S^*(t)x^* - x^*, x\rangle &= \int_0^t \langle A^*x^*, S(s)x\rangle\, ds \\ &\le \|A^*x^*\| \int_0^t \|S(s)x\|\, ds \\ &\le tM\, \|A^*x^*\|\, \|x\|\end{aligned}$$

for $0 \le t \le 1$. This implies

$$\|S^*(t)x^* - x^*\| = \sup_{x\in X\setminus\{0\}} \frac{\langle S^*(t)x^* - x^*, x\rangle}{\|x\|} \le tM\, \|A^*x^*\|$$

for $0 \le t \le 1$ and so $\lim_{t\to 0} \|S^*(t)x^* - x^*\| = 0$. Since $\mathrm{dom}(A^*)$ is dense in E and $\|S^*(t)\| = \|S(t)\| \le M$ for $0 \le t \le 1$, it follows from the Banach–Steinhaus Theorem 2.1.5 that

$$\lim_{t\to 0} \|S^*(t)x^* - x^*\| = 0 \qquad \text{for all } x^* \in E.$$

This proves Step 5.

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Step 6. *Let* $x^* \in X^*$ *such that* $\lim_{t\to 0}\|S^*(t)x^* - x^*\| = 0$. *Then* $x^* \in E$.

For $h > 0$ let $x_h^* \in X^*$ be as in Step 1. Then $x_h^* \in \mathrm{dom}(A^*)$ and

$$\langle x_h^* - x^*, x\rangle = \frac{1}{h}\int_0^h \langle x^*, S(t)x - x\rangle\, dt.$$

Now fix a constant $\varepsilon > 0$ and choose $\delta > 0$ such that

$$0 \le t < \delta \qquad \Longrightarrow \qquad \|S^*(t)x^* - x^*\| < \varepsilon.$$

Let $0 < h < \delta$. Then

$$\begin{aligned}\frac{\langle x^*, S(t)x - x\rangle}{\|x\|} &= \frac{\langle S^*(t)x^* - x^*, x\rangle}{\|x\|} \\ &\le \|S^*(t)x^* - x^*\| \\ &\le \varepsilon\end{aligned}$$

for $0 \le t \le h$ and $x \in X \setminus \{0\}$, and hence

$$\frac{\langle x_h^* - x^*, x\rangle}{\|x\|} = \frac{1}{h}\int_0^h \frac{\langle x^*, S(t)x - x\rangle}{\|x\|}\, dt \le \varepsilon.$$

Take the supremum over all $x \in X \setminus \{0\}$ to obtain the inequality

$$\|x_h^* - x^*\| = \sup_{x\in X\setminus\{0\}} \frac{\langle x_h^* - x^*, x\rangle}{\|x\|} \le \varepsilon$$

for $0 < h < \delta$. Thus we have proved that

$$\lim_{h\to 0}\|x_h^* - x^*\| = 0,$$

and hence $x^* \in E$. This proves Step 6.

Step 7. *Let* $x^* \in \mathrm{dom}(A^*)$ *such that* $y^* := A^*x^* \in E$. *Then*

$$\lim_{t\to 0}\left\|\frac{S^*(t)x^* - x^*}{t} - y^*\right\| = 0.$$

By Step 3 and Step 5, S^* restricts to a strongly continuous semigroup on the subspace E. Thus the function $[0,\infty) \to E : t \mapsto S^*(t)y^* = S^*(t)A^*x^*$ is continuous and so

$$S^*(t)x^* - x^* = \int_0^t S^*(s)y^*\, ds$$

for all $t > 0$ by Step 4. Hence

$$\left\|\frac{S^*(t)x^* - x^*}{t} - y^*\right\| = \left\|\frac{1}{t}\int_0^t \Big(S^*(s)y^* - y^*\Big)\, ds\right\| \le \sup_{0\le s\le t}\|S^*(s)y^* - y^*\|$$

and this proves Step 7.

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Step 8. *Let* $x^*, y^* \in X^*$ *such that* $\lim_{h\to 0}\left\|\frac{S^*(t)x^*-x^*}{t} - y^*\right\| = 0$. *Then*

$$x^* \in \mathrm{dom}(A^*), \qquad y^* = A^*x^* \in E.$$

It follows from the assumptions of Step 8 that $\lim_{t\to 0}\|S^*(t)x^* - x^*\| = 0$ and hence $x^* \in E$ by Step 6. This implies $t^{-1}(S^*(t)x^* - x^*) \in E$ by Step 3, and so $y^* \in E$ because E is a closed subspace of X^*. Since the function $[0,h] \to E : t \mapsto S^*(t)x^*$ is continuous by Step 3 and Step 5, the element $x_h^* \in X^*$ in Step 1 is given by

$$x_h^* = \frac{1}{h}\int_0^h S^*(t)x^*\,dt$$

and converges to x^* as h tends to zero. Moreover, by Step 1, we have that $x_h^* \in \mathrm{dom}(A^*)$ and $A^*x_h^* = h^{-1}(S^*(h)x^* - x^*)$ converges to y^* as h tends to zero. Since A^* is a closed operator, this implies $x^* \in \mathrm{dom}(A^*)$ and $A^*x^* = y^* \in E$. This proves Step 8.

Part (i) follows from Steps 5 and 6, part (ii) from Steps 3 and 5, and part (iii) from Steps 7 and 8. This proves Theorem 7.3.1. □

Corollary 7.3.2. *Let* X *be a real reflexive Banach space and let* S *be a strongly continuous semigroup on* X *with the infinitesimal generator* A. *Then the dual semigroup* $S^* : [0,\infty) \to \mathcal{L}(X^*)$ *is strongly continuous and its infinitesimal generator is the dual operator* $A^* : \mathrm{dom}(A^*) \to X^*$.

Proof. The domain of the dual operator A^* is weak* dense in X^* by part (iii) of Theorem 6.2.2, and so it is dense because X is reflexive. Hence the result follows from Theorem 7.3.1 with $E = X^*$. □

The shift group in the following example shows that Corollary 7.3.2 does not extend to nonreflexive Banach spaces. In Example 7.3.3 the subspace E is not invariant under A^* although it is invariant under $S^*(t)$ for all t.

Example 7.3.3. Let $X := L^1(\mathbb{R})$ and, for $t \in \mathbb{R}$, define the linear operator $S(t) : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ for $t \in \mathbb{R}$ by

$$(S(t)f)(s) := f(s+t) \qquad \text{for } f \in L^1(\mathbb{R}) \text{ and } s,t \in \mathbb{R}.$$

Then $X^* \cong L^\infty(\mathbb{R})$ and under this identification the dual group is given by

$$(S^*(t)g)(s) := g(s-t) \qquad \text{for } g \in L^\infty(\mathbb{R}) \text{ and } s,t \in \mathbb{R}.$$

For a general element $g \in L^\infty(\mathbb{R})$ the function $\mathbb{R} \to L^\infty(\mathbb{R}) : t \mapsto S^*(t)g$ is weak* continuous but not continuous. In this example the domain of A^* is the space of bounded Lipschitz continuous functions on $\mathbb{R}$. This space is weak* dense in $L^\infty(\mathbb{R})$ but not dense. Its closure is the space $E \subset L^\infty(\mathbb{R})$ of bounded uniformly continuous functions on $\mathbb{R}$.

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7.3.2. Self-Adjoint Semigroups. The next theorem characterizes the infinitesimal generators of self-adjoint semigroups.

Theorem 7.3.4 (**Self-Adjoint Semigroups**). *Let H be a real Hilbert space and let $A : \mathrm{dom}(A) \to H$ be a linear operator with a dense domain $\mathrm{dom}(A) \subset H$. Then the following are equivalent.*

(i) *The operator A is the infinitesimal generator of a strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}(H)$ such that $S(t) = S(t)^*$ for all $t \geq 0$.*

(ii) *The operator A is self-adjoint and*

$$\sup_{x \in \mathrm{dom}(A) \setminus \{0\}} \frac{\langle x, Ax \rangle}{\|x\|^2} < \infty.$$

If these equivalent conditions are satisfied, then

$$\frac{\log\|S(t)\|}{t} = \sup_{\substack{x \in \mathrm{dom}(A) \\ x \neq 0}} \frac{\langle x, Ax \rangle}{\|x\|^2} \tag{7.3.3}$$

for all $t > 0$.

Proof. We prove that (i) implies (ii) and

$$\sup_{\substack{x \in \mathrm{dom}(A) \\ x \neq 0}} \frac{\langle x, Ax \rangle}{\|x\|^2} \leq \frac{\log\|S(t)\|}{t} = \lim_{s \to \infty} \frac{\log\|S(s)\|}{s} \qquad \text{for all } t > 0. \tag{7.3.4}$$

For Hilbert spaces Theorem 7.3.1 asserts that the adjoint A^* of the infinitesimal generator A of a semigroup S is the infinitesimal generator of the adjoint semigroup S^*. Since $S(t)^* = S(t)$ for all $t \geq 0$ in the case at hand, it follows that the infinitesimal generator A is self-adjoint. Moreover,

$$\|S(t)\| = \|S(t)^n\|^{1/n} = \|S(nt)\|^{1/n}$$

by part (i) of Theorem 5.3.15 and hence

$$\frac{\log\|S(t)\|}{t} = \frac{\log\|S(nt)\|}{nt} \qquad \text{for all } t > 0 \text{ and all } n \in \mathbb{N}.$$

Take the limit $n \to \infty$ and use Lemma 7.1.8 to obtain

$$\frac{\log\|S(t)\|}{t} = \omega_0 := \lim_{s \to \infty} \frac{\log\|S(s)\|}{s} \qquad \text{for all } t > 0.$$

This implies $\log\|S(t)\| = t\omega_0$ and so $\|S(t)\| = e^{t\omega_0}$ for all $t > 0$. Thus

$$\langle x, S(t)x \rangle \leq e^{t\omega_0}\|x\|^2 \qquad \text{for all } x \in H \text{ and all } t \geq 0.$$

Differentiate this inequality at $t = 0$ to obtain $\langle x, Ax \rangle \leq \omega_0\|x\|^2$ for every $x \in \mathrm{dom}(A)$. This shows that (i) implies (ii) and (7.3.4).

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We prove that (ii) implies (i). Thus assume A is self-adjoint and

$$\omega := \sup_{\substack{x\in\mathrm{dom}(A)\\ x\neq 0}} \frac{\langle x, Ax\rangle}{\|x\|^2} < \infty.$$

We prove in five steps that A generates a self-adjoint semigroup.

Step 1. *If $\lambda > \omega$ and $x \in \mathrm{dom}(A)$, then $\|\lambda x - Ax\| \geq (\lambda - \omega)\|x\|$.*

Let $x \in \mathrm{dom}(A)$ and $\lambda > \omega$. Then $\langle x, Ax\rangle \leq \omega\,\|x\|^2$ and so

$$\|x\|\|\lambda x - Ax\| \geq \langle x, \lambda x - Ax\rangle \geq (\lambda - \omega)\|x\|^2.$$

This proves Step 1.

Step 2. *If $\lambda > \omega$, then $\lambda\mathbb{1} - A$ is injective and has a closed image.*

Let $\lambda > \omega$. Assume x_n is a sequence in $\mathrm{dom}(A)$ such that $y_n := \lambda x_n - Ax_n$ converges to y. Then x_n is a Cauchy sequence by Step 1 and so converges to some element $x \in H$. Hence $Ax_n = \lambda x_n - y_n$ converges to $\lambda x - y$. Since A has a closed graph by Theorem 6.2.2, this implies $x \in \mathrm{dom}(A)$ and $Ax = \lambda x - y$. Thus $y = \lambda x - Ax \in \mathrm{im}(\lambda\mathbb{1} - A)$, and so $\lambda\mathbb{1} - A$ has a closed image. That it is injective follows directly from the estimate in Step 1. This proves Step 2.

Step 3. *If $\lambda > \omega$, then $\lambda\mathbb{1} - A$ is surjective.*

Let $\lambda > \omega$ and suppose $y \in H$ is orthogonal to the image of $\lambda\mathbb{1} - A$. Then $\langle y, \lambda x\rangle = \langle y, Ax\rangle$ for all $x \in \mathrm{dom}(A)$. Hence $y \in \mathrm{dom}(A^*) = \mathrm{dom}(A)$ and $Ay = A^*y = \lambda y$. Thus $y = 0$ by Step 2. This shows that $\lambda\mathbb{1} - A$ has a dense image. Hence it is surjective by Step 2. This proves Step 3.

Step 4. *The operator A is the infinitesimal generator of a strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}(H)$ such that $\|S(t)\| \leq e^{\omega t}$ for all $t \geq 0$.*

Let $\lambda > \omega$. Then $\lambda\mathbb{1} - A : \mathrm{dom}(A) \to H$ is bijective by Step 2 and Step 3 and $\|(\lambda\mathbb{1} - A)^{-1}\| \leq (\lambda - \omega)^{-1}$ by Step 1. Hence Step 4 follows from the Hille–Yosida–Phillips Theorem 7.2.5 with $M = 1$.

Step 5. *The semigroup S in Step 4 is self-adjoint and satisfies* (7.3.3).

The operator $A = A^*$ is the infinitesimal generator of S by Step 4 and of the adjoint semigroup S^* by Theorem 7.3.1. Hence Corollary 7.2.3 asserts that $S(t) = S^*(t)$ for all $t \geq 0$. This implies that A and S satisfy (7.3.4). By (7.3.4), we have $\omega \leq t^{-1}\log\|S(t)\|$ and by Step 4 we have $\|S(t)\| \leq e^{\omega t}$ and hence $t^{-1}\log\|S(t)\| \leq \omega$ for all $t > 0$. Thus equality holds in (7.3.4). This proves (7.3.3) and Theorem 7.3.4. □

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7.3.3. Unitary Groups. On complex Hilbert spaces it is interesting to examine the infinitesimal generators of strongly continuous unitary groups. This is the content of Theorem 7.3.6 below which was proved in 1932 by M. H. Stone [**81**].

Definition 7.3.5. Let H be a complex Hilbert space. A strongly continuous group $S : \mathbb{R} \to \mathcal{L}^c(H)$ is called **unitary** if $\|S(t)x\| = \|x\|$ for all $t \in \mathbb{R}$ and all $x \in H$ or, equivalently,

$$S^*(t) = S(t)^{-1} = S(-t)$$

for all $t \in \mathbb{R}$, where $S^*(t) = S(t)^*$ denotes the adjoint operator of $S(t)$.

Theorem 7.3.6 (**Stone**). *Let H be a complex Hilbert space and suppose that $A : \mathrm{dom}(A) \to H$ is a linear operator with a dense domain $\mathrm{dom}(A) \subset H$. Then the following are equivalent.*

(i) *A is the infinitesimal generator of a unitary group.*

(ii) *The operator $\mathbf{i}A : \mathrm{dom}(A) \to H$ is self-adjoint.*

Proof. We prove that (i) implies (ii). Thus assume that A is the infinitesimal generator of a unitary group $S : \mathbb{R} \to \mathcal{L}^c(H)$. Then

$$S^*(t) = S(t)^{-1} = S(-t) \qquad \text{for all } t \in \mathbb{R}.$$

The operator $-A : \mathrm{dom}(A) \to H$ is the infinitesimal generator of the group $\mathbb{R} \to \mathcal{L}^c(H) : t \mapsto S(-t)$ by Theorem 7.2.4 and $A^* : \mathrm{dom}(A^*) \to H$ is the infinitesimal generator of the group $\mathbb{R} \to \mathcal{L}^c(H) : t \mapsto S^*(t)$ by Theorem 7.3.1. Hence

$$A^* = -A$$

and so

$$(\mathbf{i}A)^* = -\mathbf{i}A^* = \mathbf{i}A.$$

Thus $\mathbf{i}A$ is self-adjoint.

We prove that (ii) implies (i). Suppose that

$$A = \mathbf{i}B,$$

where $B : \mathrm{dom}(B) \to H$ is a complex linear self-adjoint operator. Then A has a dense domain $\mathrm{dom}(A) = \mathrm{dom}(B)$ and a closed graph. Moreover,

$$A^* = (\mathbf{i}B)^* = -\mathbf{i}B^* = -\mathbf{i}B = -A.$$

This implies

$$\mathrm{Re}\langle x, Ax\rangle = \frac{\langle x, Ax\rangle + \langle Ax, x\rangle}{2} = \frac{\langle x, (A + A^*)x\rangle}{2} = 0 \tag{7.3.5}$$

for all $x \in \mathrm{dom}(A)$.

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We prove that the operator $\mathbb{1} - A : \mathrm{dom}(A) \to H$ has a dense image. Assume that $y \in H$ is orthogonal to the image of $\mathbb{1} - A$. Then

$$0 = \langle y, x - Ax \rangle = \langle y, x \rangle - \langle y, Ax \rangle \qquad \text{for all } x \in \mathrm{dom}(A).$$

Hence it follows from the definition of the adjoint operator that

$$y \in \mathrm{dom}(A^*) = \mathrm{dom}(A), \qquad y = A^* y = -Ay.$$

This implies $\|y\|^2 = -\langle y, Ay \rangle = -\langle A^* y, y \rangle = -\|y\|^2$ and so $y = 0$. Hence the operator $\mathbb{1} - A$ has a dense image by the Hahn–Banach Theorem (Corollary 2.3.25).

Since $\mathbb{1} - A$ has a dense image it follows from (7.3.5) and the Lumer–Phillips Theorem 7.2.11 that A is the infinitesimal generator of a contraction semigroup $S : [0, \infty) \to \mathcal{L}^c(H)$. The adjoint semigroup $S^* : [0, \infty) \to \mathcal{L}^c(H)$ is also a contraction semigroup and is generated by the operator A^* by Theorem 7.3.1. Hence $-A = A^*$ is the infinitesimal generator of the semigroup S^* and so S extends to a strongly continuous group $S : \mathbb{R} \to \mathcal{L}^c(H)$ by Theorem 7.2.4. Since S^* is the group generated by $-A = A^*$ it follows that $S(t)^{-1} = S(-t) = S^*(t)$ for all $t \in \mathbb{R}$ and this proves Theorem 7.3.6. □

Example 7.3.7 (Shift Group). Consider the Hilbert space

$$H := L^2(\mathbb{R}, \mathbb{C})$$

and define the operator $A : \mathrm{dom}(A) \to H$ by

$$\begin{aligned} \mathrm{dom}(A) &:= W^{1,2}(\mathbb{R}, \mathbb{C}) \\ &:= \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is absolutely continuous} \\ \text{and } f, \frac{df}{ds} \in L^2(\mathbb{R}, \mathbb{C}) \end{array} \right\}, \\ Af &:= \frac{df}{ds} \qquad \text{for } f \in W^{1,2}(\mathbb{R}, \mathbb{C}). \end{aligned} \tag{7.3.6}$$

Here s is the variable in $\mathbb{R}$. Recall that an absolutely continuous function is almost everywhere differentiable, that its derivative is locally integrable, and that it can be written as the integral of its derivative, i.e. the fundamental theorem of calculus holds in this setting (see [**75**, Thm. 6.19]). The operator

$$\mathbf{i}A = \mathbf{i}\frac{d}{ds} : W^{1,2}(\mathbb{R}, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C})$$

is self-adjoint and hence A generates a unitary group $U : \mathbb{R} \to \mathcal{L}^c(L^2(\mathbb{R}, \mathbb{C}))$. This group is in fact the shift group in Example 7.1.4 given by

$$(U(t)f)(s) = f(s + t) \qquad \text{for } f \in L^2(\mathbb{R}, \mathbb{C}) \text{ and } s, t \in \mathbb{R}.$$

(See also Example 7.3.3 and Exercise 7.7.3.) **Exercise:** Verify the details.

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Example 7.3.8 (**Schrödinger Equation**). **(i)** Define the unbounded linear operator A on the Hilbert space $H := L^2(\mathbb{R},\mathbb{C})$ by

$$(7.3.7)\qquad \begin{aligned} \mathrm{dom}(A) &:= W^{2,2}(\mathbb{R},\mathbb{C}) \\ &:= \left\{ f:\mathbb{R}\to\mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is absolutely continuous and} \\ \frac{df}{dx} \text{ is absolutely continuous and} \\ \int_{-\infty}^{\infty}(|f|^2 + |\frac{df}{dx}|^2 + |\frac{d^2f}{dx^2}|^2)\,dx < \infty \end{array} \right\}, \\ Af &:= \mathbf{i}\hbar\frac{d^2f}{dx^2} \qquad \text{for } f\in W^{2,2}(\mathbb{R},\mathbb{C}). \end{aligned}$$

(See Example 6.1.7.) Here $\hbar$ is a positive real number (**Planck's constant**) and x is the variable in $\mathbb{R}$. The operator

$$\mathbf{i}A = -\hbar\frac{d^2}{dx^2} : W^{2,2}(\mathbb{R},\mathbb{C}) \to L^2(\mathbb{R},\mathbb{C})$$

is self-adjoint and hence A generates a unitary group $U:\mathbb{R}\to\mathcal{L}^c(L^2(\mathbb{R},\mathbb{C}))$. If $f:\mathbb{R}\to\mathbb{C}$ is a smooth function with compact support and $u:\mathbb{R}^2\to\mathbb{C}$ is defined by $u(t,x) := (U(t)f)(x)$, then u satisfies the **Schrödinger equation**

$$(7.3.8)\qquad \mathbf{i}\hbar\frac{\partial u}{\partial t} = -\hbar^2\frac{\partial^2 u}{\partial x^2}$$

with the initial condition $u(0,\cdot) = f$. **Exercise:** Prove that the operator $\mathbf{i}A$ is self-adjoint.

(ii) Another variant of the Schrödinger equation is associated to the operator $A:\mathrm{dom}(A)\to L^2(\mathbb{R},\mathbb{C})$, defined by

(7.3.9)

$$\begin{aligned} \mathrm{dom}(A) &:= \left\{ f:\mathbb{R}\to\mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is absolutely continuous and} \\ \frac{df}{dx} \text{ is absolutely continuous and} \\ \int_{-\infty}^{\infty}(|f|^2 + |-\hbar^2\frac{d^2f}{dx^2} + x^2 f|^2)\,dx < \infty \end{array} \right\}, \\ (Af)(x) &:= \mathbf{i}\hbar\frac{d^2f}{dx^2}(x) + \frac{x^2}{\mathbf{i}\hbar}f(x) \quad \text{for } f\in W^{2,2}(\mathbb{R},\mathbb{C}) \text{ and } x\in\mathbb{R}. \end{aligned}$$

The operator $\mathbf{i}A$ is again self-adjoint and hence the operator A generates a unitary group $U:\mathbb{R}\to\mathcal{L}^c(L^2(\mathbb{R},\mathbb{C}))$. If $f:\mathbb{R}\to\mathbb{C}$ is a smooth function with compact support and $u:\mathbb{R}^2\to\mathbb{C}$ is defined by $u(t,x) := (U(t)f)(x)$, then u satisfies the **Schrödinger equation with quadratic potential**

$$(7.3.10)\qquad \mathbf{i}\hbar\frac{\partial u}{\partial t}(t,x) = -\hbar^2\frac{\partial^2 u}{\partial x^2}(t,x) + x^2 u(t,x)$$

with the initial condition $u(0,\cdot) = f$. **Exercise:** Prove that the operator $\mathbf{i}A$ is self-adjoint.

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Corollary 7.3.9 (**Groups of Isometries).** *Let H be a real Hilbert space and suppose that $A : \mathrm{dom}(A) \to H$ is a linear operator with a dense domain* $\mathrm{dom}(A) \subset H$. *Then the following are equivalent.*

(i) *A is the infinitesimal generator of a group of isometries.*

(ii) *If $\lambda \in \mathbb{R} \setminus \{0\}$, then $\lambda\mathbb{1} - A$ is bijective and $\left\|(\lambda\mathbb{1} - A)^{-1}\right\| \le |\lambda|^{-1}$.*

(iii) $\mathrm{dom}(A^*) = \mathrm{dom}(A)$ *and $A^*x + Ax = 0$ for all $x \in \mathrm{dom}(A)$.*

Proof. By Theorem 7.2.4, a map $S : \mathbb{R} \to \mathcal{L}(H)$ is a strongly continuous group of isometries if and only if both $[0, \infty) \to \mathcal{L}(H) : t \mapsto S(t)$ and $[0, \infty) \to \mathcal{L}(H) : t \mapsto S(-t)$ are contraction semigroups. Hence the equivalence of (i) and (ii) follows from the Lumer–Phillips Theorem 7.2.11. The equivalence of (i) and (iii) follows from Theorem 7.3.6 for the complexified operator $A^c : \mathrm{dom}(A^c) := \mathrm{dom}(A)^c \to H^c$. □

Example 7.3.10 (**Shift Group). (i)** The formula $(L(t)f)(s) := f(s + t)$ for $s, t \in \mathbb{R}$ and $f \in H := L^2(\mathbb{R})$ defines a shift group $L : \mathbb{R} \to \mathcal{L}(H)$ of isometries. Its infinitesimal generator $A : \mathrm{dom}(A) = W^{1,2}(\mathbb{R}) \to L^2(\mathbb{R})$ is given by $Af = f'$ for $f \in W^{1,2}(\mathbb{R})$ and satisfies $A^* = -A$. (See Example 7.1.4.)

(ii) The formulas $(R(t)f)(s) := f(s - t)$ for $s \ge t \ge 0$ and $(R(t)f)(s) := 0$ for $t > s \ge 0$ and $f \in H := L^2([0, \infty))$ define a semigroup $R : [0, \infty) \to \mathcal{L}(H)$ of isometric embeddings. The infinitesimal generator $B : \mathrm{dom}(B) \to H$ has the domain $\mathrm{dom}(B) = W_0^{1,2}([0, \infty)) := \{f \in W^{1,2}([0, \infty)) \,|\, f(0) = 0\}$ and is given by $Bf = -f'$. Its adjoint has the domain $\mathrm{dom}(B^*) = W^{1,2}([0, \infty))$ and satisfies $Bf + B^*f = 0$ for $f \in \mathrm{dom}(B) \subsetneq \mathrm{dom}(B^*)$.

Example 7.3.11 (**Wave Equation). (i)** The group $\mathscr{S} : \mathbb{R} \to \mathcal{L}(\mathscr{H})$ on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}, \mathbb{R}^2)$, given by (7.1.15) in Example 7.1.7, consists of isometries and has the infinitesimal generator $\mathscr{A} = -\mathscr{A}^*$ on $\mathscr{H}$, given by $\mathrm{dom}(\mathscr{A}) = W^{1,2}(\mathbb{R}, \mathbb{R}^2)$ and $\mathscr{A}(f, g) = (g', f')$.

(ii) Fix real numbers $a < b$ and consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(t, a) = u(t, b) = 0, \tag{7.3.11}$$

on the compact interval $I := [a, b]$. Equation (7.3.11) gives rise to a strongly continuous group of isometries on the Hilbert space $H := W_0^{1,2}(I) \times L^2(I)$, where $W_0^{1,2}(I) := \left\{f \in W^{1,2}(I) \,|\, f(a) = f(b) = 0\right\}$ and

$$\|(f, g)\|_H := \sqrt{\int_a^b \left(|f'(x)|^2 + |g(x)|^2\right) dx}$$

for $f \in W_0^{1,2}(I)$ and $g \in L^2(I)$. Its infinitesimal generator is the operator

$$\mathrm{dom}(A) = (W^{2,2}(I) \cap W_0^{1,2}(I)) \times W_0^{1,2}(I), \qquad A(f, g) = (g, f'').$$

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7.4. Analytic Semigroups

7.4.1. Properties of Analytic Semigroups. For a strongly continuous semigroup

$$S : [0, \infty) \to \mathcal{L}^c(X)$$

on a complex Banach space X an important question is whether the function $t \mapsto S(t)x$ extends to a holomorphic function on a neighborhood of the positive real axis for all $x \in X$. A necessary condition for the existence of such an extension is *instant regularity*, i.e. the image of the operator $S(t)$ must be contained in the domain of the infinitesimal generator for all $t > 0$. The formal definition involves the sectors

$$\begin{aligned} U_\delta &:= \left\{ z \in \mathbb{C} \setminus \{0\} \,\middle|\, |\arg(z)| < \delta \right\} \\ &= \left\{ re^{\mathbf{i}\theta} \,\middle|\, r > 0 \text{ and } -\delta < \theta < \delta \right\} \end{aligned} \tag{7.4.1}$$

for $0 < \delta < \pi/2$.

Definition 7.4.1 (Analytic Semigroups). Let X be a complex Banach space. A strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}^c(X)$ is called **analytic** if there exist a number $0 < \delta < \pi/2$ and an extension of S to $\overline{U}_\delta$, still denoted by

$$S : \overline{U}_\delta \to \mathcal{L}^c(X),$$

such that, for every $x \in X$, the function

$$\overline{U}_\delta \to X : z \mapsto S(z)x$$

is continuous and its restriction to the interior $U_\delta \subset \mathbb{C}$ is holomorphic.

The next theorem summarizes the basic properties of analytic semigroups. In particular, it shows that the map $S_\theta : [0, \infty) \to \mathcal{L}^c(X)$, defined by

$$S_\theta(t) := S(te^{\mathbf{i}\theta}) \qquad \text{for } t \geq 0, \tag{7.4.2}$$

is a strongly continuous semigroup for $-\delta \leq \theta \leq \delta$, and that its infinitesimal generator is the operator $A_\theta : \mathrm{dom}(A) \to X$ defined by

$$A_\theta x := e^{\mathbf{i}\theta} Ax \qquad \text{for } x \in \mathrm{dom}(A). \tag{7.4.3}$$

It also shows that the semigroups S_θ satisfy an exponential estimate of the form $\|S_\theta(t)\| \leq Me^{\omega \cos(\theta)t}$, where the constants $\omega \in \mathbb{R}$ and $M \geq 1$ can be chosen independent of θ. Let ω_0 be the infimum of all $\omega \in \mathbb{R}$ for which such an estimate exists. Then the spectrum of A is contained in the sector

$$C_\delta := \left\{ \omega_0 + re^{\mathbf{i}\theta} \,|\, r \geq 0,\, \pi/2 - \delta \leq |\theta| \leq \pi \right\} \tag{7.4.4}$$

(see Figure 7.4.1).

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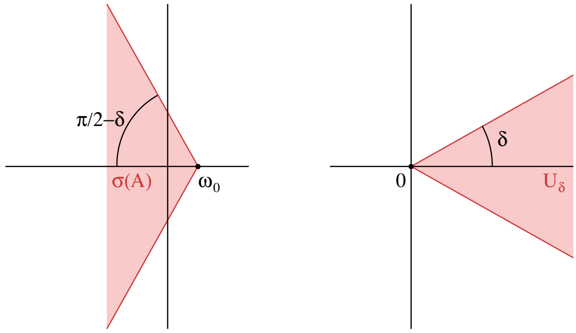

Figure 7.4.1. The spectrum of the generator of an analytic semigroup.

Theorem 7.4.2 (**Analytic Semigroups**). *Let X be a complex Banach space, let $0 < \delta < \pi/2$, let $S : \overline{U}_\delta \to \mathcal{L}^c(X)$ be an analytic semigroup, and let A be its infinitesimal generator. Then the following hold.*

(i) *$S(t+z) = S(t)S(z)$ for all $t, z \in \overline{U}_\delta$.*

(ii) *If $z \in U_\delta$, then $\mathrm{im}(S(z)) \subset \mathrm{dom}(A)$, $AS(z) \in \mathcal{L}^c(X)$, and*

$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}\setminus\{0\}}} \left\| \frac{S(z+h)-S(z)}{h} - AS(z) \right\| = 0. \tag{7.4.5}$$

Moreover, the function $U_\delta \to \mathcal{L}^c(X) : z \mapsto AS(z)$ is holomorphic.

(iii) *If $x \in \mathrm{dom}(A)$ and $z \in \overline{U}_\delta$, then*

$$S(z)x \in \mathrm{dom}(A), \qquad AS(z)x = S(z)Ax.$$

(iv) *If $z \in U_\delta$, then $\mathrm{im}(S(z)) \subset \mathrm{dom}(A^\infty)$.*

(v) *For each*

$$\omega > \omega_0 := \inf_{r>0} r^{-1} \sup\{\log\|S(z)\| \,|\, z \in \overline{U}_\delta,\ \mathrm{Re}(z) = r\} \tag{7.4.6}$$

there exists a constant $M \geq 1$ such that $\|S(z)\| \leq Me^{\omega \mathrm{Re}(z)}$ for all $z \in \overline{U}_\delta$.

(vi) *Let $x \in X$ and $z_0 \in U_\delta$. Choose $r > 0$ such that $B_r(z_0) \subset U_\delta$. Then*

$$S(z)x = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} A^n S(z_0)x \qquad \text{for all } z \in B_r(z_0). \tag{7.4.7}$$

The power series in (7.4.7) converges absolutely and uniformly on every compact subset of $B_r(z_0)$.

(vii) *For $-\delta \leq \theta \leq \delta$ the map S_θ in (7.4.2) is a strongly continuous semigroup whose infinitesimal generator is the operator A_θ in (7.4.3).*

(viii) *If ω_0 is as in (v), then $\sigma(A) \subset C_\delta$ (see equation (7.4.4)).*

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Proof. We prove part (i). Fix a number $t > 0$ and two elements $x \in X$ and $x^* \in X^*$. Define functions $u, v, w : U_\delta \to \mathbb{C}$ by

$$\begin{aligned} u(z) &:= \langle x^*, S(t+z)x\rangle, \\ v(z) &:= \langle x^*, S(z)S(t)x\rangle, \\ w(z) &:= \langle x^*, S(t)S(z)x\rangle = \langle S(t)^*x^*, S(z)x\rangle \end{aligned}$$

for $z \in U_\delta$. By assumption these functions are holomorphic and agree on the positive real axis. Hence they agree on all of U_δ by unique continuation. This shows that

$$S(t+z) = S(z)S(t) = S(t)S(z)$$

for all $t > 0$ and all $z \in \overline{U}_\delta$. Repeat the argument with $t \in U_\delta$ to obtain

$$S(t+z) = S(t)S(z)$$

for all $t, z \in \overline{U}_\delta$. This proves part (i).

We prove part (ii). Let $x \in X$ and define $f : U_\delta \to X$ by

$$f(z) := S(z)x$$

for $z \in U_\delta$. This function is holomorphic by assumption and

$$\frac{f(z+h) - f(z)}{h} = \frac{S(h)S(z)x - S(z)x}{h} \qquad \text{for all } h > 0$$

by part (i). The difference quotient on the left converges to $f'(z)$ as h tends to zero because f is holomorphic. Hence it follows from the definition of the infinitesimal generator that

$$S(z)x \in \mathrm{dom}(A), \qquad AS(z)x = f'(z)$$

for all $z \in U_\delta$. Since f' is holomorphic by Exercise 5.1.13, and every weakly holomorphic operator valued function is holomorphic by Lemma 5.1.12, this proves part (ii).

We prove part (iii). Let $x \in \mathrm{dom}(A)$ and define $f, g : U_\delta \to X$ by

$$f(z) := S(z)Ax, \qquad g(z) := AS(z)x \qquad \text{for } z \in U_\delta.$$

Then f is holomorphic by assumption and g is holomorphic by part (ii). Moreover, the functions agree on the positive real axis by Lemma 7.1.13. Hence they agree on all of U_δ by unique continuation. This proves part (iii) for $z \in U_\delta$. Now let $z \in \overline{U}_\delta$ and choose a sequence $z_n \in U_\delta$ that converges to z. Then it follows from the strong continuity of the map $S : \overline{U}_\delta \to \mathcal{L}^c(X)$ and from what we have just proved that

$$\lim_{n\to\infty} S(z_n)x = S(z)x, \qquad \lim_{n\to\infty} AS(z_n)x = \lim_{n\to\infty} S(z_n)Ax = S(z)Ax.$$

Since A is closed, it follows that $S(z)x \in \mathrm{dom}(A)$ and $AS(z)x = S(z)Ax$. This proves part (iii).

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We prove part (iv). We prove by induction on n that $S(z)x \in \mathrm{dom}(A^n)$ for all $z \in U_\delta$ and all $x \in X$. For $n = 1$ this was established in part (ii). Assume by induction that $S(z)x \in \mathrm{dom}(A^n)$ for all $z \in U_\delta$ and all $x \in X$. Fix two elements $x \in X$ and $z \in U_\delta$. Then it follows from parts (i), (ii), (iii) and the induction hypothesis that

$$AS(z)x = AS(z/2)S(z/2)x = S(z/2)AS(z/2)x \in \mathrm{dom}(A^n)$$

and hence $S(z)x \in \mathrm{dom}(A^{n+1})$. This completes the induction argument and the proof of part (iv).

We prove part (v). The function $\overline{U}_\delta \to [0,\infty) : z \mapsto \|S(z)x\|$ is bounded on every compact subset of U_δ and for every $x \in X$ by strong continuity. Hence it follows from the Uniform Boundedness Theorem 2.1.1 and the analyticity of the semigroup that, for every real number $r > 0$, there exists a constant $c \geq 1$ such that $c^{-1} \leq \|S(z)\| \leq c$ for all $z \in \overline{U}_\delta$ with $\mathrm{Re}(z) \leq r$. Define

$$\omega_0 := \inf_{r>0} \frac{\omega(r)}{r}, \qquad \omega(r) := \sup\left\{\log\|S(z)\| \,\middle|\, z \in \overline{U}_\delta,\, \mathrm{Re}(z) = r\right\}, \tag{7.4.8}$$

and define the functions $g : \overline{U}_\delta \to \mathbb{R}$ and $M : [0,\infty) \to [0,\infty)$ by

$$g(z) := \log\|S(z)\|, \qquad M(r) := \sup_{z\in\overline{U}_\delta,\, \mathrm{Re}(z)\leq r} |g(z)| \tag{7.4.9}$$

for $z \in \overline{U}_\delta$ and $r \geq 0$. Then it follows from part (i) that

$$g(t+z) \leq g(t) + g(z)$$

for all $t, z \in \overline{U}_\delta$. Fix a real number $r > 0$ and let $z \in \overline{U}_\delta \setminus \{0\}$. Then there exist an integer $k \geq 0$ and a number $0 \leq s < r$ such that $\mathrm{Re}(z) = kr + s$. Define $\zeta := \mathrm{Re}(z)^{-1}z$. Then $g(z) = g(kr\zeta + s\zeta)$ and hence

$$\frac{g(z)}{\mathrm{Re}(z)} \leq \frac{kg(r\zeta) + g(s\zeta)}{\mathrm{Re}(z)} = \frac{g(r\zeta)}{r} - \frac{sg(r\zeta)}{r\mathrm{Re}(z)} + \frac{g(s\zeta)}{\mathrm{Re}(z)} \leq \frac{\omega(r)}{r} + \frac{2M(r)}{\mathrm{Re}(z)}.$$

Now fix a constant $\omega > \omega_0$, choose $r > 0$ such that $r^{-1}\omega(r) < \omega$, and then choose $R > 0$ such that $r^{-1}\omega(r) + 2R^{-1}M(r) \leq \omega$. Then each $z \in \overline{U}_\delta$ with $|z| \geq R$ satisfies $|z|^{-1}g(z) \leq \omega$ and hence $\|S(z)\| = e^{g(z)} \leq e^{\omega\mathrm{Re}(z)}$. This proves part (v) with $M := \sup_{z\in\overline{U}_\delta,\,\mathrm{Re}(z)\leq R} e^{-\omega\mathrm{Re}(z)}\|S(z)\|$.

We prove part (vi). Let $x \in X$ and $x^* \in X^*$ and define $f : U_\delta \to \mathbb{C}$ by $f(z) := \langle x^*, S(z)x\rangle$. By parts (ii), (iii), and (iv) the derivatives of f are given by $f^{(n)}(z) = \langle x^*, A^nS(z)x\rangle$ for $n \in \mathbb{N}$ and $z \in U_\delta$. Hence part (vi) follows by carrying over the familiar result in complex analysis about the convergence of power series (e.g. [**1**, p. 179] or [**74**, Thm. 3.43]) to operator valued holomorphic functions. (See also Exercises 5.1.13 and 5.1.14.) This proves part (vi).

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We prove part (vii). Fix a real number $-\delta \le \theta \le \delta$. That S_θ is strongly continuous follows directly from the definition and that it is a semigroup follows from part (i). We must prove that its infinitesimal generator is the operator $A_\theta = e^{\mathbf{i}\theta}A : \mathrm{dom}(A) \to X$ in (7.4.3). To see this, fix an element $x_0 \in \mathrm{dom}(A)$ and define the function

$$x : [0,\infty) \to X$$

by

$$x(t) := S_\theta(t)x_0 = S(te^{\mathbf{i}\theta})x_0 \qquad \text{for } t \ge 0.$$

This function is continuous by assumption and takes values in the subspace $\mathrm{dom}(A_\theta) = \mathrm{dom}(A)$ by part (ii). Moreover, it follows from part (ii) that x is differentiable and

$$\begin{aligned} \frac{d}{dt}S_\theta(t)x &= \lim_{h\to 0}\frac{S(te^{\mathbf{i}\theta} + he^{\mathbf{i}\theta})x - S(te^{\mathbf{i}\theta})x}{h} \\ &= e^{\mathbf{i}\theta}AS(te^{\mathbf{i}\theta})x \\ &= S_\theta(t)A_\theta x \end{aligned}$$

for all $t \ge 0$. Here the last equality follows from part (iii). Thus x is continuously differentiable and satisfies the differential equation $\dot{x} = A_\theta x$. Hence S_θ and A_θ satisfy condition (iii) in Lemma 7.1.17 and so A_θ is the infinitesimal generator of S_θ. This proves part (vii)

We prove part (viii). Recall the definition of the spectrum of a closed unbounded operator in (6.1.9). Let $\lambda \in \sigma(A)$ and fix a real number $-\delta \le \theta \le \delta$. Then $e^{\mathbf{i}\theta}\lambda \in \sigma(A_\theta)$. Let $\omega > \omega_0$. Then part (v) asserts that there is a constant $M \ge 1$ such that $\|S_\theta(t)\| \le Me^{\omega\cos(\theta)t}$ for all $t \ge 0$. By Theorem 7.2.5 this implies that $\mathrm{Re}(e^{\mathbf{i}\theta}\lambda) \le \omega\cos(\theta)$. Since $\omega > \omega_0$ was chosen arbitrarily, this implies $\mathrm{Re}(e^{\mathbf{i}\theta}\lambda) \le \omega_0\cos(\theta)$, i.e.

$$\mathrm{Re}(e^{\mathbf{i}\theta}(\lambda - \omega_0)) \le 0 \qquad \text{for } -\delta \le \theta \le \delta.$$

Thus $\lambda \in C_\delta$. This proves part (viii) and Theorem 7.4.2. □

Example 7.4.3. This elementary example shows that the number ω_0 in (7.4.6) may depend on the domain U_δ on which the semigroup is (chosen to be) defined. Let $\lambda \in \mathbb{C}$ and consider the analytic semigroup $S : \overline{U}_\delta \to \mathcal{L}^c(X)$ on one-dimensional complex Banach space $X = \mathbb{C}$, given by

$$S(z)x = e^{\lambda z}x$$

for $z \in \overline{U}_\delta$ and $x \in X = \mathbb{C}$. This semigroup extends to a holomorphic function on the entire complex plane, so the number $0 < \delta < \pi/2$ can be chosen arbitrarily. We have $\log\|S(z)\| = \log|e^{\lambda z}| = \mathrm{Re}(\lambda z)$ for all $z \in \overline{U}_\delta$ and hence

$$\sup\left\{\log\|S(z)\| \,\middle|\, z \in \overline{U}_\delta,\, \mathrm{Re}(z) = r\right\} = r\big(\mathrm{Re}(\lambda) + \tan(\delta)\,|\mathrm{Im}(\lambda)|\big).$$

Thus $\omega_0 = \mathrm{Re}(\lambda) + \tan(\delta)\,|\mathrm{Im}(\lambda)|$ and so $\sigma(A) = \{\lambda\} \subset C_\delta$.

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7.4.2. Generators of Analytic Semigroups. The next theorem is the main result of this section. It characterizes the infinitesimal generators of analytic semigroups.

Theorem 7.4.4 (**Generators of Analytic Semigroups**). *Let X be a complex Banach space and let $A : \mathrm{dom}(A) \to X$ be a complex linear operator with a dense domain and a closed graph. Fix a real number ω_0. Then the following are equivalent.*

(i) *There exists a number $0 < \delta < \pi/2$ such that A generates an analytic semigroup $S : \overline{U}_\delta \to \mathcal{L}^c(X)$ that satisfies*

$$\lim_{t\to\infty} \frac{\log\|S(t)\|}{t} = \inf_{t>0} \frac{\log\|S(t)\|}{t} \le \omega_0. \tag{7.4.10}$$

(ii) *For each $\omega > \omega_0$ there exists a constant $M \ge 1$ such that*

$$\|(\lambda 1\!\!1 - A)^{-1}\| \le \frac{M}{|\lambda - \omega|} \qquad \text{for all } \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda > \omega. \tag{7.4.11}$$

If these equivalent conditions are satisfied, then $\mathrm{im}(S(t)) \subset \mathrm{dom}(A)$ for all $t > 0$ and, for each $\omega > \omega_0$, there exists a constant $M \ge 1$ such that

$$\|AS(t)x\| \le \frac{M}{t} e^{\omega t}\|x\| \qquad \text{for all } t > 0 \text{ and all } x \in X. \tag{7.4.12}$$

Proof. We prove that (i) implies the last assertion. Thus assume part (i). Then $\mathrm{im}(S(t)) \subset \mathrm{dom}(A)$ for all $t > 0$ by Theorem 7.4.2. Let $\omega > \omega_0$ and assume $\omega_1 := \inf_{r>0} \sup\{\frac{\log\|S(z)\|}{r} \mid z \in \overline{U}_\delta, \mathrm{Re}(z) = r\} < \omega$. (Shrink δ if necessary.) Choose $r > 0$ so small that

$$\overline{B_r(1)} \subset U_\delta, \qquad \omega_1 < \frac{\omega}{1+r}, \qquad \omega_1 < \frac{\omega}{1-r}.$$

(Note that ω might be negative.) Let $t > 0$ and define $\gamma_t : [0,1] \to U_\delta$ by $\gamma_t(s) := t + rte^{2\pi \mathbf{i} s}$ for $0 \le s \le 1$. Fix an element $x \in X$. Then $AS(t)x$ is the derivative at $z = t$ of the holomorphic function $U_\delta \to X : z \mapsto S(z)x$ by Theorem 7.4.2. Hence the Cauchy integral formula asserts that

$$AS(t)x = \frac{1}{2\pi\mathbf{i}} \int_{\gamma_t} \frac{S(z)x}{(z-t)^2}\,dz = \frac{1}{rt}\int_0^1 e^{-2\pi\mathbf{i}s} S(t + rte^{2\pi\mathbf{i}s})x\,ds.$$

Choose $M \ge 1$ such that $\|S(z)\| \le Me^{\frac{\omega\mathrm{Re}(z)}{1+r}}$ and $\|S(z)\| \le Me^{\frac{\omega\mathrm{Re}(z)}{1-r}}$ for $z \in \overline{U}_\delta$. Since $(1-r)t \le \mathrm{Re}(t + rte^{2\pi\mathbf{i}s}) \le (1+r)t$ this implies

$$\|AS(t)x\| \le \frac{1}{rt} \sup_{s\in\mathbb{R}} \left\|S(t + rte^{2\pi\mathbf{i}s})x\right\| \le \frac{M}{rt} e^{\omega t}\|x\|.$$

This shows that (i) implies (7.4.12).

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We prove that (i) implies (ii). Thus assume part (i). Let $\omega > \omega_0$ and assume

$$\omega_1 := \inf_{r>0} \sup \left\{ \frac{\log\|S(z)\|}{r} \,\middle|\, z \in \overline{U}_\delta, \, \mathrm{Re}(z) = r \right\} < \omega.$$

(Shrink δ if necessary.) Then, by part (v) of Theorem 7.4.2, there exists a constant $M \geq 1$ such that

$$\|S(z)\| \leq Me^{\omega \mathrm{Re}(z)} \qquad \text{for all } z \in \overline{U}_\delta.$$

Thus the semigroup $S_{-\delta}$ in (7.4.2) satisfies the inequality

$$\|S_{-\delta}(t)\| = \|S(te^{-\mathbf{i}\delta})\| \leq Me^{\omega\cos(\delta)t}$$

for all $t \geq 0$. Since the operator $A_{-\delta} = e^{-\mathbf{i}\delta}A$ in (7.4.3) is the infinitesimal generator of $S_{-\delta}$, it follows from Corollary 7.2.8 that every complex number λ' with $\mathrm{Re}(\lambda') > \omega\cos(\delta)$ belongs to the resolvent set of $A_{-\delta}$ and satisfies

$$\|(\lambda'\mathbb{1} - e^{-\mathbf{i}\delta}A)^{-1}\| \leq \frac{M}{\mathrm{Re}(\lambda') - \omega\cos(\delta)}. \tag{7.4.13}$$

Define

$$c := \sqrt{\frac{1}{\sin^2(\delta)} + \frac{1}{\cos^2(\delta)}}. \tag{7.4.14}$$

Let $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > \omega$ and $\mathrm{Im}\lambda \geq 0$. Define $\lambda' := e^{-\mathbf{i}\delta}\lambda$. Then

$$\mathrm{Re}(\lambda') - \omega\cos(\delta) = \cos(\delta)(\mathrm{Re}(\lambda) - \omega) + \sin(\delta)\mathrm{Im}(\lambda) > 0,$$

hence

$$\mathrm{Re}(\lambda) - \omega \leq \frac{\mathrm{Re}(\lambda') - \omega\cos(\delta)}{\cos(\delta)}, \qquad \mathrm{Im}(\lambda) \leq \frac{\mathrm{Re}(\lambda') - \omega\cos(\delta)}{\sin(\delta)},$$

and so

$$|\lambda - \omega| \leq c\left(\mathrm{Re}(\lambda') - \omega\cos(\delta)\right). \tag{7.4.15}$$

Since $\mathrm{Re}\lambda' > \omega\cos(\delta)$, the operator $\lambda\mathbb{1} - A = e^{\mathbf{i}\delta}(\lambda'\mathbb{1} - e^{-\mathbf{i}\delta}A)$ is invertible and, by (7.4.13), (7.4.14), and (7.4.15), it satisfies the estimate

$$\|(\lambda\mathbb{1} - A)^{-1}\| = \|(\lambda'\mathbb{1} - e^{-\mathbf{i}\delta}A)^{-1}\| \leq \frac{M}{\mathrm{Re}\lambda' - \omega\cos(\delta)} \leq \frac{cM}{|\lambda - \omega|}.$$

This shows that A satisfies (7.4.11) whenever $\mathrm{Im}(\lambda) \geq 0$. When $\mathrm{Im}(\lambda) \leq 0$ repeat this argument with $A_{-\delta}$ replaced by A_δ and $\lambda' := e^{\mathbf{i}\delta}\lambda$ to obtain that A satisfies (7.4.11). This shows that (i) implies (ii).

We prove that (ii) implies (i). Thus assume part (ii). We prove in eight steps that A generates an analytic semigroup satisfying (7.4.10).

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Step 1. *Let* $\omega > \omega_0$ *and choose* $M \geq 1$ *such that* (7.4.11) *holds. Choose the real number* $0 < \varepsilon_0 \leq \pi/2$ *such that* $\sin(\varepsilon_0) = 1/M$ *and define*

$$M_\varepsilon := \frac{M}{1 - M\sin(\varepsilon)} \qquad \text{for } 0 < \varepsilon < \varepsilon_0. \tag{7.4.16}$$

Then

$$\sigma(A) \subset \{\omega + re^{\mathbf{i}\theta} \,|\, r \geq 0,\, \pi/2 + \varepsilon_0 \leq |\theta| \leq \pi\}$$

and, if $0 < \varepsilon < \varepsilon_0$, *then*

$$\left\|(\lambda\mathbb{1} - A)^{-1}\right\| \leq \frac{M_\varepsilon}{|\lambda - \omega|} \tag{7.4.17}$$

for all $\lambda = \omega + re^{\mathbf{i}\theta}$ *with* $r > 0$ *and* $|\theta| \leq \pi/2 + \varepsilon$.

We prove first that, for all $\lambda \in \mathbb{C}$,

$$\mathrm{Re}\lambda \geq \omega,\ \lambda \neq \omega \quad \Longrightarrow \quad \left\|(\lambda\mathbb{1} - A)^{-1}\right\| \leq \frac{M}{|\lambda - \omega|}. \tag{7.4.18}$$

If $\mathrm{Re}\lambda > \omega$, this holds by assumption. Thus assume $\lambda = \omega + \mathbf{i}t$ for $t \in \mathbb{R}\setminus\{0\}$ and define $\lambda_s := \omega + s + \mathbf{i}t$ for $s > 0$. Then $\|(\lambda_s\mathbb{1} - A)^{-1}\| \leq M/|t|$ for all $s > 0$. With $0 < s < |t|/M$ this implies

$$|\lambda - \lambda_s|\|(\lambda_s\mathbb{1} - A)^{-1}\| \leq \frac{sM}{|t|} < 1$$

and so it follows from Lemma 6.1.10 that $\lambda \in \rho(A)$ and $\|(\lambda\mathbb{1} - A)^{-1}\| \leq \frac{M}{|t| - sM}$. Take the limit $s \to 0$ to obtain the estimate (7.4.18).

Now let $0 < \varepsilon < \varepsilon_0$ and let $\lambda = \omega \pm \mathbf{i}re^{\pm\mathbf{i}\theta}$ with $r > 0$ and $0 < \theta \leq \varepsilon$. Consider the number $\mu := \omega \pm \mathbf{i}r/\cos(\theta)$. It satisfies $|\lambda - \mu| = r\tan(\theta)$ and

$$\|(\mu\mathbb{1} - A)^{-1}\| \leq \frac{M}{|\mu - \omega|} = \frac{M\cos(\theta)}{r} \leq \frac{M}{r}$$

by (7.4.18). Hence

$$|\lambda - \mu|\|(\mu\mathbb{1} - A)^{-1}\| \leq \frac{M\cos(\theta)}{r}|\lambda - \mu| = M\sin(\theta) \leq M\sin(\varepsilon) < 1.$$

Thus $\lambda \in \rho(A)$ and

$$(\lambda\mathbb{1} - A)^{-1} = \sum_{k=0}^{\infty}(\mu - \lambda)^k(\mu\mathbb{1} - A)^{-k-1}$$

by Lemma 6.1.10. Hence

$$\|(\lambda\mathbb{1} - A)^{-1}\| \leq \frac{\|(\mu\mathbb{1} - A)^{-1}\|}{1 - |\lambda - \mu|\|(\mu\mathbb{1} - A)^{-1}\|} \leq \frac{M/r}{1 - M\sin(\varepsilon)} = \frac{M_\varepsilon}{|\lambda - \omega|}.$$

Here the last step uses the equation $|\lambda - \omega| = r$. This proves Step 1.

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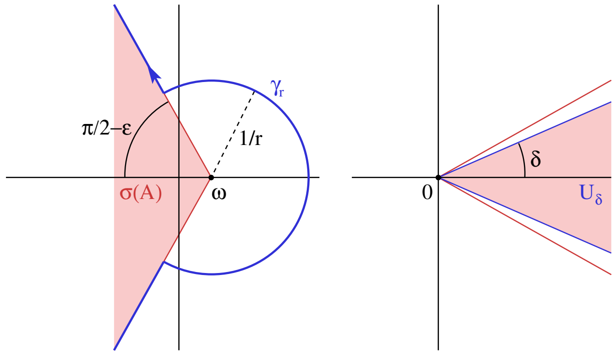

Figure 7.4.2. Integration along γ_r.

Step 2. *Let $\omega > \omega_0$ and $0 < \varepsilon < \varepsilon_0 \le \pi/2$ be as in Step 1. For $r > 0$ define the curve $\gamma_r = \gamma_{r,\varepsilon} : \mathbb{R} \to \mathbb{C}$ by*

$$\gamma_r(t) := \begin{cases} \omega + \frac{1}{r} e^{\mathbf{i} rt(\frac{\pi}{2}+\varepsilon)}, & \text{for } -1/r \le t \le 1/r, \\ \omega + \mathbf{i}te^{-\mathbf{i}\varepsilon}, & \text{for } t \le -1/r, \\ \omega + \mathbf{i}te^{\mathbf{i}\varepsilon}, & \text{for } t \ge 1/r \end{cases} \tag{7.4.19}$$

(see Figure 7.4.2). Then the formula

$$S(z) := \frac{1}{2\pi\mathbf{i}} \int_{\gamma_r} e^{z\zeta}(\zeta\mathbb{1} - A)^{-1}\, d\zeta \qquad \text{for } z \in U_\varepsilon \tag{7.4.20}$$

defines a holomorphic map $S : U_\varepsilon \to \mathcal{L}^c(X)$, which is independent of r.

Step 1 asserts that $\omega + \mathbf{i}te^{\mathbf{i}\varepsilon} \in \rho(A)$ and $\omega - \mathbf{i}te^{-\mathbf{i}\varepsilon} \in \rho(A)$ for $t > 0$ and

$$\left\| ((\omega \pm \mathbf{i}te^{\pm\mathbf{i}\varepsilon})\mathbb{1} - A)^{-1} \right\| \le \frac{M_\varepsilon}{t} \qquad \text{for all } t > 0.$$

Let $z = |z|e^{\mathbf{i}\theta} \in U_\varepsilon$ with $|\theta| < \varepsilon$. Then

$$\operatorname{Re}(z\mathbf{i}e^{\mathbf{i}\varepsilon}) = -|z|\sin(\varepsilon + \theta) < 0, \qquad \operatorname{Re}(-z\mathbf{i}e^{-\mathbf{i}\varepsilon}) = -|z|\sin(\varepsilon - \theta) < 0.$$

Hence

$$\left\| \frac{e^{\pm\mathbf{i}\varepsilon}}{2\pi} e^{z(\omega \pm \mathbf{i}te^{\pm\mathbf{i}\varepsilon})}((\omega \pm \mathbf{i}te^{\pm\mathbf{i}\varepsilon})\mathbb{1} - A)^{-1} \right\| \le \frac{M_\varepsilon e^{|z|\omega\cos(\theta)}}{2\pi} \frac{e^{-t|z|\sin(\varepsilon\pm\theta)}}{t}$$

for all $t \ge 1/r$. This shows that the integrals

$$S^\pm(z) := \frac{e^{\pm\mathbf{i}\varepsilon}}{2\pi} \int_{1/r}^\infty e^{z(\omega\pm\mathbf{i}te^{\pm\mathbf{i}\varepsilon})}((\omega \pm \mathbf{i}te^{\pm\mathbf{i}\varepsilon})\mathbb{1} - A)^{-1}\, dt$$

converge in $\mathcal{L}^c(X)$. That the map $S : U_\varepsilon \to \mathcal{L}^c(X)$ is holomorphic follows from the definition and the convergence of the integrals. That it is independent of the choice of r follows from Step 1 and the Cauchy integral formula. This proves Step 2.

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Step 3. *Let ε and S be as in Step 2 and let $0 < \delta < \varepsilon$. Then there exists a constant $M_{\delta,\varepsilon} \geq 1$ such that*

$$\|S(z)\| \leq M_{\delta,\varepsilon} e^{\omega \mathrm{Re}(z)}$$

for all $z \in \overline{U}_\delta \setminus \{0\}$.

Let $z \in \overline{U}_\delta \setminus \{0\}$ and choose $r := |z|$ in (7.4.19). Then $z = re^{\mathbf{i}\theta}$ with $|\theta| \leq \delta$. Hence, by Step 2,

$$\begin{aligned}
S(z) &= \frac{1}{2\pi\mathbf{i}} \int_{\gamma_r} e^{z\zeta} (\zeta \mathbb{1} - A)^{-1}\, d\zeta \\
&= \frac{1}{2\pi\mathbf{i}} \int_{-\infty}^{\infty} e^{z\gamma_r(t)} \dot{\gamma}_r(t) (\gamma_r(t)\mathbb{1} - A)^{-1}\, dt \\
&= \frac{\pi + 2\varepsilon}{4\pi} \int_{-1/r}^{1/r} e^{z(\omega + \frac{1}{r} e^{\mathbf{i}rt(\frac{\pi}{2}+\varepsilon)})} e^{\mathbf{i}rt(\frac{\pi}{2}+\varepsilon)} \left(\left(\omega + \frac{e^{\mathbf{i}rt(\frac{\pi}{2}+\varepsilon)}}{r} \right) \mathbb{1} - A \right)^{-1} dt \\
&\quad + \frac{e^{-\mathbf{i}\varepsilon}}{2\pi} \int_{-\infty}^{-1/r} e^{z(\omega + \mathbf{i}te^{-\mathbf{i}\varepsilon})} ((\omega + \mathbf{i}te^{-\mathbf{i}\varepsilon})\mathbb{1} - A)^{-1}\, dt \\
&\quad + \frac{e^{\mathbf{i}\varepsilon}}{2\pi} \int_{1/r}^{\infty} e^{z(\omega + \mathbf{i}te^{\mathbf{i}\varepsilon})} ((\omega + \mathbf{i}te^{\mathbf{i}\varepsilon})\mathbb{1} - A)^{-1}\, dt \\
&=: S^0(z) + S^-(z) + S^+(z).
\end{aligned}$$

By Step 1, $\|((\omega + r^{-1}e^{\mathbf{i}rt(\frac{\pi}{2}+\varepsilon)})\mathbb{1} - A)^{-1}\| \leq M_\varepsilon r$ and hence

$$\left\|S^0(z)\right\| \leq \frac{\pi + 2\varepsilon}{2\pi r} e^{\omega r \cos(\theta) + 1} M_\varepsilon r \leq M_\varepsilon e^{\omega r \cos(\theta) + 1}.$$

Now use the fact that $\mathrm{Re}(\pm z \mathbf{i} e^{\pm \mathbf{i}\varepsilon}) = -r\sin(\varepsilon \pm \theta) < 0$ to obtain

$$\begin{aligned}
\left\|S^\pm(z)\right\| &\leq \frac{M_\varepsilon e^{\omega r \cos(\theta)}}{2\pi} \int_{1/r}^{\infty} \frac{e^{-tr\sin(\varepsilon \pm \theta)}}{t}\, dt \\
&\leq \frac{M_\varepsilon e^{\omega r \cos(\theta)}}{2\pi} \int_{1/r}^{\infty} \frac{e^{-tr\sin(\varepsilon - \delta)}}{t}\, dt \\
&= \frac{M_\varepsilon e^{\omega r \cos(\theta)}}{2\pi} \int_{1}^{\infty} \frac{e^{-s\sin(\varepsilon - \delta)}}{s}\, ds \\
&\leq \frac{M_\varepsilon e^{\omega r \cos(\theta)}}{2\pi \sin(\varepsilon - \delta)}.
\end{aligned}$$

Since $r\cos(\theta) = \mathrm{Re}(z)$, the last two estimates imply

$$\|S(z)\| \leq M_\varepsilon \left(e + \frac{1}{\pi \sin(\varepsilon - \delta)} \right) e^{\omega \mathrm{Re}(z)} \qquad \text{for all } z \in \overline{U}_\delta \setminus \{0\}.$$

This proves Step 3.

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Step 4. *Let $0 < \delta < \varepsilon < \pi/2$ and let $z \in \overline{U}_\delta$. Choose a real number $r > 0$ and let*

$$\gamma_r = \gamma_{r,\varepsilon} : \mathbb{R} \to \mathbb{C}$$

be given by (7.4.19) *as in Step 2. Then*

$$\frac{1}{2\pi\mathbf{i}} \int_{\gamma_r} \frac{e^{z\zeta}}{\zeta - \omega}\, d\zeta = e^{\omega z}.$$

The loop obtained from $\gamma_r|_{[-T,T]}$ by joining the endpoints with a straight line encircles the number ω with winding number one for $T \geq 1/r$. Moreover, the straight line

$$\beta_T : [-1, 1] \to \mathbb{C}$$

joining the endpoints (from top to bottom) is given by

$$\beta_T(s) := \omega - T\sin(\varepsilon) - \mathbf{i}sT\cos(\varepsilon)$$

and so

$$\begin{aligned} \left| \frac{1}{2\pi\mathbf{i}} \int_{\beta_T} \frac{e^{z\zeta}}{\zeta - \omega}\, d\zeta \right| &= \left| \frac{-T\cos(\varepsilon)}{2\pi} \int_{-1}^{1} \frac{e^{z(\omega - T\sin(\varepsilon) - \mathbf{i}sT\cos(\varepsilon))}}{-T\sin(\varepsilon) - \mathbf{i}sT\cos(\varepsilon)}\, ds \right| \\ &\leq \frac{\cos(\varepsilon)e^{\omega\mathrm{Re}(z)}}{\sin(\varepsilon)\pi} e^{-T\sin(\varepsilon)\mathrm{Re}(z) + T\cos(\varepsilon)|\mathrm{Im}(z)|}. \end{aligned}$$

Since $z \in \overline{U}_\delta$, the last factor is bounded above by $e^{-|z|T\sin(\varepsilon-\delta)}$ and so converges exponentially to zero as T tends to infinity. This proves Step 4.

Step 5. *For $0 < \delta < \varepsilon < \varepsilon_0$ the map*

$$S : \overline{U}_\delta \setminus \{0\} \to \mathcal{L}^c(X)$$

in Step 2 satisfies

$$\lim_{r \to 0} \sup \left\{ \|S(z)x - x\| \,\middle|\, z \in \overline{U}_\delta,\, |z| = r \right\} = 0$$

for all $x \in X$.

Assume first that $x \in \mathrm{dom}(A)$. Let $z \in \overline{U}_\delta \setminus \{0\}$, define $r := |z|$, and let the curve $\gamma_r : \mathbb{R} \to \mathbb{C}$ be given by equation (7.4.19). Then, by Step 2 and Step 4,

$$\begin{aligned} S(z)x - e^{\omega z}x &= \frac{1}{2\pi\mathbf{i}} \int_{\gamma_r} e^{z\zeta} \left((\zeta\mathbb{1} - A)^{-1}x - (\zeta - \omega)^{-1}x \right) d\zeta \\ &= \frac{1}{2\pi\mathbf{i}} \int_{\gamma_r} \frac{e^{z\zeta}}{\zeta - \omega} (\zeta\mathbb{1} - A)^{-1}(Ax - \omega x)\, d\zeta \\ &= \frac{1}{2\pi\mathbf{i}} \int_{-\infty}^{\infty} \frac{\dot\gamma_r(t)e^{z\gamma_r(t)}}{\gamma_r(t) - \omega} (\gamma_r(t)\mathbb{1} - A)^{-1}(Ax - \omega x)\, dt. \end{aligned}$$

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Since

$$\|(\gamma_r(t)\mathbb{1} - A)^{-1}\| \le \frac{M_\varepsilon}{|\gamma_r(t) - \omega|}$$

by Step 1 and

$$|\gamma_r(t) - \omega| \ge \frac{1}{r}$$

by (7.4.19), it follows that

$$\begin{aligned}\|S(z)x - e^{\omega z}x\| &\le \frac{M_\varepsilon}{2\pi}\int_{-\infty}^{\infty} \frac{|\dot\gamma_r(t)|e^{\mathrm{Re}(z\gamma_r(t))}}{|\gamma_r(t)-\omega|^2}\,dt\,\|Ax - \omega x\| \\ &\le \frac{M_\varepsilon}{2\pi}\int_{-\infty}^{\infty} |\dot\gamma_r(t)|e^{\mathrm{Re}(z\gamma_r(t))}\,dt\,\|Ax - \omega x\|\,r^2.\end{aligned}$$

Now

$$\gamma_r(t) - \omega = \frac{1}{r}e^{\mathbf{i}rt(\frac{\pi}{2}+\varepsilon)} \qquad \text{for } |t| \le \frac{1}{r}$$

and

$$\gamma_r(t) - \omega = te^{\mathbf{i}(\frac{\pi}{2}+\varepsilon)} \qquad \text{for } t \ge \frac{1}{r}$$

and

$$\gamma_r(t) - \omega = -te^{-\mathbf{i}(\frac{\pi}{2}+\varepsilon)} \qquad \text{for } t \le -\frac{1}{r}.$$

Thus $|\dot\gamma_r(t)| \le \pi$ for $|t| < 1/r$ and $|\dot\gamma_r(t)| = 1$ for $|t| > 1/r$. Write $z = re^{\mathbf{i}\theta}$ with $|\theta| \le \delta < \varepsilon$ and use the inequality

$$\mathrm{Re}\left(tze^{\mathbf{i}(\frac{\pi}{2}+\varepsilon)}\right) = tr\cos\left(\frac{\pi}{2}+\varepsilon+\theta\right) = -tr\sin(\varepsilon+\theta) \le -tr\sin(\varepsilon-\delta)$$

to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} |\dot\gamma_r(t)|\,e^{\mathrm{Re}(z\gamma_r(t))}\,dt &= e^{\omega\mathrm{Re}(z)}\int_{-1/r}^{1/r} |\dot\gamma_r(t)|\,e^{\mathrm{Re}(\frac{z}{r}e^{\mathbf{i}rt(\frac{\pi}{2}+\varepsilon)})}\,dt \\ &\quad + 2e^{\omega\mathrm{Re}(z)}\int_{1/r}^{\infty} e^{\mathrm{Re}(tze^{\mathbf{i}(\frac{\pi}{2}+\varepsilon)})}\,dt \\ &\le e^{\omega\mathrm{Re}(z)}\left(\frac{2\pi e}{r} + 2\int_{1/r}^{\infty} e^{-tr\sin(\varepsilon-\delta)}\,dt\right) \\ &\le e^{\omega\mathrm{Re}(z)}\left(\frac{2\pi e}{r} + \frac{2}{r\sin(\varepsilon-\delta)}\right).\end{aligned}$$

Combine these inequalities to obtain

$$\begin{aligned}\|S(z)x - e^{\omega z}x\| &\le \frac{M_\varepsilon}{2\pi}\int_{-\infty}^{\infty} |\dot\gamma_r(t)|e^{\mathrm{Re}(z\gamma_r(t))}\,dt\,\|Ax - \omega x\|\,r^2 \\ &\le M_\varepsilon e^{\omega\mathrm{Re}(z)}\left(e + \frac{1}{\pi\sin(\varepsilon-\delta)}\right)\|Ax - \omega x\|\,r\end{aligned}$$

for all $z \in \overline{U}_\delta \setminus \{0\}$ with $|z| = r$. This proves Step 5 in the case $x \in \mathrm{dom}(A)$. The general case follows from the special case by Step 3 and Theorem 2.1.5.

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Step 6. *Let $0 < \varepsilon < \varepsilon_0$, let S be as in Step 2, and let $0 < \delta < \varepsilon$. Extend the map $S : \overline{U}_\delta \setminus \{0\} \to \mathcal{L}^c(X)$ to all of $\overline{U}_\delta$ by setting $S(0) := 1\!\!1$. Then*

$$S : \overline{U}_\delta \to \mathcal{L}^c(X)$$

is strongly continuous and satisfies

$$\frac{S(z+h)x - S(z)x}{h} = \int_0^1 S(z+th)Ax\,dt \tag{7.4.21}$$

for all $x \in \mathrm{dom}(A)$ and all $z, h \in \overline{U}_\delta$.

Strong continuity follows from Step 5. To prove (7.4.21), let $x \in \mathrm{dom}(A)$ and $z, h \in \overline{U}_\delta$. Assume first that $z \neq 0$. Define the curve $\gamma_r = \gamma_{r,\varepsilon} : \mathbb{R} \to \mathbb{C}$ by (7.4.19) as in Step 2. Then

$$\begin{aligned}
\int_0^1 S(z+th)Ax\,dt &= \frac{1}{2\pi\mathbf{i}} \int_0^1 \int_{\gamma_r} e^{(z+th)\zeta} (\zeta 1\!\!1 - A)^{-1} Ax\,d\zeta\,dt \\
&= \frac{1}{2\pi\mathbf{i}} \int_{\gamma_r} \int_0^1 e^{(z+th)\zeta}\,dt (\zeta 1\!\!1 - A)^{-1} Ax\,d\zeta \\
&= \frac{1}{2\pi\mathbf{i}} \int_{\gamma_r} \frac{e^{(z+h)\zeta} - e^{z\zeta}}{h\zeta} (\zeta 1\!\!1 - A)^{-1} Ax\,d\zeta \\
&= \frac{1}{2\pi\mathbf{i}} \int_{\gamma_r} \frac{e^{(z+h)\zeta} - e^{z\zeta}}{h} \left((\zeta 1\!\!1 - A)^{-1} x - \frac{x}{\zeta} \right) d\zeta \\
&= \frac{S(z+h)x - S(z)x}{h}.
\end{aligned}$$

Here the last assertion follows from the fact that, by the same argument as in Step 4, we have

$$\frac{1}{2\pi\mathbf{i}} \int_{\gamma_{r,\varepsilon}} \frac{e^{z\zeta}(e^{h\zeta} - 1)}{h\zeta} d\zeta = 1$$

whenever $r > 0$, $0 < \delta < \varepsilon < \pi/2$, and $z, h \in \overline{U}_\delta$. This proves (7.4.21) in the case $z \neq 0$. In the case $z = 0$ the equation then follows from strong continuity. This proves Step 6.

Step 7. *The map $S : \overline{U}_\delta \to \mathcal{L}^c(X)$ in Step 2 and Step 6 satisfies*

$$S(w+z) = S(w)S(z) \tag{7.4.22}$$

for all $z, w \in \overline{U}_\delta$.

By strong continuity it suffices to prove equation (7.4.22) for $z, w \in U_\delta$. Fix two elements $w, z \in U_\delta$. Choose two numbers $0 < \rho < r$, define the curve $\gamma = \gamma_{r,\varepsilon} : \mathbb{R} \to \mathbb{C}$ by equation (7.4.19) as in Step 2, and define the curve $\beta := \beta_{\rho,\delta} : \mathbb{R} \to \mathbb{C}$ by the same formula with ε replaced by δ and r replaced by ρ (see Figure 7.4.3).

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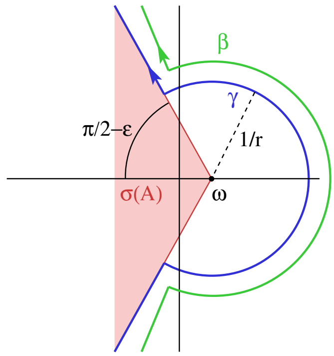

Figure 7.4.3. Integration along β and γ.

With this notation in place, the argument in the proof of Step 4 yields

$$\frac{1}{2\pi\mathbf{i}}\int_\gamma \frac{e^{z\eta}}{\eta-\beta(s)}d\eta = 0, \qquad \frac{1}{2\pi\mathbf{i}}\int_\beta \frac{e^{w\xi}}{\xi-\gamma(t)}d\xi = e^{w\gamma(t)}$$

for all $s,t\in\mathbb{R}$ and all $z,w\in\mathbb{C}$. The key observation is that the integrals along the relevant vertical straight lines converge to zero as in Step 4, and that in the first case the resulting γ-loops have winding number zero about $\beta(s)$, while in the second case the resulting β-loops have winding number one about $\gamma(t)$ for sufficiently large T. Hence

$$\begin{aligned}
S(w)S(z) &= \frac{1}{2\pi\mathbf{i}}\int_\beta e^{w\xi}\,(\xi\mathbb{1}-A)^{-1}\,S(z)\,d\xi \\
&= \frac{1}{2\pi\mathbf{i}}\int_\beta e^{w\xi}\,(\xi\mathbb{1}-A)^{-1}\left(\frac{1}{2\pi\mathbf{i}}\int_\gamma e^{z\eta}\,(\eta\mathbb{1}-A)^{-1}\,d\eta\right)d\xi \\
&= \frac{1}{2\pi\mathbf{i}}\int_\beta \left(\frac{1}{2\pi\mathbf{i}}\int_\gamma e^{w\xi+z\eta}\,(\xi\mathbb{1}-A)^{-1}\,(\eta\mathbb{1}-A)^{-1}\,d\eta\right)d\xi \\
&= \frac{1}{2\pi\mathbf{i}}\int_\beta \left(\frac{1}{2\pi\mathbf{i}}\int_\gamma \frac{e^{w\xi+z\eta}}{\eta-\xi}\Big((\xi\mathbb{1}-A)^{-1}-(\eta\mathbb{1}-A)^{-1}\Big)\,d\eta\right)d\xi \\
&= \frac{1}{2\pi\mathbf{i}}\int_\beta \left(\frac{1}{2\pi\mathbf{i}}\int_\gamma \frac{e^{w\xi+z\eta}}{\eta-\xi}\,d\eta\right)(\xi\mathbb{1}-A)^{-1}\,d\xi \\
&\quad +\frac{1}{2\pi\mathbf{i}}\int_\gamma \left(\frac{1}{2\pi\mathbf{i}}\int_\beta \frac{e^{w\xi+z\eta}}{\xi-\eta}\,d\xi\right)(\eta\mathbb{1}-A)^{-1}\,d\eta \\
&= \frac{1}{2\pi\mathbf{i}}\int_\gamma e^{(w+z)\eta}\,(\eta\mathbb{1}-A)^{-1}\,d\eta \\
&= S(w+z).
\end{aligned}$$

This proves Step 7.

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Step 8. *The map* $S : \overline{U}_\delta \to \mathcal{L}^c(X)$ *is an analytic semigroup. It satisfies* (7.4.10) *and its infinitesimal generator is the operator* A.

That S is an analytic semigroup follows from Step 6 and Step 7, and the estimate (7.4.10) follows from Step 3 by taking the limit $\omega \to \omega_0$. Now let $x \in \mathrm{dom}(A)$ and $t > 0$. Then the integral

$$S(t)x = \frac{1}{2\pi\mathbf{i}} \int_{\gamma_r} e^{t\zeta}(\zeta\mathbb{1} - A)^{-1}x\, d\zeta$$

in (7.4.20) converges in the Banach space $\mathrm{dom}(A)$ with the graph norm. Hence we have $S(t)x \in \mathrm{dom}(A)$ and

$$AS(t)x = \frac{1}{2\pi\mathbf{i}} \int_{\gamma_r} e^{t\zeta}(\zeta\mathbb{1} - A)^{-1}Ax\, d\zeta = S(t)Ax.$$

Moreover,

$$S(t)x - x = \int_0^t S(s)Ax\, ds$$

by Step 6. Hence A and S satisfy condition (ii) in Lemma 7.1.17 and so A is the infinitesimal generator of S. This proves Step 8 and Theorem 7.4.4. □

7.4.3. Examples of Analytic Semigroups. By Theorem 7.4.2 an analytic semigroup $S : [0,\infty) \to \mathcal{L}^c(X)$ on a complex Banach space X with infinitesimal generator $A : \mathrm{dom}(A) \to X$ satisfies $\mathrm{im}(S(t)) \subset \mathrm{dom}(A)$ for all $t > 0$. Hence a group of operators $S : \mathbb{R} \to \mathcal{L}^c(X)$ cannot be analytic unless its infinitesimal generator is a bounded operator (see Lemma 7.1.18 and Theorem 7.2.4).

Example 7.4.5 (Self-Adjoint Semigroups). Let H be a complex Hilbert space and let $A : \mathrm{dom}(A) \to H$ be a self-adjoint operator such that

$$\omega_0 := \sup_{x \in \mathrm{dom}(A)\setminus\{0\}} \frac{\langle x, Ax\rangle}{\|x\|^2} < \infty.$$

By Theorem 7.3.4 the operator A is the infinitesimal generator of a strongly continuous self-adjoint semigroup $S : [0,\infty) \to \mathcal{L}^c(H)$. Moreover, if $\lambda \in \mathbb{C}$ satisfies $\mathrm{Re}\lambda > \omega_0$, then $\lambda \in \rho(A)$ and

$$|\lambda - \omega_0|\, \|x\|^2 = \left|\lambda\|x\|^2 - \omega_0\|x\|^2\right| \le \left|\lambda\|x\|^2 - \langle x, Ax\rangle\right| \le \|x\|\|\lambda x - Ax\|$$

for all $x \in X$. This implies

$$\|(\lambda\mathbb{1} - A)^{-1}\| \le \frac{1}{|\lambda - \omega_0|} \qquad \text{for all } \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda > \omega_0.$$

Hence it follows from Theorem 7.4.4 that S is an analytic semigroup. In fact, the proof of Theorem 7.4.4 with $M = 1$ and $\varepsilon_0 = \pi/2$ shows that S extends to a holomorphic function $S : \{z \in \mathbb{C} \,|\, \mathrm{Re}z > \omega_0\} \to \mathcal{L}^c(H)$ on an open half-space and that the spectrum of A is contained in the half-axis $(-\infty, \omega_0]$.

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Example 7.4.6 (Heat Equation). The solutions of the heat equation

$$\partial_t u = \Delta u, \qquad \Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \tag{7.4.23}$$

determine a contraction semigroup on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, given by

$$S(t)f := K_t * f, \qquad K_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \tag{7.4.24}$$

for $t > 0$ and $f \in L^2(\mathbb{R}^n)$ (see Example 7.1.6). Its infinitesimal generator is the Laplace operator $\Delta : W^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ in Example 6.1.6. In the case $p = 2$ the semigroup S is self-adjoint, and so is analytic by Example 7.4.5. In general, one can verify directly that the formula (7.4.24) is well-defined for every complex number t with positive real part and defines a holomorphic function on the right half-plane.

Example 7.4.7. This example shows that every closed subset of a sector of the form C_δ in (7.4.4) is the spectrum of the infinitesimal generator of an analytic semigroup on a Hilbert space. Let H be a separable complex Hilbert space, let $(e_i)_{i\in\mathbb{N}}$ be a complex orthonormal basis of H, and let $(\lambda_i)_{i\in\mathbb{N}}$ be a sequence of complex numbers. Define the operator $A_\lambda : \mathrm{dom}(A_\lambda) \to H$ by

$$\begin{aligned} \mathrm{dom}(A_\lambda) &:= \left\{ x \in H \,\middle|\, \sum_{i=1}^{\infty} |\lambda_i|^2 |\langle e_i, x\rangle|^2 < \infty \right\}, \\ A_\lambda x &:= \sum_{i=1}^{\infty} \lambda_i \langle e_i, x\rangle e_i \qquad \text{for } x \in \mathrm{dom}(A_\lambda). \end{aligned} \tag{7.4.25}$$

By Example 7.1.12 this operator generates a strongly continuous semigroup if and only if $\sup_{i\in\mathbb{N}} \mathrm{Re}\lambda_i < \infty$. In this case the semigroup is given by

$$S_\lambda(t)x := \sum_{i=1}^{\infty} e^{\lambda_i t} \langle e_i, x\rangle e_i \qquad \text{for } t \geq 0 \text{ and } x \in H. \tag{7.4.26}$$

(See Example 7.1.3.) The semigroup (7.4.26) is analytic if and only if

$$\sup_{i\in\mathbb{N}} \frac{|\mathrm{Im}\lambda_i|}{\omega - \mathrm{Re}\lambda_i} < \infty \qquad \text{for } \omega > \omega_0 := \sup_{i\in\mathbb{N}} \mathrm{Re}\lambda_i. \tag{7.4.27}$$

Exercise: Show that this condition holds for some $\omega > \omega_0$ if and only if it holds for all $\omega > \omega_0$. Assuming (7.4.27), let $\omega > \omega_0$, choose $0 < \delta < \pi/2$ such that $\sin(\delta)|\mathrm{Im}\lambda_i| \leq \cos(\delta)(\omega - \mathrm{Re}\lambda_i)$ for all i, and define $M := 1/\sin(\delta)$. Show that, for all $\mu \in \mathbb{C}$,

$$\mathrm{Re}(\mu) \geq \omega \quad \Longrightarrow \quad \left\|(\mu\mathbb{1} - A_\lambda)^{-1}\right\| = \sup_{i\in\mathbb{N}} \frac{1}{|\mu - \lambda_i|} \leq \frac{M}{|\mu - \omega|}.$$

Show that $\sigma(A_\lambda) = \overline{\{\lambda_i \,|\, i \in \mathbb{N}\}} \subset \{\omega + re^{\mathbf{i}\theta} \,|\, r \geq 0,\, \pi/2 + \delta \leq |\theta| \leq \pi\}$.

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7.5. Banach Space Valued Measurable Functions

This is a preparatory section. It studies measurable functions on an interval with values in a Banach space, a subject with many applications and of interest in its own right. The first subsection introduces the concept of a strongly measurable function and proves Pettis' Theorem. The next four subsections deal with the Banach space $L^p(I,X)$, the Radon–Nikodým property of a Banach space, the dual space of $L^p(I,X)$, and the Sobolev space $W^{1,p}(I,X)$. All these results will be used in Section 7.6 on the inhomogeneous equation $\dot{x} = Ax + f$ associated to a semigroup.

7.5.1. Measurable Functions. The following definition summarizes the different notions of measurability for functions with values in a Banach space. Although these definitions and many of the results carry over to functions on general measure spaces, in this book we will only use Banach space valued functions on an interval and restrict the discussion to that case.

Definition 7.5.1. Let X be a real Banach space and let $I \subset \mathbb{R}$ be an interval. A function $f : I \to X$ is called

• **weakly continuous** if the function

$$\langle x^*, f\rangle : I \to \mathbb{R}$$

is continuous for all $x^* \in X^*$,

• **weakly measurable** if the function

$$\langle x^*, f\rangle : I \to \mathbb{R}$$

is Borel measurable for all $x^* \in X^*$,

• **measurable** if $f^{-1}(B) \subset I$ is a Borel set for every Borel set $B \subset X$,

• a **measurable step function** if it is measurable and $f(I)$ is a finite set,

• **strongly measurable** if there exists a sequence of measurable step functions $f_n : I \to X$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ for almost all $t \in I$.

The basic example, which illustrates the subtlety of this story is the function $[0,1] \to L^\infty([0,1]) : t \mapsto f_t$, defined by $f_t := \chi_{[0,t]}$, i.e. $f_t(s) = 1$ for $0 \le s \le t$ and $f_t(s) = 0$ for $t < s \le 1$. This function is weakly measurable, but not strongly measurable and is everywhere discontinuous. The same function, understood with values in the Banach space $L^1([0,1])$, is an example of a Lipschitz continuous function which is nowhere differentiable.

It follows directly from the definition that the image of a strongly measurable function $f : I \to X$ is contained in a separable subspace of X. Example 7.5.3 below shows that weakly measurable functions need not satisfy this condition.

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Theorem 7.5.2 (Pettis). *Let X be a real Banach space. Fix two numbers $a < b$ and a function $f : [a,b] \to X$. Then the following hold.*

(i) *Assume X is separable and let $E \subset X^*$ be a linear subspace such that*

$$\|x\| = \sup_{x^* \in E \setminus \{0\}} \frac{|\langle x^*, x\rangle|}{\|x^*\|} \qquad \text{for all } x \in X. \tag{7.5.1}$$

If $\langle x^*, f\rangle$ is measurable for all $x^* \in E$, then f is strongly measurable.

(ii) *If X is separable and f is weakly measurable, then f is strongly measurable.*

(iii) *If f is weakly continuous, then f is strongly measurable.*

(iv) *If f is strongly measurable, then the function $[a,b] \to \mathbb{R} : t \mapsto \|f(t)\|$ is Borel measurable.*

Proof. We prove part (i). Thus assume X is separable and $E \subset X^*$ is a linear subspace that satisfies (7.5.1). Abbreviate $I := [a,b]$ and let $f : I \to X$ be a function such that $\langle x^*, f\rangle : I \to \mathbb{R}$ is measurable for all $x^* \in E$. We prove in three steps that f is strongly measurable.

Step 1. *Let $\xi \in X$ and $r > 0$. Then $f^{-1}(\overline{B_r(\xi)})$ is a Borel subset of I.*

Choose a dense sequence $x_n \in X \setminus \overline{B_r(\xi)}$ and define

$$\varepsilon_n := \frac{1}{2}\big(\|x_n - \xi\| - r\big) > 0 \qquad \text{for } n \in \mathbb{N}.$$

Then $X \setminus \overline{B_r(\xi)} = \bigcup_{n=1}^\infty B_{\varepsilon_n}(x_n)$. For $n \in \mathbb{N}$ choose $x_n^* \in E$ such that

$$\|x_n^*\| = 1, \qquad \langle x_n^*, x_n - \xi\rangle > \|x_n - \xi\| - \varepsilon_n.$$

Then, for all $n \in \mathbb{N}$, all $\eta \in \overline{B_r(\xi)}$, and all $x \in B_{\varepsilon_n}(x_n)$, we have

$$\langle x_n^*, \eta\rangle \le \langle x_n^*, \xi\rangle + r = \langle x_n^*, \xi\rangle + \|x_n - \xi\| - 2\varepsilon_n < \langle x_n^*, x_n\rangle - \varepsilon_n < \langle x_n^*, x\rangle.$$

This implies $\overline{B_r(\xi)} = \bigcap_{n=1}^\infty \{y \in X \,|\, \langle x_n^*, y\rangle \le \langle x_n^*, \xi\rangle + r\}$. Hence

$$f^{-1}(\overline{B_r(\xi)}) = \bigcap_{n=1}^\infty \big\{t \in I \,\big|\, \langle x_n^*, f(t)\rangle \le \langle x_n^*, \xi\rangle + r\big\}$$

is a Borel set. This proves Step 1.

Step 2. *f is measurable.*

Let $U \subset X$ be open. Since X is separable, there exist a sequence $x_n \in X$ and a sequence of real numbers $\varepsilon_n > 0$ such that $U = \bigcup_{n=1}^\infty \overline{B_{\varepsilon_n}(x_n)}$. Hence it follows from Step 1 that $f^{-1}(U) = \bigcup_{n=1}^\infty f^{-1}(\overline{B_{\varepsilon_n}(x_n)})$ is a Borel subset of I. This shows that f is Borel measurable by [**75**, Thm. 1.20].

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Step 3. *f is strongly measurable.*

Since X is separable there exists a dense sequence $x_k \in X$. For $k, n \in \mathbb{N}$ define the set

$$\Sigma_{k,n} := \left\{ t \in I \,\middle|\, \begin{array}{l} \|f(t) - x_k\| < 1/n \text{ and} \\ \|f(t) - x_i\| \geq 1/n \text{ for } i = 1, \dots, k-1 \end{array} \right\}. \tag{7.5.2}$$

This is a Borel subset of I by Step 2. Moreover $\Sigma_{k,n} \cap \Sigma_{\ell,n} = \emptyset$ for $k \neq \ell$ and $\bigcup_{k=1}^{\infty} \Sigma_{k,n} = I$. Hence, for each $n \in \mathbb{N}$, there is an $N_n \in \mathbb{N}$ such that

$$\mu\left(\bigcup_{k=N_n+1}^{\infty} \Sigma_{k,n} \right) < 2^{-n}. \tag{7.5.3}$$

Here μ denotes the restriction of the Lebesgue measure to the Borel σ-algebra of I. Define the functions $f_n : I \to X$ by

$$f_n(t) := \begin{cases} x_k, & \text{for } t \in \Sigma_{k,n} \text{ and } k = 1, \dots, N_n, \\ 0, & \text{for } t \in \bigcup_{k=N_n+1}^{\infty} \Sigma_{k,n}. \end{cases} \tag{7.5.4}$$

These are measurable step functions. Define

$$\Omega := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k=N_n+1}^{\infty} \Sigma_{k,n}, \qquad I \setminus \Omega = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \bigcup_{k=1}^{N_n} \Sigma_{k,n}.$$

Then $\mu(\Omega) = 0$ by (7.5.3) and $\|f_n(t) - f(t)\| < 1/n$ for all $t \in \bigcup_{k=1}^{N_n} \Sigma_{k,n}$ by (7.5.2) and (7.5.4). If $t \in I \setminus \Omega$, then there exists an integer $m \in \mathbb{N}$ such that $t \in \bigcap_{n=m}^{\infty} \bigcup_{k=1}^{N_n} \Sigma_{k,n}$ and hence $\|f_n(t) - f(t)\| < 1/n$ for every integer $n \geq m$. Thus

$$\lim_{n \to \infty} f_n(t) = f(t) \qquad \text{for all } t \in I \setminus \Omega.$$

This proves Step 3 and part (i).

Part (ii) follows from (i) with $E = X^*$ and the Hahn–Banach Theorem (Corollary 2.3.4).

We prove part (iii). Assume $f : I \to X$ is weakly continuous and define

$$X_0 := \overline{\text{span}\{f(t) \,|\, t \in I \cap \mathbb{Q}\}}.$$

If $t \in I$ and $x^* \in X_0^{\perp}$, then $\langle x^*, f(t) \rangle = 0$ by weak continuity. Hence it follows from Corollary 2.3.24 that $f(I) \subset X_0$. Since X_0 is separable by definition, it follows from (ii) that f is strongly measurable.

We prove part (iv). Assume $f : I \to X$ is strongly measurable and choose a sequence of measurable step functions $f_n : I \to X$ that converges almost everywhere to f. Then the sequence $\|f_n\| : I \to \mathbb{R}$ of measurable step functions converges almost everywhere to $\|f\| : I \to \mathbb{R}$ and hence the function $\|f\| : I \to \mathbb{R}$ is measurable. This proves Theorem 7.5.2. □

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The next example shows that the hypothesis that X is separable cannot be removed in part (ii) of Theorem 7.5.2.

Example 7.5.3. **(i)** Let H be a nonseparable real Hilbert space, equipped with an uncountable orthonormal basis

$$\{e_t\}_{0\le t\le 1}.$$

Thus the vectors $e_t \in H$ are parametrized by the elements of the unit interval $[0,1]\subset\mathbb{R}$ and satisfy $\langle e_s, e_t\rangle = 0$ for $s\ne t$ and $\|e_t\|=1$ for all t. The function $f:[0,1]\to H$ defined by $f(t) := e_t$ is not strongly measurable because every Borel set $\Omega\subset[0,1]$ of measure zero has an uncountable complement, so $f([0,1]\setminus\Omega)$ is not contained in a separable subspace of H. However, the function f is weakly measurable because each $x\in H$ has the form $x=\sum_{i=1}^\infty \lambda_i e_{s_i}$ for a sequence $\lambda_i\in\mathbb{R}$ such that $\sum_{i=1}^\infty \lambda_i^2<\infty$ and a sequence of pairwise distinct elements $s_i\in[0,1]$; thus $\langle x, f(t)\rangle = \lambda_i$ for $t=s_i$ and $\langle x, f(t)\rangle = 0$ for $t\notin\{s_i\,|\,i\in\mathbb{N}\}$.

(ii) Let $X := L^\infty([0,1])$ and define the function $f:[0,1]\to L^\infty([0,1])$ by

$$(f(t))(x) := f(t,x) := \begin{cases} 1, & \text{if } 0\le x\le t,\\ 0, & \text{if } t<x\le 1.\end{cases}$$

This function satisfies $\|f(s)-f(t)\|_{L^\infty}=1$ for all $s\ne t$ and the same argument as in part (i) shows that f is not strongly measurable. However, when the same function is considered with values in the Banach space $L^p([0,1])$ for $1\le p<\infty$, it is continuous and hence strongly measurable.

Theorem 7.5.4. *Let X be a Banach space. Fix real numbers $1\le p<\infty$ and $a<b$ and a function $f: I := [a,b]\to X$. The following are equivalent.*

(i) *f is strongly measurable and*

$$\int_a^b \|f(t)\|^p\,dt<\infty.$$

(ii) *For every $\varepsilon>0$ there exists a measurable step function $g: I\to X$ such that the function $I\to\mathbb{R}: t\mapsto \|f(t)-g(t)\|$ is Borel measurable and*

$$\int_a^b \|f(t)-g(t)\|^p\,dt<\varepsilon.$$

(iii) *For every $\varepsilon>0$ there exists a continuous function $g: I\to X$ such that the function $I\to\mathbb{R}: t\mapsto \|f(t)-g(t)\|$ is Borel measurable and*

$$\int_a^b \|f(t)-g(t)\|^p\,dt<\varepsilon.$$

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Proof. We prove that (i) implies (ii). Choose a sequence of measurable step functions $g_n : I \to X$ that converges almost everywhere to f. For $n \in \mathbb{N}$ define the function $f_n : I \to X$ by

$$f_n(t) := \begin{cases} g_n(t), & \text{if } \|g_n(t)\| < \|f(t)\| + 1, \\ 0, & \text{if } \|g_n(t)\| \geq \|f(t)\| + 1, \end{cases} \qquad \text{for } t \in I.$$

Then f_n is a measurable step function for every $n \in \mathbb{N}$ by part (iv) of Theorem 7.5.2. Moreover, $\lim_{n\to\infty} \|f_n(t) - f(t)\|^p = 0$ for almost all $t \in I$ and

$$\|f(t) - f_n(t)\|^p \leq \big(2\,\|f(t)\| + 1\big)^p \leq 4^p\,\|f(t)\|^p + 2^p$$

for all $t \in I$ and all $n \in \mathbb{N}$. The function on the right is integrable by (i). Hence $\lim_{n\to\infty} \int_a^b \|f(t) - f_n(t)\|^p\,dt = 0$ by the Lebesgue Dominated Convergence Theorem. This shows that (i) implies (ii).

We prove that (ii) implies (i). Choose a sequence of measurable step functions $f_n : I \to X$ such that the function $\|f - f_n\| : I \to \mathbb{R}$ is Borel measurable and $\lim_{n\to\infty} \int_a^b \|f(t) - f_n(t)\|^p\,dt = 0$. Then there exists a subsequence f_{n_i} such that $\lim_{i\to\infty} \|f(t) - f_{n_i}(t)\| = 0$ for almost every $t \in I$ by [**75**, Cor. 4.10]. Hence f is strongly measurable. Now choose an integer n such that

$$\int_a^b \|f(t) - f_n(t)\|^p\,dt < 1.$$

Then, by Minkowski's inequality,

$$\left(\int_a^b \|f(t)\|^p\,dt\right)^{1/p} \leq \left(\int_a^b \|f_n(t)\|^p\,dt\right)^{1/p} + 1 < \infty.$$

Hence (ii) implies (i) and the same argument shows that (iii) implies (i).

We prove that (i) implies (iii). For this it suffices to assume that f is a measurable step function with precisely one nonzero value. Let $B \subset I$ be a Borel set and let $x \in X \setminus \{0\}$ and assume $f = \chi_B x$. Fix a constant $\varepsilon > 0$. Since the Lebesgue measure is regular by [**75**, Thm. 2.13], there exists a compact set $K \subset I$ and an open set $U \subset I$ such that

$$K \subset B \subset U, \qquad \mu(U \setminus K) < \frac{\varepsilon}{\|x\|^p}.$$

By Urysohn's Lemma there exists a continuous function $\psi : I \to [0,1]$ such that $\psi(t) = 1$ for all $t \in K$ and $\psi(t) = 0$ for all $t \in I \setminus U$. Define the function $g : I \to X$ by $g := \psi x$. Then $|\psi - \chi_B| \leq \chi_{U\setminus K}$ and hence

$$\int_a^b \|f(t) - g(t)\|^p\,dt \leq \int_{U\setminus K} \|x\|^p\,dt = \mu(U \setminus K)\|x\|^p < \varepsilon.$$

This proves Theorem 7.5.4. □

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The next lemma is a direct consequence of Theorem 7.5.4. It will play a central role in Exercise 7.7.11.

Lemma 7.5.5. *Let X be a Banach space and fix real numbers $1 \le p < \infty$ and $a < b$. Let $f : [a,b] \to X$ be a strongly measurable function such that*

$$\int_a^b \|f(t)\|^p \, dt < \infty.$$

Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $h \in \mathbb{R}$,

$$0 < h < \delta \qquad \Longrightarrow \qquad \int_a^{b-h} \|f(t+h) - f(t)\|^p \, dt < \varepsilon.$$

Proof. Exercise. **Hint:** Prove this first when f is continuous and then use Theorem 7.5.4. □

7.5.2. The Banach Space $L^p(I,X)$. The remainder of this section begins with a discussion of Banach space valued L^p functions on an interval, and then moves on to the Radon–Nikodým property, the dual space of L^p, and the Sobolev space $W^{1,p}$. These are important topics with many applications. In particular, this material will be used in Section 7.6 on the inhomogeneous equation associated to a semigroup.

Let X be a real Banach space, fix real numbers $1 \le p < \infty$ and $a < b$, and abbreviate $I := [a,b]$. Define $L^p(I,X) := \mathcal{L}^p(I,X)/\sim$, where

$$\mathcal{L}^p(I,X) := \left\{ f : I \to X \;\middle|\; \begin{array}{l} f \text{ is strongly measurable} \\ \text{and } \int_a^b \|f(t)\|^p \, dt < \infty \end{array} \right\} \tag{7.5.5}$$

and the equivalence relation is equality almost everywhere. It is often convenient to abuse notation and use f to denote an equivalence class in $L^p(I,X)$ as well as a representative of this class in $\mathcal{L}^p(I,X)$. For $f \in \mathcal{L}^p(I,X)$ define

$$\|f\|_{L^p} := \left(\int_a^b \|f(t)\|^p \, dt \right)^{1/p}. \tag{7.5.6}$$

By the Minkowski inequality $L^p(I,X)$ is a normed vector space. For $p = \infty$ we define $L^\infty(I,X) := \mathcal{L}^\infty(I,X)/\sim$, where

$$\mathcal{L}^\infty(I,X) := \left\{ f : I \to X \;\middle|\; \begin{array}{l} f \text{ is strongly measurable} \\ \text{and bounded} \end{array} \right\} \tag{7.5.7}$$

and the equivalence relation is again given by equality almost everywhere. The norm on $L^\infty(I,X)$ is the essential supremum

$$\|f\|_{L^\infty} := \inf \left\{ \sup_{t \in I \setminus E} \|f(t)\| \;\middle|\; \begin{array}{l} E \subset I \text{ is a Borel set} \\ \text{of Lebesgue measure zero} \end{array} \right\} \tag{7.5.8}$$

for $f \in \mathcal{L}^\infty(I,X)$. We emphasize that these definitions have been chosen such that the functions in $\mathcal{L}^p(I,X)$ are all strongly measurable.

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Theorem 7.5.6. *Let X be a Banach space, let $I \subset \mathbb{R}$ be a compact interval, and let $1 \le p \le \infty$. Then the following hold.*

(i) *Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p(I,X)$. If $p=\infty$, then the sequence $(f_n(t))_{n\in\mathbb{N}}$ converges in X for almost every $t \in I$. If $1 \le p < \infty$, then there exists a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ such that the sequence $(f_{n_i}(t))_{i\in\mathbb{N}}$ converges in X for almost every $t \in I$.*

(ii) *$L^p(I,X)$ is a Banach space.*

(iii) *For $1 \le p < \infty$, the subspace $C_0^\infty(I,X)$ of smooth functions $f : I \to X$ that vanish near the boundary is a dense subset of $L^p(I,X)$.*

(iv) *There exists a unique linear operator*

$$L^p(I,X) \to X : f \mapsto \int_a^b f(t)\,dt,$$

called the **integral**, *such that*

$$\left\langle x^*, \int_a^b f(t)\,dt \right\rangle = \int_a^b \langle x^*, f(t)\rangle\,dt \tag{7.5.9}$$

for all $f \in \mathcal{L}^p(I,X)$ and all $x^* \in X^*$.

Proof. We prove the assertions only for $p < \infty$. The case $p = \infty$ is left to the reader. Let $f_n \in L^p(I,X)$ be a Cauchy sequence. Choose a subsequence f_{n_i} such that $\left\|f_{n_i} - f_{n_{i+1}}\right\|_{L^p} < 2^{-i}$ for all $i \in \mathbb{N}$. Then the same argument as in [**75**, p. 139] shows that f_{n_i} converges almost everywhere to a function $f : I \to X$. Namely, the sequence of Borel measurable functions $\phi_k := \sum_{i=1}^k \left\|f_{n_{i+1}} - f_{n_i}\right\| : I \to [0,\infty)$ is monotonically increasing and satisfies $\|\phi_k\|_{L^p} < 1$ for all k. Hence, by the Lebesgue Monotone Convergence Theorem, the sequence $\phi_k^p : I \to [0,\infty)$ converges to a Borel measurable function $\psi : I \to [0,\infty]$ and

$$\int_a^b \psi(t)\,dt = \lim_{k\to\infty} \int_a^b \phi_k(t)^p\,dt \le 1.$$

Thus there is a Borel set $E \subset I$ of Lebesgue measure zero such that $\psi(t) < \infty$ for all $t \in I \setminus E$ (see [**75**, Lemma 1.47]). Hence the sequence $(f_{n_i}(t))_{i\in\mathbb{N}}$ converges in X for all $t \in I \setminus E$ by Lemma 1.5.1. Define $f : I \to X$ by

$$f(t) := \begin{cases} \lim_{i\to\infty} f_{n_i}(t), & \text{for } t \in I \setminus E, \\ 0, & \text{for } t \in E. \end{cases}$$

By Theorem 7.5.4 and the axiom of countable choice, there exists a sequence of measurable step functions $g_i : I \to X$ such that $\|g_i - f_{n_i}\|_{L^p} < 2^{-i}$ for all $i \in \mathbb{N}$. Use the same argument as above, pass to a further subsequence, and enlarge the Borel set E of Lebesgue measure zero, if necessary, to obtain

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that the sequence $(g_i(t) - f_{n_i}(t))_{i\in\mathbb{N}}$ converges to zero for every $t \in I \setminus E$. Then g_i converges to f almost everywhere, and so f is strongly measurable.

We must prove that $f \in \mathcal{L}^p(I, X)$ and $\lim_{n\to\infty} \|f - f_n\|_{L^p} = 0$. To see this, fix a constant $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\|f_n - f_m\|_{L^p} < \varepsilon$ for all integers $n, m \geq n_0$. Then, by the Lemma of Fatou [**75**, Thm. 1.41],

$$\begin{aligned}\int_a^b \|f_n(t) - f(t)\|^p \, dt &= \int_a^b \liminf_{k\to\infty} \left\|f_n(t) - f_{n_k}(t)\chi_{I\setminus E}(t)\right\|^p \, dt \\ &\leq \liminf_{k\to\infty} \int_a^b \left\|f_n(t) - f_{n_k}(t)\chi_{I\setminus E}(t)\right\|^p \, dt \\ &= \liminf_{k\to\infty} \int_a^b \|f_n(t) - f_{n_k}(t)\|^p \, dt \\ &\leq \varepsilon^p\end{aligned}$$

for all $n \geq n_0$. Hence $\|f\|_{L^p} \leq \|f_{n_0}\|_{L_p} + \varepsilon < \infty$, and so $f \in L^p(I, X)$ and the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^p(I, X)$. Hence $L^p(I, X)$ is a Banach space and this proves (i) and (ii).

We prove (iii). That $C(I, X)$ is dense in $L^p(I, X)$ follows directly from Theorem 7.5.4. Hence multiplication with smooth cutoff functions that vanish near the boundary shows that the space $C_c(I, X)$ of continuous functions with support in the interior of I is also dense in $L^p(I, X)$. Now fix a function $f \in C_c(I, X)$ and choose a smooth function $\rho : \mathbb{R} \to [0, \infty)$ with support in the interval $[-1, 1]$ and mean value 1, and define $\rho_\delta(t) := \delta^{-1}\rho(\delta^{-1}t)$ for $\delta > 0$ and $t \in \mathbb{R}$. Then the function $f_\delta : I \to \mathbb{R}$, defined by

$$f_\delta(t) := (\rho_\delta * f)(t) := \int_{\mathbb{R}} \rho_\delta(t - s) f(s) \, ds$$

for $t \in \mathbb{R}$, is smooth for every $\delta > 0$ and vanishes near the boundary of I for $\delta > 0$ sufficiently small. Moreover, f_δ converges to f uniformly, because

$$\begin{aligned}\sup_{t\in I} \|f_\delta(t) - f(t)\| &= \sup_{t\in I} \left\|\int_{\mathbb{R}} \rho_\delta(t - s)(f(s) - f(t)) \, ds\right\| \\ &\leq \sup \{\|f(s) - f(t)\| \mid s, t \in I,\, |s - t| \leq \delta\}\end{aligned}$$

and f is uniformly continuous. Since $\|f_\delta - f\|_{L^p} \leq |I|^{1/p} \|f_\delta - f\|_{L^\infty}$, this implies $\lim_{\delta\to 0} \|f_\delta - f\|_{L^p} = 0$ and this proves part (iii).

Next observe that the operator $C(I, X) \to X : f \mapsto \int_a^b f(t) \, dt$ in Lemma 5.1.8 is bounded with respect to the L^p norm on $C(I, X)$ by part (vi) of Lemma 5.1.10 and the Hölder inequality. Since the subspace $C(I, X)$ is dense in $L^p(I, X)$ by part (i), the integral extends uniquely to a bounded linear functional on $L^p(I, X)$. Since every linear operator satisfying (7.5.9) is necessarily bounded, this proves part (iv) and Theorem 7.5.6. □

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7.5.3. The Radon–Nikodým Property. The next goal is to examine the dual space of $L^p(I,X)$. This is a surprisingly delicate topic and many mathematicians have worked on this problem, starting with Bochner [**15, 16**]. It has led to the question of whether an absolutely continuous function on an interval with values in a Banach space is almost everywhere differentiable. We begin this discussion by examining the derivative of a continuous function on the domain where it exists.

Lemma 7.5.7. *Let X be a Banach space, let $I=[0,1]$ be the unit interval, and let $F:I\to X$ be a continuous function. Then the set*

$$Z := \{t\in I\,|\,f \text{ is not differentiable at } t\} \tag{7.5.10}$$

is a Borel set, and the function $f:I\to X$ defined by

$$f(t) := \begin{cases} 0, & \text{for } t\in Z,\\ F'(t), & \text{for } t\in I\setminus Z,\end{cases} \tag{7.5.11}$$

is strongly measurable.

Proof. Let $\varepsilon>0$. Then the set

$$E(\varepsilon,h,h') := \left\{ t\in I \;\middle|\; \begin{array}{l}\text{if } t+h\in I \text{ and } t+h'\in I, \text{ then}\\ \left\|\frac{F(t+h)-F(t)}{h} - \frac{F(t+h')-F(t)}{h'}\right\| \le \varepsilon\end{array}\right\}$$

is a Borel set for all $h,h'\in\mathbb{R}\setminus\{0\}$ and hence so is the set

$$E_{\varepsilon,\delta} := \bigcap_{\substack{h,h'\in\mathbb{Q}\\ 0<|h|,|h'|<\delta}} E(\varepsilon,h,h') = \bigcap_{\substack{h,h'\in\mathbb{R}\\ 0<|h|,|h'|<\delta}} E(\varepsilon,h,h')$$

for all $\delta>0$. Here the second equality holds because F is continuous. Thus

$$E := \bigcap_{\substack{\varepsilon\in\mathbb{Q}\\ \varepsilon>0}} \bigcup_{\substack{\delta\in\mathbb{Q}\\ \delta>0}} E_{\varepsilon,\delta} = \bigcap_{\varepsilon>0}\bigcup_{\delta>0} E_{\varepsilon,\delta}$$

is a Borel set. Now the function F is differentiable at an element $t\in I$ if and only if $t\in E$. Thus $Z=I\setminus E$ is a Borel set.

For each $n\in\mathbb{N}$ define the function $f_n:I\to X$ by

$$f_n(t) := \begin{cases} 0, & \text{if } t\in Z,\\ 2^n\big(F(t+2^{-n})-F(t)\big), & \text{if } t\in E \text{ and } 0\le t\le 1/2,\\ 2^n\big(F(t)-F(t-2^{-n})\big), & \text{if } t\in E \text{ and } 1/2<t\le 1.\end{cases} \tag{7.5.12}$$

Let $X_0\subset X$ be the smallest closed subspace that contains the image of F. Then X_0 is a separable subspace of X. For each n the function f_n takes values in X_0 and is weakly measurable, and hence is strongly measurable by part (ii) of Theorem 7.5.2. Moreover, $f(t)=\lim_{t\to\infty} f_n(t)$ for every $t\in I$. Hence f takes values in X_0 and is weakly measurable, and so is strongly measurable by part (ii) of Theorem 7.5.2. This proves Lemma 7.5.7. □

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Let $I \subset \mathbb{R}$ be a compact interval and let $F : I \to X$ be a continuous function with values in a Banach space. Recall that F is called **Lipschitz continuous** if there exists a $c \geq 0$ such that $\|F(s) - F(t)\| \leq c\,|s-t|$ for all $s, t \in I$. Recall that F is called **absolutely continuous** if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every sequence $s_1 \leq t_1 \leq \cdots \leq s_N \leq t_N$ in I with $\sum_i |s_i - t_i| < \delta$ satisfies $\sum_i \|F(s_i) - F(t_i)\| < \varepsilon$.

Lemma 7.5.8. *Let X be a Banach space, let $I = [0,1]$, and let $F : I \to X$ be a Lipschitz continuous function that is almost everywhere differentiable. Then the function $f : I \to X$ defined by* (7.5.10) *and* (7.5.11) *is bounded and strongly measurable and satisfies*

$$F(t) - F(0) = \int_0^t f(s)\,ds \qquad \text{for all } t \in I.$$

Proof. Choose $c > 0$ such that $\|F(s) - F(t)\| \leq c\,|s-t|$ for all $s, t \in I$. Then the functions $f_n : I \to X$ in (7.5.12) satisfy $\|f_n(t)\| \leq c$ for all $t \in I$ and all $n \in \mathbb{N}$. Hence $\|f(t)\| \leq c$ for all $t \in I$. Second, f is strongly measurable by Lemma 7.5.7. Third, the set $Z \subset I$ in Lemma 7.5.7 has Lebesgue measure zero by assumption. Hence for each $x^* \in X^*$ the function $\langle x^*, F\rangle : I \to \mathbb{R}$ is absolutely continuous and its derivative agrees almost everywhere with the function $\langle x^*, f\rangle : I \to \mathbb{R}$. By [**75**, Thm. 6.19], this implies

$$\langle x^*, F(t) - F(0)\rangle = \int_0^t \langle x^*, f(s)\rangle\,ds$$

for all $t \in I$ and all $x^* \in X^*$. This proves Lemma 7.5.8. □

Lemma 7.5.9. *Let X be a Banach space, let $I = [0,1]$, let $f : I \to X$ be a strongly measurable function with $\int_0^1 \|f(t)\|\,dt < \infty$, and define $F : I \to X$ by $F(t) := \int_0^t f(s)\,ds$ for $t \in I$. Then F is absolutely continuous and almost everywhere differentiable with $F'(t) = f(t)$ for almost every $t \in I$.*

Proof. The absolute continuity of F follows as in [**75**, Thm. 6.29]. That F is almost everywhere differentiable with $F' = f$ follows from the Lebesgue Differentiation Theorem [**75**, Thm. 6.14] whose proof carries over verbatim to Banach space valued functions. This proves Lemma 7.5.9. □

With these preparations in place we are now ready to formulate the main problem of this subsection, namely whether or not every Lipschitz continuous function with values in a given Banach space X is almost everywhere differentiable. If it is, then Lemma 7.5.7 shows that its derivative is strongly measurable and Lemma 7.5.8 shows that it is the integral of its derivative. Lemma 7.5.9 shows that the integrals of bounded measurable functions are necessarily almost everywhere differentiable. The next lemma relates this problem to the differentiability of absolutely continuous functions.

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Lemma 7.5.10. *Let X be a Banach space and let $I := [0,1]$ be the unit interval. Then the following are equivalent.*

(i) *Every Lipschitz continuous function $F : I \to X$ is almost everywhere differentiable.*

(ii) *Every absolutely continuous function $F : I \to X$ is almost everywhere differentiable.*

If these equivalent conditions are satisfied, and $F : I \to X$ is an absolutely continuous function, then its derivative $f := F' : I \to X$ is strongly measurable, $\int_0^1 \|f(s)\|\, ds < \infty$, and $F(t) - F(0) = \int_0^t f(s)\, ds$ for all $t \in I$.

Proof. That (ii) implies (i) is obvious, because every Lipschitz continuous function is absolutely continuous. Hence assume (i) and let $F : I \to X$ be an absolutely continuous function. Define $\Phi : [0,1] \to [0,\infty)$ by

$$\Phi(t) := \mathrm{Var}(F|_{[0,t]}) = \sup_{0=t_0<t_1<\cdots<t_N=t} \sum_{i=1}^{N} \|F(t_i) - F(t_{i-1})\|.$$

Then Φ is absolutely continuous and monotone. Denote

$$c := \Phi(1) = \mathrm{Var}(F).$$

Since $\|F(t) - F(s)\| \leq \Phi(t) - \Phi(s)$ for all $0 \leq s \leq t \leq 1$, there is a unique function $G : [0,c] \to X$ such that $G(\Phi(t)) = F(t)$ for all $t \in [0,1]$, and G is Lipschitz continuous with Lipschitz constant 1. Hence, by part (i), G is almost everywhere differentiable and so, by Lemma 7.5.8, there exists a strongly measurable $g : I \to X$ such that

$$\sup_{0\leq\tau\leq c} \|g(\tau)\| \leq 1, \qquad G(\theta) = G(0) + \int_0^\theta g(\tau)\, d\tau$$

for all $\theta \in [0,c]$. Moreover, by [**75**, Thm. 6.19], there exists a Borel measurable function $\phi : I \to [0,\infty)$ with $\int_0^1 |\phi(s)|\, ds < \infty$ such that

$$\Phi(t) = \int_0^t \phi(s)\, ds$$

for all $t \in I$. Hence the function $f := \phi(g \circ \Phi) : I \to X$ is strongly measurable and satisfies $\int_0^1 \|f(s)\|\, ds \leq \int_0^1 \phi(s)\, ds < \infty$ and

$$\int_0^t f(s)\, ds = \int_0^t \phi(s) g(\Phi(s))\, ds = \int_0^{\Phi(t)} g(\tau)\, d\tau = F(t) - F(0)$$

for all $t \in I$. Here the second step uses the fact that $C(I)$ is dense in $L^1(I)$ and so there exists a sequence of continuous functions $\phi_i : [0,1] \to [0,\infty)$ with $\int_0^1 \phi_i(t)\, dt = c$ and $\lim_{i\to\infty} \int_0^1 |\phi_i(t) - \phi(t)| = 0$. Now it follows from Lemma 7.5.9 that F is differentiable almost everywhere and $F' = f$. This proves Lemma 7.5.10. □

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Definition 7.5.11. A Banach space X is said to have the **Radon–Nikodým property** if every Lipschitz continuous function $f : [0,1] \to X$ is almost everywhere differentiable or, equivalently, every absolutely continuous function $f : [0,1] \to X$ is almost everywhere differentiable.

Remark 7.5.12. The reason for this terminology lies in the fact that a Banach space X has the Radon–Nikodým property if and only if it satisfies the following for every measurable space $(M, \mathcal{A})$. *Let $\nu : \mathcal{A} \to X$ be a* **countably additive map***, i.e. if $A_i \in \mathcal{A}$ is a sequence of pairwise disjoint measurable sets, then $\nu\left(\bigcup_{i=1}^\infty A_i\right) = \lim_{n\to\infty} \sum_{i=1}^n \nu(A_i) = \sum_{i=1}^\infty \nu(A_i)$. Assume ν has* **bounded variation***, i.e.*

$$\mu(A) := \sup \left\{ \sum_{i=1}^N \|\nu(A_i)\| \;\middle|\; \begin{array}{l} A_1, \dots, A_N \in \mathcal{A}, \\ A_i \cap A_j = \emptyset \text{ for } i \neq j, \\ A_1 \cup \cdots \cup A_N = A \end{array} \right\} < \infty$$

for all $A \in \mathcal{A}$. Then there exists a strongly $\mathcal{A}$-measurable map $f : M \to X$ with $\int_M \|f\|\, d\mu < \infty$ and $\nu(A) = \int_A f\, d\mu$ for all $A \in \mathcal{A}$.

That this condition is indeed equivalent to the Radon–Nikodým property in Definition 7.5.11 was proved by Bochner–Taylor [**17**] in the late 1930s. For other expositions see [**10, 21, 22**].

Theorem 7.5.13 (**Dunford–Pettis**). **(i)** *If X is a Banach space and its dual space X^* is separable, then X^* has the Radon–Nikodým property.*

(ii) *Every reflexive Banach space has the Radon–Nikodým property.*

Proof. See page 416. □

Remark 7.5.14. **(i)** Part (i) of Theorem 7.5.13 was proved by Gelfand [**29**] using the notion in Definition 7.5.11, and then by Dunford–Pettis [**25**] using the notion in Remark 7.5.12. That Hilbert spaces have the Radon–Nikodým property was first proved by Birkhoff [**14**], and this was extended to all reflexive spaces by Dunford–Pettis [**25**].

(ii) By part (i) of Theorem 7.5.13 the Banach space $X = \ell^1$ has the Radon–Nikodým property. This was first noted by Clarkson [**20**] and was extended by Dunford–Morse [**24**] to all Banach spaces with boundedly complete Schauder bases. Clarkson [**20**] also proved that all uniformly convex Banach spaces have the Radon–Nikodým property.

(iii) Banach spaces that do not have the Radon–Nikodým property include the examples $X = L^\infty([0,1])$ (Bochner [**16**]) and $X = c_0$ and $X = L^1([0,1])$ (Clarkson [**20**]). Hence c_0 and $L^1([0,1])$ cannot be isomorphic to the dual space of any Banach space.

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Proof of Theorem 7.5.13. We prove part (i), following the exposition by Kreuter [**50**]. Let X be a real Banach space with a separable dual space X^* and let $G : I = [0,1] \to X^*$ be a Lipschitz continuous function with $G(0) = 0$ and Lipschitz constant 1. Since X^* is separable, so is X by Theorem 2.4.6. Hence there exists a linearly independent sequence $(x_k)_{k\in\mathbb{N}}$ in X such that

$$X = \overline{Y}, \qquad Y := \left\{ \sum_{k=1}^{N} \lambda_k x_k \,\middle|\, N \in \mathbb{N},\, \lambda_1, \dots, \lambda_N \in \mathbb{R} \right\}.$$

For each $x \in X$ the function $\langle G, x\rangle : I \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $\|x\|$. Hence $\langle G, x\rangle$ is almost everywhere differentiable. For each $k \in \mathbb{N}$ let $Z_k \subset I$ be the set of all $t \in I$ such that $\langle G, x_k\rangle$ is not differentiable at t. Then Z_k is a Borel set by Lemma 7.5.7 and it has Lebesgue measure zero. Hence the Borel set $Z := \bigcup_{k=1}^\infty Z_k$ has Lebesgue measure zero and $\langle G, y\rangle$ is differentiable on $I \setminus Z$ for each $y \in Y$. For $y \in Y$ define the function $g_y : I \to \mathbb{R}$ by

$$g_y(t) := \begin{cases} \lim_{h\to 0} h^{-1}\langle G(t+h) - G(t), y\rangle, & \text{if } t \in I \setminus Z, \\ 0, & \text{if } t \in Z. \end{cases} \tag{7.5.13}$$

This function is measurable and satisfies

$$\langle G(t), y\rangle = \int_0^t g_y(s)\, ds, \qquad \|g_y(t)\| \le \|y\| \qquad \text{for all } t \in I. \tag{7.5.14}$$

Moreover, for each $t \in I$ the functional $Y \to \mathbb{R} : y \mapsto g_y(t)$ is linear and bounded by (7.5.14), and so extends uniquely to a bounded linear functional on all of X. Thus there exists a unique function $g : I \to X^*$ such that

$$\langle g(t), y\rangle = g_y(t) \quad \text{for } t \in I \text{ and } y \in Y, \qquad \sup_{0\le t\le 1} \|g(t)\| \le 1. \tag{7.5.15}$$

By (7.5.14) we have

$$\langle G(t), x\rangle = \int_0^t \langle g(s), x\rangle\, ds \tag{7.5.16}$$

for all $x \in Y$ and all $t \in I$. By continuity in x the function $\langle g, x\rangle$ is Borel measurable for all $x \in X$ and equation (7.5.16) continues to hold for all $x \in X$ and all $t \in I$. Since X^* is separable, it follows from part (i) of Theorem 7.5.2 (with X replaced by X^* and $E := \iota(X) \subset X^{**}$) that the function $g : I \to X^*$ is strongly measurable. Hence $G(t) = \int_0^t g(s)\, ds$ for all $t \in I$ by (7.5.16), and so it follows from the Lebesgue Differentiation Theorem that G is almost everywhere differentiable (see Lemma 7.5.9). This proves part (i).

We prove part (ii). Let X be a reflexive Banach space and let $G : I \to X$ be a Lipschitz continuous function. Denote by $Y \subset X$ the smallest closed subspace of X that contains the image of G. Then Y is separable and is reflexive by Theorem 2.4.4. Hence it follows from part (i) that G is almost everywhere differentiable, and this proves Theorem 7.5.13. □

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7.5.4. The Dual Space of $L^p(I,X)$. It is a natural question to ask how the dual space of $L^p(I,X)$ can be characterized. The obvious candidate for the dual space is $L^q(I,X^*)$ with $1/p+1/q=1$.

Lemma 7.5.15. *Let X be a real Banach space, let $I=[a,b]$ be a compact interval, let $1\le p,q\le\infty$ with $1/p+1/q=1$, and let $g\in\mathcal{L}^q(I,X^*)$. Then the map $\Lambda_g:L^p(I,X)\to\mathbb{R}$, defined by*

$$\Lambda_g(f):=\int_a^b\langle g(t),f(t)\rangle\,dt\qquad\text{for } f\in\mathcal{L}^p(I,X), \tag{7.5.17}$$

is a bounded linear functional with $\|\Lambda_g\|=\|g\|_{L^q}$.

Proof. The function $I\to\mathbb{R}:t\mapsto\langle g(t),f(t)\rangle$ is measurable because f and g are strongly measurable. Moreover, by the Hölder inequality, this function is integrable and satisfies $|\int_a^b\langle g(t),f(t)\rangle\,dt|\le\|g\|_{L^q}\|f\|_{L^p}$. Hence the map $\Lambda_g:L^p(I,X)\to\mathbb{R}$ is a bounded linear functional with $\|\Lambda_g\|\le\|g\|_{L^q}$. Thus the map $L^q(I,X^*)\to L^p(I,X)^*$ is a bounded linear operator of norm less than or equal to one. To prove that it is an isometry, it suffices to prove the equation $\|\Lambda_g\|=\|g\|_{L^q}$ for all elements g of a dense subset of $L^q(I,X^*)$. Such a dense subset is the set of measurable step functions by Theorem 7.5.4, provided that $q<\infty$. Here we focus on the case $1<p,q<\infty$ and leave the remaining cases to the reader. Consider a function of the form

$$g=\sum_{i=1}^m\chi_{B_i}x_i^*$$

for $x_1^*,\dots,x_m^*\in X^*\setminus\{0\}$ and pairwise disjoint Borel sets $B_1,\dots,B_m\subset I$. Fix a number $\varepsilon>0$ and choose elements $x_1,\dots,x_m\in X$ such that $\|x_i\|=1$ and $\langle x_i^*,x_i\rangle>(1-\varepsilon)\|x_i^*\|$ for all i. Define the function $f:I\to X$ by

$$f:=\sum_i\chi_{B_i}\|x_i^*\|^{q-1}x_i.$$

Then

$$\int_I\langle g,f\rangle=\sum_i\mu(B_i)\|x_i^*\|^{q-1}\langle x_i^*,x_i\rangle>(1-\varepsilon)\|g\|_{L^q}^q$$

and

$$\|f\|_{L^p}=\left(\sum_i\mu(B_i)\|x_i^*\|^{p(q-1)}\right)^{1/p}=\left(\sum_i\mu(B_i)\|x_i^*\|^q\right)^{1-1/q}=\|g\|_{L^q}^{q-1}.$$

This implies $\|\Lambda_g\|\ge\|f\|_{L_p}^{-1}\int_I\langle g,f\rangle>(1-\varepsilon)\|g\|_{L_q}$. Since $\varepsilon>0$ was chosen arbitrarily, we find that $\|\Lambda_g\|=\|g\|_{L_q}$ for every measurable step function $g:I\to X^*$ and this proves Lemma 7.5.15. □

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The central question is now under which conditions the isometric embedding $L^q(I, X^*) \to L^p(I, X)^*$ in Lemma 7.5.15 is surjective. The answer depends on the Banach space X and is surprisingly subtle. It was first noted by Bochner [**15, 16**] that a positive answer requires that every absolutely continuous function with values in the dual space X^* is almost everywhere differentiable.

Theorem 7.5.16 (Bochner). *Let X be a Banach space, let $I := [0, 1]$, and let $p, q > 1$ with $1/p + 1/q = 1$. Then the following are equivalent.*

(i) *The isometric embedding $L^q(I, X^*) \to L^p(I, X)^*$ is surjective.*

(ii) *The isometric embedding $L^\infty(I, X^*) \to L^1(I, X)^*$ is surjective.*

(iii) *The dual space X^* has the Radon–Nikodým property.*

Proof. We prove that (i) implies (ii). Let $\Lambda : L^1(I, X) \to \mathbb{R}$ be a bounded linear functional and denote

$$c := \|\Lambda\|.$$

Then Λ restricts to a bounded linear functional on $L^p(I, X)$. Hence by part (i) there is a function $g \in \mathcal{L}^q(I, X^*)$ such that

$$\int_I \langle g, f\rangle = \Lambda(f) \le c\,\|f\|_{L^1}$$

for all $f \in \mathcal{L}^p(I, X)$. We claim that $\|g\|_{L^\infty} \le c$. Otherwise, there exists a constant $\delta > 0$ such that the set

$$A := \{t \in I \mid \|g(t)\| > c + \delta\}$$

has positive measure. By Theorem 7.5.6 there is a sequence of measurable step functions $g_i : I \to X^* \setminus \{0\}$ that converges in L^q and almost everywhere to g. For each i let $f_i : I \to X$ be a measurable step function that satisfies $\langle g_i(t), f_i(t)\rangle \ge (1 - \frac{1}{i})\,\|g_i(t)\|$ and $\|f_i(t)\| = 1$ for all i and t. Then

$$\langle g(t), f_i(t)\rangle \ge \left(1 - \frac{1}{i}\right)\|g_i(t)\| - \|g_i(t) - g(t)\|$$

for all i and t, and hence

$$\liminf_{i\to\infty} \int_I \langle g, \chi_A f_i\rangle \ge \lim_{i\to\infty} \int_A \|g_i\| = \int_A \|g\| \ge (c + \delta)\mu(A).$$

Thus

$$\int_I \langle g, \chi_A f_i\rangle > c\mu(A) = c\,\|\chi_A f_i\|_{L^1}$$

for i sufficiently large. This contradiction shows that $\|g\|_{L^\infty} \le c$ as claimed. This proves that (i) implies (ii).

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We prove that (ii) implies (iii). Let $G : I \to X^*$ be a Lipschitz continuous function with Lipschitz constant c so that

$$\|G(s) - G(t)\| \le c\,|s - t|$$

for all $s, t \in I$. For a step function $f : I \to X$ of the form

$$f = \sum_{i=0}^{N} \chi_{[t_{i-1},t_i)} x_i$$

with $0 = t_0 < t_1 < \cdots < t_N = 1$ and $x_i \in X$ define

$$\Lambda(f) := \sum_{i=1}^{N} \langle G(t_i) - G(t_{i-1}), x_i \rangle.$$

Then

$$|\Lambda(f)| \le c \sum_{i=1}^{N} (t_i - t_{i-1})\,\|x_i\| = c\,\|f\|_{L^1}\,.$$

Thus Λ is a bounded linear functional on a dense subset of $L^1(I, X)$, by Theorem 7.5.6, and hence extends uniquely to a bounded linear functional on $L^1(I, X)$ which will still be denoted by

$$\Lambda : L^1(I, X) \to \mathbb{R}.$$

By part (ii), there exists a bounded strongly measurable function $g : I \to X^*$ such that

$$\int_0^1 \langle g(t), f(t) \rangle\, dt = \Lambda(f)$$

for all $f \in \mathcal{L}^1(I, X)$. Take $f := \chi_{[0,t)} x$ to obtain

$$\begin{aligned} \left\langle \int_0^t g(s)\, ds, x \right\rangle &= \int_0^t \langle g(s), x \rangle\, ds \\ &= \Lambda(\chi_{[0,t)} x) \\ &= \langle G(t) - G(0), x \rangle \end{aligned}$$

for all $t \in I$ and all $x \in X$. This implies

$$\int_0^t g(s)\, ds = G(t) - G(0)$$

for all $t \in I$. Hence it follows from the Lebesgue Differentiation Theorem (see for example [**75**, Thm. 6.14]) that the function G is almost everywhere differentiable and

$$G' = g.$$

This shows that X^* has the Radon–Nikodým property.

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We prove that (iii) implies (i). Let $\Lambda : L^p(I, X) \to \mathbb{R}$ be a bounded linear functional and let $\mathcal{B} \subset 2^I$ be the Borel σ-algebra. Define the map $\nu : \mathcal{B} \to X^*$ by

$$\langle \nu(B), x \rangle := \Lambda(\chi_B x) \qquad \text{for } B \in \mathcal{B} \text{ and } x \in X. \tag{7.5.18}$$

More precisely, the linear functional $X \to \mathbb{R} : x \mapsto \Lambda(\chi_B x)$ is bounded because $|\Lambda(\chi_B x)| \le \|\Lambda\| \, \|\chi_B x\|_{L^1} \le \|\Lambda\| \, \mu(B)^{1/p} \, \|x\|$. We prove that every finite sequence of pairwise disjoint Borel sets $B_1, \dots, B_N \in \mathcal{B}$ satisfies

$$\sum_{i=1}^N \|\nu(B_i)\| \le \|\Lambda\| \, \mu\left(\bigcup_{i=1}^N B_i\right)^{1/p}. \tag{7.5.19}$$

To see this, fix a constant $\varepsilon > 0$ and, for each i, choose a vector $x_i \in X$ such that $\|x_i\| = 1$ and $\langle \nu(B_i), x_i \rangle \ge (1 - \varepsilon) \, \|\nu(B_i)\|$. Define $f := \sum_i \chi_{B_i} x_i$. Then

$$\sum_i \|\nu(B_i)\| \le \sum_i \frac{\langle \nu(B_i), x_i \rangle}{1 - \varepsilon} = \frac{\Lambda(f)}{1 - \varepsilon} \le \frac{\|\Lambda\| \, \|f\|_{L^p}}{1 - \varepsilon} = \frac{\|\Lambda\| \, \mu(\bigcup_i B_i)^{1/p}}{1 - \varepsilon}.$$

This proves (7.5.19).

Now define the function $G : I \to X^*$ by

$$G(t) := \nu([0, t]) \qquad \text{for } t \in I. \tag{7.5.20}$$

This function satisfies $G(0) = 0$ and is absolutely continuous by (7.5.19). Hence, by (iii) there exists a function $g \in \mathcal{L}^1(I, X^*)$ such that

$$G(t) = \int_0^t g(s) \, ds \qquad \text{for all } t \in I. \tag{7.5.21}$$

For each $x \in X$ consider the bounded linear functional $\Lambda_x : L^p(I) \to \mathbb{R}$ defined by $\Lambda_x(\phi) := \Lambda(\phi x)$ for $\phi \in \mathcal{L}^p(I)$. By [**75**, Thm. 4.35] there exists a function $g_x \in \mathcal{L}^q(I)$ such that

$$\int_I g_x \phi = \Lambda_x(\phi) = \Lambda(\phi x) \qquad \text{for all } \phi \in \mathcal{L}^p(I). \tag{7.5.22}$$

Then, for each $t \in I$ and each $x \in X$, we have

$$\int_0^t g_x(s) \, ds = \Lambda(\chi_{[0,t]} x) = \langle \nu([0, t]), x \rangle = \langle G(t), x \rangle = \int_0^t \langle g(s), x \rangle \, ds.$$

Here the first equality follows from the definition of g_x in (7.5.22), the second from the definition of ν in (7.5.18), the third from the definition of G in (7.5.20), and the last from (7.5.21). This shows that

$$g_x(t) = \langle g(t), x \rangle \tag{7.5.23}$$

for every $x \in X$ and almost every $t \in I$.

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We prove that every $f \in \mathcal{L}^p(I,X)$ satisfies

$$(7.5.24)\quad \langle g,f\rangle \in \mathcal{L}^1(I), \qquad \int_I \langle g,f\rangle = \Lambda(f), \qquad \int_I |\langle g,f\rangle| \le \|\Lambda\|\,\|f\|_{L^p}.$$

First let $f : I \to X$ be a measurable step function of the form $f = \sum_i \chi_{B_i} x_i$, where the $B_i \subset I$ are pairwise disjoint Borel sets and $x_i \in X \setminus \{0\}$. Then

$$\langle g,f\rangle = \sum_i \chi_{B_i}\langle g,x_i\rangle \overset{a.e.}{=} \sum_i \chi_{B_i} g_{x_i},$$

where the last equation follows from (7.5.23). Thus $\langle g,f\rangle$ is integrable because χ_{B_i} is bounded and $g_{x_i} \in \mathcal{L}^q(I)$ for each i. Moreover, it follows from the definition of the functions g_{x_i} in (7.5.22) that

$$\int_I \langle g,f\rangle = \sum_i \int_I g_{x_i}\chi_{B_i} = \sum_i \Lambda(\chi_{B_i}x_i) = \Lambda(f).$$

Now define the function $\phi_i : I \to \mathbb{R}$ by $\phi_i(t) := 0$ for $t \in I \setminus B_i$, by $\phi_i(t) := 1$ for $t \in B_i$ with $g_{x_i}(t) \ge 0$, and by $\phi_i(t) := -1$ for $t \in B_i$ with $g_{x_i}(t) < 0$. Let $\widetilde{f} := \sum_i \chi_{B_i}\phi_i x_i$. Then

$$|\langle g,f\rangle| = \sum_i \chi_{B_i}|g_{x_i}| = \sum_i \chi_{B_i}\phi_i g_{x_i} \overset{a.e.}{=} \left\langle g, \sum_i \chi_{B_i}\phi_i x_i\right\rangle = \left\langle g, \widetilde{f}\right\rangle$$

and hence

$$\int_I |\langle g,f\rangle| = \int_I \left\langle g,\widetilde{f}\right\rangle = \Lambda(\widetilde{f}) \le \|\Lambda\|\,\left\|\widetilde{f}\right\|_{L^p} = \|\Lambda\|\,\|f\|_{L^p}.$$

This proves (7.5.24) for measurable step functions $f : I \to X$.

Now let $f \in \mathcal{L}^p(I,X)$ and choose a sequence of measurable step functions $f_i : I \to X$ that converges in L^p and almost everywhere to f. Then

$$\int_I |\langle g,f_i\rangle - \langle g,f_j\rangle| = \int_I |\langle g, f_i - f_j\rangle| \le \|\Lambda\|\,\|f_i - f_j\|_{L^p}$$

and so $(\langle g,f_i\rangle)_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^1(I)$. Thus it converges to a function $h \in L^1(I)$. Passing to a suitable subsequence, we may assume the sequence converges almost everywhere to h. Hence

$$h(t) \overset{a.e.}{=} \lim_{i\to\infty}\langle g(t), f_i(t)\rangle \overset{a.e.}{=} \langle g(t), f(t)\rangle.$$

Hence $\langle g,f\rangle$ is integrable and

$$\int_I \langle g,f\rangle = \int_I h = \lim_{i\to\infty}\int_I \langle g,f_i\rangle = \lim_{i\to\infty}\Lambda(f_i) = \Lambda(f).$$

Moreover,

$$\|\langle g,f\rangle\|_{L^1} = \|h\|_{L^1} = \lim_{i\to\infty}\|\langle g,f_i\rangle\|_{L^1} \le \lim_{i\to\infty}\|\Lambda\|\,\|f_i\|_{L^p} = \|\Lambda\|\,\|f\|_{L^p}$$

and this proves (7.5.24).

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With the help of (7.5.24) we are now able to prove that $g \in \mathcal{L}^q(I, X^*)$. For $n \in \mathbb{N}$ define the function $g_n : I \to X^*$ by

$$g_n(t) := \begin{cases} g(t), & \text{if } \|g(t)\| \le n, \\ 0, & \text{if } \|g(t)\| > n, \end{cases} \qquad \text{for } t \in I.$$

These functions are strongly measurable and satisfy $\lim_{n\to\infty} g_n(t) = g(t)$ for all $t \in I$. Moreover, it follows from (7.5.24) that

$$\int_I |\langle g_n, f\rangle| \le \int_I |\langle g, f\rangle| \le \|\Lambda\| \, \|f\|_{L^p}$$

for every $n \in \mathbb{N}$ and every $f \in \mathcal{L}^p(I, X)$. Since each function g_n is bounded, and hence an element of the space $\mathcal{L}^q(I, X^*)$, this implies

$$\|g_n\|_{L^q} = \sup_{f \in L^p(I,X)\setminus\{0\}} \frac{\left|\int_I \langle g_n, f\rangle\right|}{\|f\|_{L^p}} \le \sup_{f \in L^p(I,X)\setminus\{0\}} \frac{\int_I |\langle g_n, f\rangle|}{\|f\|_{L^p}} \le \|\Lambda\|\,.$$

Here the equality follows from Lemma 7.5.15. By the Lebesgue Monotone Convergence Theorem, this implies

$$\int_I \|g(t)\|^q \; dt = \lim_{n\to\infty} \int_I \|g_n(t)\|^q \; dt \le \|\Lambda\|^q\,.$$

Thus $g \in \mathcal{L}^q(I, X^*)$, $\|g\|_{L^q} \le \|\Lambda\|$, and $\int_I \langle g, f\rangle = \Lambda(f)$ for all $f \in \mathcal{L}^p(I, X)$ by (7.5.24). This completes the proof of Theorem 7.5.16. □

The following result was proved by R. S. Phillips [**66**] in 1943.

Corollary 7.5.17 (Phillips). *Fix a constant $1 < p < \infty$, let X be a reflexive Banach space, and let $I \subset \mathbb{R}$ be a compact interval. Then the Banach space $L^p(I, X)$ is reflexive.*

Proof. Choose the real number $1 < q < \infty$ such that $1/p + 1/q = 1$. The dual space X^* is reflexive by Theorem 2.4.4, and so has the Radon–Nikodým property by part (ii) of Theorem 7.5.13. Hence Theorem 7.5.16 asserts that the isometric embeddings

$$L^q(I, X^*) \to L^p(I, X)^* \tag{7.5.25}$$

and

$$L^p(I, X^{**}) \to L^q(I, X^*)^* \tag{7.5.26}$$

are isomorphisms. Now the canonical inclusion $\iota : L^p(I, X) \to L^p(I, X)^{**}$ is the composition

$$L^p(I, X) \to L^p(I, X^{**}) \to L^q(I, X^*)^* \to L^p(I, X)^{**}, \tag{7.5.27}$$

where the first map is induced by the canonical isomorphism $\iota : X \to X^{**}$, the second map is the isomorphism (7.5.26), and the third map is the inverse of the dual operator of (7.5.25) (see Corollary 4.1.18). This proves Corollary 7.5.17. □

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7.5.5. The Sobolev Space $W^{1,p}(I,X)$. Let X be a Banach space, fix a real number $1 \le p < \infty$, and let $I = [a,b] \subset \mathbb{R}$ be a compact interval. The **Sobolev space** $W^{1,p}(I,X)$ can be defined as the completion of the space of continuously differentiable functions $f : I \to X$ with respect to the norm

$$\|f\|_{W^{1,p}} := \left(\int_a^b \left(\|f(t)\|^p + \|f'(t)\|^p\right) dt\right)^{1/p}. \tag{7.5.28}$$

Alternatively, $W^{1,p}(I,X)$ is the space of all functions $f : I \to \mathbb{R}$ that can be expressed as the integrals of L^p functions, i.e.

$$W^{1,p}(I,X) := \left\{ f : I \to X \;\middle|\; \begin{array}{l} \text{there exists a strongly} \\ \text{measurable function } g : I \to X \\ \text{such that } \int_a^b \|g(t)\|^p \, dt < \infty \\ \text{and } f(t) - f(a) = \int_a^t g(s)\, ds \\ \text{for all } t \in I \end{array} \right\}. \tag{7.5.29}$$

The Lebesgue Differentiation Theorem asserts that the function $g : I \to X$ in (7.5.29) is uniquely determined by f up to equality almost everywhere and agrees with the derivative of f (Lemma 7.5.9). The norm is again given by equation (7.5.28). With this definition the functions in $W^{1,p}(I,X)$ are absolutely continuous, are almost everywhere differentiable, have derivatives in $\mathcal{L}^p(I,X)$, and can be expressed as the integrals of their derivatives. If X has the Radon–Nikodým property, then every absolutely continuous function $f : I \to X$ is almost everywhere differentiable and we have

$$W^{1,p}(I,X) := \left\{ f : I \to X \;\middle|\; \begin{array}{l} f \text{ is absolutely continuous} \\ \text{and } f' \in \mathcal{L}^p(I,X) \end{array} \right\}.$$

If X does not have the Radon–Nikodým property, this last definition does not even make sense, because absolutely continuous functions need not be differentiable. Thus we will work with the definition (7.5.29). However, in all the relevant examples in this book the Banach space in question is reflexive and therefore does have the Radon–Nikodým property by Theorem 7.5.13. The next theorem asserts that the Sobolev space $W^{1,p}(I,X)$ is a Banach space and that the space $C^\infty(I,X)$ of smooth functions $f : I \to X$ is dense in $W^{1,p}(I,X)$.

Theorem 7.5.18. *Let X be a Banach space, let $I = [a,b] \subset \mathbb{R}$ be a compact interval, and fix a constant $1 \le p < \infty$. Then the following hold.*

(i) *There exists a $c > 0$ such that $\|f\|_{L^\infty} \le c\,\|f\|_{W^{1,p}}$ for all $f \in W^{1,p}(I,X)$.*

(ii) *The Sobolev space $W^{1,p}(I,X)$ is complete with the norm (7.5.28).*

(iii) *The subspace $C^\infty(I,X)$ is dense in $W^{1,p}(I,X)$.*

(iv) *If X is reflexive and $1 < p < \infty$, then $W^{1,p}(I,X)$ is reflexive.*

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Proof. We prove part (i). Let $f \in W^{1,p}$ and choose $g \in \mathcal{L}^p(I,X)$ such that $\int_a^t g(s)\,ds = f(t) - f(a)$ for all $t \in I$. Then, by Hölder's inequality,

$$\|f(t) - f(s)\| = \left\|\int_s^t g(r)\,dr\right\| \le \left|\int_s^t \|g(r)\|\,dr\right| \le (b-a)^{1/q}\|g\|_{L^p}$$

for all $s,t \in [a,b]$. Here $1 < q \le \infty$ is chosen such that $1/p + 1/q = 1$. In the case $q = \infty$ we use the standard convention $(b-a)^{1/q} = (b-a)^0 := 1$. Now raise this inequality to the power p and integrate to obtain

$$\int_a^b \|f(t) - f(s)\|^p\,ds \le (b-a)^{1+p/q}\|g\|_{L^p}^p \le (b-a)^p\|g\|_{L^p}^p .$$

Take the pth root of this estimate to obtain

$$\left(\int_a^b \|f(t) - f(s)\|^p\,ds\right)^{1/p} \le (b-a)\,\|g\|_{L^p} .$$

Hence $(b-a)^{1/p}\|f(t)\| \le \|f\|_{L^p} + (b-a)\|g\|_{L^p}$ for all $t \in I$ by Minkowski's inequality. This proves part (i).

We prove part (ii). Let $f_n : I \to X$ be a Cauchy sequence in $W^{1,p}(I,X)$ and choose a sequence $g_n \in \mathcal{L}^p(I,X)$ such that $\int_a^t g_n(s)\,ds = f_n(t) - f_n(a)$ for all $t \in I$ and all $n \in \mathbb{N}$. Then $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $C(I,X)$ of continuous functions with the supremum norm, and $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $L^p(I,X)$. Hence the sequence f_n converges uniformly to a continuous function $f : I \to X$ and g_n converges to a function $g \in L^p(I,X)$ by Theorem 7.5.6. The limit functions satisfy

$$f(t) - f(a) = \lim_{n\to\infty}(f_n(t) - f_n(a)) = \lim_{n\to\infty}\int_a^t g_n(s)\,ds = \int_a^t g(s)\,ds$$

for all $t \in I$. Thus $f \in W^{1,p}(I,X)$ and

$$\lim_{n\to\infty}\|f - f_n\|_{W^{1,p}} = \lim_{n\to\infty}\left(\|f - f_n\|_{L^p}^p + \|g - g_n\|_{L^p}^p\right)^{1/p} = 0.$$

This proves part (ii).

We prove part (iii) by a standard mollifier argument. Let $f \in W^{1,p}(I,X)$ and extend f to all of $\mathbb{R}$ by $f(t) := f(b)$ for $t > b$ and $f(t) := f(a)$ for $t < a$. Choose a smooth function $\rho : \mathbb{R} \to [0,\infty)$ with compact support and mean value 1 and define $\rho_\delta(t) := \delta^{-1}\rho(\delta^{-1}t)$ for $\delta > 0$ and $t \in \mathbb{R}$. Then the function $f_\delta : I \to \mathbb{R}$, defined by $f_\delta(t) := (\rho_\delta * f)(t) := \int_{\mathbb{R}} \rho_\delta(t-s)f(s)\,ds$ for $t \in \mathbb{R}$, is smooth for every $\delta > 0$, and f_δ converges to f uniformly, and hence also in the L^p-norm. Moreover, $f_\delta' = \rho_\delta * f'$ converges to f' in the L^p-norm and thus $\lim_{\delta\to 0}\|f - f_\delta\|_{W^{1,p}} = 0$. This proves part (iii).

We prove part (iv). The map $W^{1,p}(I,X) \to L^p(I, X \times X) : f \mapsto (f, f')$ is an isometric embedding. The target space is reflexive by Corollary 7.5.17, so $W^{1,p}(I,X)$ is reflexive by Theorem 2.4.4. This proves Theorem 7.5.18. □

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7.6. Inhomogeneous Equations

Let X be a real Banach space and let $A : \mathrm{dom}(A) \to X$ be the infinitesimal generator of a strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}(X)$. This section is devoted to the study of the solutions of the inhomogeneous equation

$$\dot{x} = Ax + f, \qquad x(0) = x_0. \tag{7.6.1}$$

Here we assume that the function $f : [0, \infty) \to \mathbb{R}$ is strongly measurable and locally integrable. In this situation we consider the function $x : [0, \infty) \to X$, defined by the variation of constants formula

$$x(t) := S(t)x_0 + \int_0^t S(t-s)f(s)\,ds \tag{7.6.2}$$

for $t \geq 0$. If $x_0 \in \mathrm{dom}(A)$ and $f : [0, \infty) \to X$ is continuously differentiable, then, by Lemma 7.1.14, the function $x : [0, \infty) \to X$ in (7.6.2) is continuously differentiable, takes values in the domain of A, and satisfies equation (7.6.1). While this is a rather crude general observation, it is the starting point for any more refined study of the solutions of (7.6.1).

7.6.1. Weak Solutions. As a first step we use the concepts developed in Section 7.5 to introduce the notion of a weak solution. This notion uses test functions $g : I \to X^*$ on a compact interval $I \subset \mathbb{R}$ that take values in $\mathrm{dom}((A^*)^\infty)$ and have the property that the function $(A^*)^k g : I \to X^*$ is smooth for every $k \in \mathbb{N}$. The space of such functions will be denoted by $C^\infty(I, \mathrm{dom}(A^*)^\infty)$.

Definition 7.6.1 (Weak Solution). Let X be a real Banach space and let $A : \mathrm{dom}(A) \to X$ be the infinitesimal generator of a strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}(X)$. Fix a compact interval $I = [0, T]$, a strongly measurable function $f : I \to X$ with $\int_0^T \|f(t)\|\,dt < \infty$, and an element $x_0 \in X$. A **weak solution of equation** (7.6.1) is a strongly measurable function $x : I \to X$ with $\int_0^T \|x(t)\|\,dt < \infty$ that satisfies the condition

$$\langle g(0), x_0 \rangle + \int_0^T \langle g(s), f(s) \rangle\,ds + \int_0^T \langle \dot{g}(s) + A^* g(s), x(s) \rangle\,ds = 0 \tag{7.6.3}$$

for every test function $g \in C^\infty(I, \mathrm{dom}(A^*)^\infty)$ with $g(T) = 0$.

The next theorem shows that equation (7.6.1) admits a unique (almost everywhere) weak solution and that it is given by (7.6.2).

Theorem 7.6.2 (Existence and Uniqueness). *Let X, I, S, A, f, x_0 be as in Definition* 7.6.1 *and let $x \in \mathcal{L}^1(I, X)$. The following are equivalent.*

(i) *x is a weak solution of* (7.6.1).

(ii) *x is given by* (7.6.2) *for almost every $t \in I$.*

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Proof. We prove that (ii) implies (i). Let $g \in C^\infty(I, \mathrm{dom}(A^*)^\infty)$ be a test function with $g(T) = 0$. Recall from Theorem 7.3.1 that the restriction of the dual semigroup $S^*(t)$ to the strong closure $E \subset X^*$ of the domain of A^* is a strongly continuous semigroup, whose infinitesimal generator is the restriction of the operator A^* to the subspace $\{x^* \in \mathrm{dom}(A^*) \mid A^*x^* \in E\}$. This implies that the function $I \to X^* : t \mapsto S^*(t)g(t)$ is continuously differentiable with the derivative

$$\frac{d}{dt}S^*(t)g(t) = S^*(t)(\dot{g}(t) + A^*g(t))$$

for $t \in I$. Hence the function $x_0(t) := S(t)x_0$ satisfies

$$\begin{aligned}
\int_0^T \langle \dot{g}(t) + A^*g(t), x_0(t)\rangle\, dt &= \int_0^T \langle \dot{g}(t) + A^*g(t), S(t)x_0\rangle\, dt \\
&= \int_0^T \langle S^*(t)(\dot{g}(t) + A^*g(t)), x_0\rangle\, dt \\
&= \int_0^T \frac{d}{dt}\langle S^*(t)g(t), x_0\rangle\, dt \\
&= \langle S^*(T)g(T), x_0\rangle - \langle g(0), x_0\rangle \\
&= -\langle g(0), x_0\rangle
\end{aligned}$$

and for $x_1(t) := \int_0^t S(t-s)f(s)\, ds$ we obtain

$$\begin{aligned}
&\int_0^T \langle \dot{g}(t) + A^*g(t), x_1(t)\rangle\, dt \\
&\quad = \int_0^T \int_0^t \langle \dot{g}(t) + A^*g(t), S(t-s)f(s)\rangle\, dsdt \\
&\quad = \int_0^T \int_0^t \langle S^*(t-s)(\dot{g}(t) + A^*g(t)), f(s)\rangle\, dsdt \\
&\quad = \int_0^T \int_s^T \langle S^*(t-s)(\dot{g}(t) + A^*g(t)), f(s)\rangle\, dtds \\
&\quad = \int_0^T \int_0^{T-s} \langle S^*(t)(\dot{g}(s+t) + A^*g(s+t)), f(s)\rangle\, dtds \\
&\quad = \int_0^T \int_0^{T-s} \frac{d}{dt}\langle S^*(t)g(s+t), f(s)\rangle\, dtds \\
&\quad = \int_0^T \Big(\langle S^*(T-s)g(T), f(s)\rangle - \langle g(s), f(s)\rangle\Big)\, ds \\
&\quad = -\int_0^T \langle g(s), f(s)\rangle\, ds.
\end{aligned}$$

Take the sum of these equations to obtain that $x := x_0 + x_1 : I \to X$ is a weak solution of (7.6.1).

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We prove that (i) implies (ii). Thus assume that $x : I \to X$ is a weak solution of (7.6.1) and define the function $y : I \to X$ by

$$y(t) := x(t) - S(t)x_0 - \int_0^t S(t-s)f(s)\,ds \qquad \text{for } 0 \le t \le T.$$

Then, by what we have just proved, y is a weak solution of equation (7.6.1) with $x_0 = 0$ and $f = 0$, i.e. $y \in \mathcal{L}^1(I, X)$ and

$$\int_0^T \langle \dot{g}(s) + A^* g(s), y(s)\rangle\,ds = 0 \tag{7.6.4}$$

for all $g \in C^\infty(I, \mathrm{dom}(A^*)^\infty)$ with $g(T) = 0$. We must prove that $y(t) = 0$ for almost every $t \in I$. To see this, fix an element $x^* \in \mathrm{dom}((A^*)^\infty)$ and a smooth function $\phi : I \to \mathbb{R}$, and define the function $g : I \to X^*$ by

$$g(s) := \int_0^{T-s} \phi(r) S^*(T-s-r)x^*\,dr \qquad \text{for } 0 \le s \le T.$$

Then $g(T) = 0$. Moreover, it follows from Theorem 7.3.1 by induction that, for $k \in \mathbb{N}_0$, the restriction of $S^*(t)$ to $E_k := \big\{\xi^* \in \mathrm{dom}(A^*)^k \,|\, (A^*)^k \in E\big\}$ is a strongly continuous semigroup whose infinitesimal generator is the restriction $B_k := A^*|_{E_{k+1}} : E_{k+1} \to E_k$. Apply Lemma 7.1.14 to this semigroup to deduce that, for every integer $k \ge 0$, the function $g : I \to X^*$ takes values in E_{k+1}, is continuously differentiable as a function with values in E_k, and satisfies $\frac{d}{ds} g(T-s) = A^* g(T-s) + \phi(T-s)x^*$ or equivalently

$$\dot{g}(s) + A^* g(s) = \phi(s)x^* \qquad \text{for } 0 \le s \le T. \tag{7.6.5}$$

This implies $g \in C^\ell(I, E_k)$ for all $k, \ell \in \mathbb{N}_0$ and so $g \in C^\infty(I, \mathrm{dom}((A^*)^\infty))$. Thus it follows from (7.6.4) and (7.6.5) that

$$\int_0^T \phi(s)\langle x^*, y(s)\rangle\,ds = 0 \tag{7.6.6}$$

for all $x^* \in \mathrm{dom}((A^*)^\infty)$ and all $\phi \in C^\infty(I)$. Choose a sequence of smooth functions $\phi_i : [0,1] \to [0,1]$ converging pointwise to the characteristic function of the subinterval $[0,t]$ and use Lebesgue dominated convergence and equation (7.6.6) to obtain $\int_0^t \langle x^*, y(s)\rangle\,ds = 0$ and hence

$$\left\langle x^*, \int_0^t y(s)\,ds \right\rangle = 0 \tag{7.6.7}$$

for all $x^* \in \mathrm{dom}((A^*)^\infty)$ and all $t \in I$. Since $\mathrm{dom}((A^*)^\infty)$ is dense in E by Lemma 7.1.16, equation (7.6.7) continues to hold for all $x^* \in E$ and all $t \in I$. Since E contains the domain of A^*, it is weak* dense in X^* by part (iii) of Theorem 6.2.2. This implies $\int_0^t y(s)\,ds = 0$ for $0 \le t \le T$. Now it follows from Lebesgue differentiation that $y(t) = 0$ for almost every $t \in I$ (Lemma 7.5.9). This proves Theorem 7.6.2. □

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7.6.2. Regular Solutions. The next theorem examines the properties of weak solutions of (7.6.1) that belong to the Sobolev space $W^{1,1}(I,X)$.

Theorem 7.6.3 (Regular Solutions). *Let X, I, S, A, f, x_0 be as in Definition 7.6.1 and let $x : I \to X$ be a strongly measurable function. Then the following are equivalent.*

(i) *$x \in W^{1,1}(I,X)$ and x is a weak solution of equation (7.6.1).*

(ii) *There exists a Borel set $Z \subset I$ of Lebesgue measure zero such that*

- *$x(t) \in \mathrm{dom}(A)$ for every $t \in I \setminus Z$,*
- *the function $x : I \to X$ is differentiable on $I \setminus Z$ and*
$$\dot{x}(t) = Ax(t) + f(t) \qquad \text{for all } t \in I \setminus Z,$$
- *the function $y : I \to X$, defined by*
$$y(t) := \begin{cases} \dot{x}(t), & \text{for } t \in I \setminus Z, \\ 0, & \text{for } t \in Z, \end{cases}$$
is strongly measurable and satisfies $\int_0^T \|y(s)\|\,ds < \infty$ and
$$x(t) = x_0 + \int_0^t y(s)\,ds \qquad \text{for all } t \in I.$$

Proof. We prove that (i) implies (ii). Thus assume that $x \in W^{1,1}(I,X)$ is a weak solution of equation (7.6.1). Then x is continuous and
$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)\,ds \qquad \text{for } 0 \le t \le T$$
by Theorem 7.6.2. For $0 \le t < t+h \le T$ this implies
$$\begin{aligned} \frac{S(h)x(t) - x(t)}{h} &= \frac{x(t+h) - x(t)}{h} - \frac{1}{h}\int_0^h S(s)f(t)\,ds \\ &\quad - \frac{1}{h}\int_0^h S(h-s)\Big(f(t+s) - f(t)\Big)\,ds. \end{aligned} \tag{7.6.8}$$
Moreover, by definition of $W^{1,1}(I,X)$ there exists a function $\xi \in \mathcal{L}^1(I,X)$ such that $x(t) = x_0 + \int_0^t \xi(s)\,ds$ for all $t \in I$. Hence, by Lebesgue differentiation, there exists a Borel set $Z \subset I$ of Lebesgue measure zero such that

- x is differentiable on $I \setminus Z$ and $\dot{x}(t) = \xi(t)$ for all $t \in I \setminus Z$,
- $\lim_{h \searrow 0} \frac{1}{h}\int_0^h \|f(t+s) - f(t)\|\,ds = 0$ for all $t \in I \setminus Z$.

For $t \in I \setminus Z$ this implies that the right-hand side of (7.6.8) converges to $\xi(t) - f(t)$ as h tends to zero. Thus $x(t) \in \mathrm{dom}(A)$ and $Ax(t) = \xi(t) - f(t)$ for all $t \in I \setminus Z$. This shows that x satisfies (ii) with this Borel set Z and the function $y : I \to X$ defined by $y(t) := \xi(t) = \dot{x}(t)$ for $t \in I \setminus Z$ and by $y(t) = 0$ for $t \in Z$.

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We prove that (ii) implies (i). Thus assume that $Z \subset I$ is a Borel set of Lebesgue measure zero that satisfies the requirements of part (ii). Let $E \subset X^*$ be the strong closure of the domain of the dual operator A^*. Fix an element $x^* \in \mathrm{dom}(A^*)$ with $A^*x^* \in E$ and a real number $0 < t \le T$. Then, by Theorem 7.3.1, the function $[0,t] \to X^* : s \mapsto S^*(t-s)x^*$ is continuously differentiable and has the derivative

$$\frac{d}{ds}S^*(t-s)x^* = -S^*(t-s)A^*x^* = -A^*S^*(t-s)x^*.$$

Moreover, by assumption, the function $x : I \to X$ is absolutely continuous and differentiable in $I \setminus Z$. This implies that the function

$$[0,t] \to \mathbb{R} : s \mapsto \langle S^*(t-s)x^*, x(s)\rangle \tag{7.6.9}$$

is absolutely continuous and differentiable in $[0,t] \setminus Z$. Since $x(t) \in \mathrm{dom}(A)$ and $\dot{x}(t) = Ax(t) + f(t)$ for $t \in I \setminus Z$, the function (7.6.9) has the derivative

$$\begin{aligned}\frac{d}{ds}\langle S^*(t-s)x^*, x(s)\rangle &= \langle S^*(t-s)x^*, \dot{x}(s)\rangle - \langle A^*S^*(t-s)x^*, x(s)\rangle \\ &= \langle S^*(t-s)x^*, \dot{x}(s) - Ax(s)\rangle \\ &= \langle x^*, S(t-s)f(s)\rangle\end{aligned}$$

for $s \in [0,t] \setminus Z$. Since Z has measure zero and (7.6.9) is absolutely continuous, this implies $\langle x^*, x(t)\rangle - \langle S^*(t)x^*, x(0)\rangle = \int_0^t \langle x^*, S(t-s)f(s)\rangle\, ds$ and hence

$$\left\langle x^*, x(t) - S(t)x_0 - \int_0^t S(t-s)f(s)\, ds \right\rangle = 0 \tag{7.6.10}$$

for all $x^* \in \mathrm{dom}(A^*)$ with $A^*x^* \in E$. By Theorem 7.3.1 the set of all such x^* is the domain of the infinitesimal generator of the strongly continuous semigroup $[0,\infty) \to \mathcal{L}(E) : t \mapsto S^*(t)|_E$ and so is dense in E by Lemma 7.1.16. Hence equation (7.6.10) continues to hold for all $x^* \in E$ and hence, in particular, for all $x^* \in \mathrm{dom}(A^*)$. Since the domain of A^* is weak* dense in X^*, by Theorem 6.2.2, it follows that $x(t)$ is given by equation (7.6.2) for every $t \in [0,T]$. This proves Theorem 7.6.3. □

Theorem 7.6.3 leads to the question under which conditions on x_0 and f the weak solution (7.6.2) of (7.6.1) belongs to the Sobolev space $W^{1,1}(I,X)$. This is the fundamental **regularity problem for semigroups**. It has two parts, one for the inhomogeneous term f when $x_0 = 0$ (see Subsection 7.6.3) and one for the initial condition x_0 when $f = 0$ (see Subsection 7.6.4). By Lemma 7.1.14, the weak solution (7.6.2) belongs to $W^{1,1}(I,X)$ whenever $x_0 \in \mathrm{dom}(A)$ and $f : I \to X$ is continuously differentiable. By Exercise 7.7.12 this continues to hold for $f \in W^{1,1}(I,X)$.

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7.6.3. Maximal Regularity. In applications one is interested in a refined regularity problem associated to a number $1 \le q < \infty$, which asks for weak solutions in the Sobolev space $W^{1,q}(I,X)$ when $f \in L^q(I,X)$. The sharp answer would be that, for every $f \in L^q(I,X)$, the formula (7.6.2) with $x_0 = 0$ defines a weak solution $x : I \to X$ of (7.6.1) in the Sobolev space $W^{1,q}(I,X)$, i.e. both $\dot{x}$ and Ax, and not just their difference, belong to the space $L^q(I,X)$. This property is called *maximal q-regularity.*

Definition 7.6.4 (Maximal Regularity). Let X be a Banach space, let $A : \mathrm{dom}(A) \to X$ be the infinitesimal generator of a strongly continuous semigroup $S : [0,\infty) \to \mathcal{L}(X)$, and fix a real number $q \ge 1$. The semigroup S is called **maximal q-regular** if, for every $T > 0$, there exists a $c_T > 0$ such that every continuously differentiable function $f : [0,T] \to X$ satisfies

$$\left(\int_0^T \left\| A \int_0^t S(t-s)f(s)\,ds \right\|^q dt\right)^{1/q} \le c_T \left(\int_0^T \|f(t)\|^q\,dt\right)^{1/q}. \tag{7.6.11}$$

This condition is independent of T. The semigroup S is called **uniformly maximal q-regular** if it is maximal q-regular and the constant in (7.6.11) can be chosen independent of T.

Lemma 7.6.5 (Maximal Regularity). *Let X be a Banach space and let $A : \mathrm{dom}(A) \to X$ be the infinitesimal generator of a strongly continuous semigroup $S : [0,\infty) \to \mathcal{L}(X)$. Fix two real numbers $q \ge 1$ and $T > 0$ and abbreviate $I := [0,T]$. Then the following are equivalent.*

(i) *For every strongly measurable function $f : I \to X$ with $\int_I \|f\|^q < \infty$ equation* (7.6.1) *has a weak solution $x \in W^{1,q}(I,X)$ with $x(0) = x_0 = 0$.*

(ii) *For every strongly measurable function $f : I \to X$ with $\int_I \|f\|^q < \infty$ the continuous function $x : I \to X$ defined by* (7.6.2) *with $x_0 = 0$ belongs to the Sobolev space $W^{1,q}(I,X)$.*

(iii) *The semigroup S is maximal q-regular.*

Proof. That (i) implies (ii) follows directly from Theorem 7.6.2. To prove that (ii) implies (iii), denote by $\iota : W^{1,q}(I,X) \to C(I,X)$ the obvious inclusion and define the linear operator $\mathscr{S} : L^q(I,X) \to C(I,X)$ by

$$(\mathscr{S}f)(t) := \int_0^t S(t-s)f(s)\,ds$$

for $f \in L^q(I,X)$ and $t \in I$. Then ι is a bounded linear operator by part (i) of Theorem 7.5.18. To prove that $\mathscr{S}$ is a bounded linear operator, choose a constant $M \ge 1$ such that $\|S(t)\| \le M$ for $0 \le t \le T$ (Lemma 7.1.8). Then

$$\|(\mathscr{S}f)(t)\| \le \int_0^t \|S(t-s)f(s)\|\,ds \le M \int_0^T \|f(s)\|\,ds \le MT^{1-1/q}\,\|f\|_{L^q}$$

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for all $t \in I$ and all $f \in \mathcal{L}^q(I, X)$. Moreover, $\mathrm{im}(\mathscr{S}) \subset \mathrm{im}(\iota)$ by (ii). Since ι is injective, Corollary 2.2.17 (Douglas factorization) asserts that the linear operator $\iota^{-1} \circ \mathscr{S} : L^q(I, X) \to W^{1,q}(I, X)$ is bounded. Thus there exists a constant $C > 0$ such that $\|\mathscr{S} f\|_{W^{1,q}} \leq C \|f\|_{L^q}$ for all $f \in L^q(I, X)$. For $f \in C^1(I, X)$ this is equivalent to the estimate (7.6.11). Thus S is maximal q-regular.

We prove that (iii) implies (i). Assume S is maximal q-regular and let $f : I \to X$ be a strongly measurable function with $\int_0^T \|f(t)\|^q \, dt < \infty$. By part (iii) of Theorem 7.5.4, there exists a sequence of smooth functions $f_i : I \to X$ such that $\lim_{i\to\infty} \|f_i(t) - f(t)\|_{L^q} = 0$. Define the functions $x : I \to X$ and $x_i : I \to X$, $i \in \mathbb{N}$, by

$$x(t) := \int_0^t S(t-s) f(s)\, ds, \qquad x_i(t) := \int_0^t S(t-s) f_i(s)\, ds$$

for $t \in I$. Then $\lim_{i\to\infty} \sup_{t\in I} \|x_i(t) - x(t)\| = 0$. By Lemma 7.1.14, we have $x_i(t) \in \mathrm{dom}(A)$ and $\dot{x}_i(t) = Ax_i(t) + f_i(t) =: y_i(t)$ for all t and i. Moreover, $\|Ax_i - Ax_j\|_{L^q} \leq c_T \|f_i - f_j\|_{L^q}$ for all i, j by maximal regularity. Thus $(y_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^q(I, X)$ and so, by Theorem 7.5.6, there exists a function $y \in \mathcal{L}^q(I, X)$ with $\lim_{i\to\infty} \|y_i - y\|_{L^q} = 0$. Hence

$$x(t) = \lim_{i\to\infty} x_i(t) = \lim_{i\to\infty} \int_0^t y_i(s)\, ds = \int_0^t y(s)\, ds$$

for all $t \in I$ and so $x \in W^{1,q}(I, X)$. Since x is a weak solution of (7.6.1) by Theorem 7.6.2, this proves Lemma 7.6.5. □

The next lemma shows that there are many semigroups that cannot be maximal q-regular for any $q \geq 1$. Such examples include all strongly continuous groups generated by unbounded operators.

Lemma 7.6.6. *Let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup on a Banach space X that is maximal q-regular for some $q \geq 1$. Then*

$$\mathrm{im}(S(t)) \subset \mathrm{dom}(A) \qquad \textit{for all } t > 0.$$

Proof. Assume that there exists a $T > 0$ such that $\mathrm{im}(S(T)) \not\subset \mathrm{dom}(A)$, abbreviate $I := [0, T]$, and choose $\xi \in X$ such that $S(T)\xi \in X \setminus \mathrm{dom}(A)$. Define the function $f : I \to X$ by $f(t) := S(t)\xi$ for $0 \leq t \leq T$. Then we have $f \in C(I, X) \subset L^q(I, X)$ and

$$x(t) := \int_0^t S(t-s) f(s)\, ds = tS(t)\xi \in X \setminus \mathrm{dom}(A)$$

for $0 < t \leq T$. Hence the function $x : I \to X$ cannot belong to the Sobolev space $W^{1,1}(I, X)$ by Theorem 7.6.3. This shows that the semigroup S violates condition (i) in Lemma 7.6.5 for any $q \geq 1$ and hence cannot be maximal q-regular. This proves Lemma 7.6.6. □

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Remark 7.6.7. Let X be a reflexive Banach space. Let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup with infinitesimal generator A such that

$$\operatorname{im}(S(t)) \subset \operatorname{dom}(A^2), \qquad \left\|A^2 S(t)x\right\| \le ct^{-2} \|x\| \tag{7.6.12}$$

for all $t > 0$, all $x \in X$, and some $c > 0$. Under these assumptions it was proved by Benedek–Calderón–Panzone [**9**] that S is (uniformly) maximal q-regular for some $q > 1$ if and only if it is (uniformly) maximal q-regular for all $q > 1$. Another exposition of their theorem can be found in [**76**]. Note that analytic contraction semigroups satisfy (7.6.12) by Theorem 7.4.4.

Remark 7.6.8. Let $(M, \mathcal{A}, \mu)$ be a measure space and let

$$S : [0, \infty) \to \mathcal{L}(L^2(\mu))$$

be an analytic semigroup that satisfies the estimate

$$\|S(t)f\|_{L^p} \le \|f\|_{L^p} \tag{7.6.13}$$

for all $p \ge 1$, all $t \ge 0$, and all $f \in L^p(\mu) \cap L^2(\mu)$. Under this assumption a theorem of Lamberton [**54**] asserts that the induced contraction semigroup on $L^p(\mu)$ is uniformly maximal q-regular for all $p, q > 1$. For the heat flow in Example 7.1.6 an exposition can be found in [**76**]. The proof goes far beyond the scope of the present book. However, for $p = q = 2$ the result follows from an elementary abstract observation that is explained below.

For the study of maximal regularity it is convenient to introduce a Banach space that contains all the regular solutions of equation (7.6.1). For each $q \ge 1$ and each interval $I = [0, T]$ this is the space

$$\begin{aligned}\mathscr{W}_A^{1,q}(I, X) &:= W^{1,q}(I, X) \cap L^q(I, \operatorname{dom}(A)) \\ &:= \left\{ x \in W^{1,q}(I, X) \;\middle|\; \begin{array}{l} \text{there is a Borel set } Z \subset I \\ \text{of measure zero such that} \\ x(t) \in \operatorname{dom}(A) \text{ for } t \in I \setminus Z, \\ \text{the function } Ax : I \to X \\ \text{is strongly measurable,} \\ \text{and } \int_0^T \|Ax(t)\|^q \, dt < \infty \end{array} \right\}\end{aligned} \tag{7.6.14}$$

equipped with the norm

$$\|x\|_{\mathscr{W}_A^{1,q}} := \left(\int_0^T \left(\|x(t)\|^q + \|\dot{x}(t)\|^q + \|Ax(t)\|^q \right) dt \right)^{1/q}. \tag{7.6.15}$$

In this definition the function $Ax : I \to X$ is understood to be zero for $t \in Z$. The next lemma summarizes some basic properties of this space. In particular, it is a Banach space and is reflexive when X is reflexive and $1 < q < \infty$.

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Lemma 7.6.9. *Let $A : \mathrm{dom}(A) \to X$ be the infinitesimal generator of a strongly continuous semigroup on a Banach space X, let $1 \le q < \infty$, and let $T > 0$ and $I := [0, T]$. Then the following hold.*

(i) *Let $x : I \to \mathrm{dom}(A)$ be any function. Then x is strongly measurable in the Banach space $\mathrm{dom}(A)$ with the graph norm if and only if both functions $x : I \to X$ and $Ax : I \to X$ are strongly measurable in X.*

(ii) *$\mathscr{W}_A^{1,q}(I, X)$ is a Banach space with the norm (7.6.15).*

(iii) *The space*

$$C^\infty(I, \mathrm{dom}(A)) := \left\{ x : I \to \mathrm{dom}(A) \;\middle|\; \begin{array}{l} \text{the functions } x : I \to X \\ \text{and } Ax : I \to X \text{ are smooth} \end{array} \right\}$$

is dense in $\mathscr{W}_A^{1,q}(I, X)$.

(iv) *If X is reflexive and $1 < q < \infty$, then $\mathscr{W}_A^{1,q}(I, X)$ is reflexive.*

Proof. We prove part (i). Let us temporarily denote by $\iota : \mathrm{dom}(A) \to X$ the obvious inclusion and think of $x : I \to \mathrm{dom}(A)$ solely as a function with values in the Banach space $\mathrm{dom}(A)$, equipped with the graph norm. Then the operator $\lambda\iota - A : \mathrm{dom}(A) \to X$ is invertible for $\lambda > 0$ sufficiently large and for such a λ we have $x = (\lambda\iota - A)^{-1} \circ (\lambda\iota \circ x - A \circ x)$. Thus, if $x : I \to \mathrm{dom}(A)$ is strongly measurable, so are $\iota \circ x, A \circ x : I \to X$, and conversely if those two are strongly measurable, so is $\lambda\iota \circ x - A \circ x$ and hence also x.

We prove part (ii). Let $(x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{W}_A^{1,q}(I, X)$. Then $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,q}(I, X)$ and hence converges to a function $x \in W^{1,q}(I, X)$, both with respect to the $W^{1,q}$-norm and with respect to the supremum norm by Theorem 7.5.18. Moreover, the functions $y_i := Ax_i : I \to X$ form a Cauchy sequence in $L^q(I, X)$. Hence Theorem 7.5.6 asserts that there exists a strongly measurable function $y : I \to X$ such that $\int_I \|y\|^q < \infty$ and $\lim_{i\to\infty} \|y_i - y\|_{L^q} = 0$, and that a subsequence of y_i converges almost everywhere to y. Since A is closed, this implies that there exists a Borel set $Z \subset I$ of measure zero such that $x(t) \in \mathrm{dom}(A)$ and $Ax(t) = y(t)$ for all $t \in I \setminus Z$. Hence $x \in \mathscr{W}_A^{1,q}(I, X)$ and

$$\lim_{i\to\infty} \|x - x_i\|_{\mathscr{W}_A^{1,q}} = \lim_{i\to\infty} \left(\|x - x_i\|_{W^{1,q}}^q + \|y - y_i\|_{L^q}^q\right)^{1/q} = 0.$$

This proves part (ii). Part (iii) follows from the same mollifier argument as in the proof of Theorem 7.5.18, and part (iv) follows from the fact that the map $\mathscr{W}_A^{1,q}(I, X) \to W^{1,q}(I, X \times X \times X) : x \mapsto (x, \dot{x}, Ax)$ is an isometric embedding, by definition, and the target space is reflexive whenever X is reflexive and $1 < q < \infty$, by Corollary 7.5.17. This proves Lemma 7.6.9. □

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It follows from Theorem 7.6.3 that the weak $W^{1,q}$ solutions of equation (7.6.1) with $f \in L^q(I,X)$ are elements of the space $\mathscr{W}_A^{1,q}(I,X)$ and that the inhomogeneous term in the equation can be recovered from the element $x \in \mathscr{W}_A^{1,q}(I,X)$ via the formula $f = \dot{x} - Ax$. Thus the semigroup generated by A is maximal q-regular if and only if the map

$$\left\{x \in \mathscr{W}_A^{1,q}(I,X) \,\middle|\, x(0) = 0\right\} \to L^q(I,X) : x \mapsto \dot{x} - Ax \tag{7.6.16}$$

is a Banach space isomorphism. If that holds, then the bounded linear operator $\iota^{-1} \circ \mathscr{S} : L^q(I,X) \to W^{1,q}(I,X)$ in the proof of Lemma 7.6.5 is the inverse of the operator (7.6.16).

7.6.4. Regular Initial Conditions. With these preparations we are ready to formulate the second regularity problem for equation (7.6.1). The question is, which initial conditions $x_0 \in X$ give rise to solutions of the homogeneous equation in the space $\mathscr{W}_A^{1,q}(I,X)$. Define the normed vector space

$$X_{A,q} := \left\{ x \in X \,\middle|\, \begin{array}{l} S(t)x \in \mathrm{dom}(A) \text{ for all } t > 0 \\ \text{and } \int_0^T \|AS(t)x\|_X^q \, dt < \infty \end{array} \right\}, \tag{7.6.17}$$

$$\|x\|_{A,q} := \|x\|_X + \left(\int_0^T \|AS(t)x\|_X^q \, dt \right)^{1/q} \qquad \text{for } x \in X_{A,q}. \tag{7.6.18}$$

Lemma 7.6.10. *Let $A : \mathrm{dom}(A) \to X$ be the infinitesimal generator of a strongly continuous semigroup $S : [0,\infty) \to \mathcal{L}(X)$ on a Banach space X, let $I = [0,T]$, and let $1 \le q < \infty$. Then the following hold.*

(i) *$X_{A,q}$ is a Banach space with the norm* (7.6.18) *and* $\mathrm{dom}(A) \subset X_{A,q} \subset X$ *with continuous dense inclusions.*

(ii) *The subspace $X_{A,q} \subset X$ is invariant under the operator $S(t)$ for all $t \ge 0$ and $S(t)$ restricts to a strongly continuous semigroup on the space $X_{A,q}$.*

(iii) *Let $x_0 \in X$ and define the function $x : I \to X$ by $x(t) := S(t)x_0$ for $0 \le t \le T$. Then $x_0 \in X_{A,q}$ if and only if $x \in \mathscr{W}_A^{1,q}(I,X)$.*

(iv) *Assume S is maximal q-regular. Then there exists a $c > 0$ such that every continuously differentiable function $x : I \to \mathrm{dom}(A)$ satisfies*

$$\sup_{0 \le t \le T} \|x(t)\|_{A,q} \le c \, \|x\|_{\mathscr{W}_A^{1,q}}. \tag{7.6.19}$$

Thus there is a continuous inclusion $\mathscr{W}_A^{1,q}(I,X) \hookrightarrow C(I,X_{A,q})$.

(v) *Assume S is maximal q-regular. Then the map*

$$\mathscr{W}_A^{1,q}(I,X) \to X_{A,q} \times L^q(I,X) : x \mapsto (x(0), \dot{x} - Ax) \tag{7.6.20}$$

is a Banach space isomorphism. If this holds, then the inverse of (7.6.20) *is the operator $X_{A,q} \times L^q(I,X) \to \mathscr{W}_A^{1,q}(I,X) : (x_0,f) \mapsto x$ defined by* (7.6.2).

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Proof. Let $(x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $X_{A,q}$ and define the functions $y_i : I \to X$ by $y_i(0) := 0$ and

$$y_i(t) := AS(t)x_i$$

for $0 < t \le T$ and $i \in \mathbb{N}$. Then $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in X and $(y_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^q(I, X)$. Thus there exist an element $x \in X$ and, by Theorem 7.5.6, a strongly measurable function $y : I \to X$ with

$$\int_0^T \|y(t)\|_X^q \, dt < \infty$$

such that

$$\lim_{i\to\infty} \|x - x_i\|_X = 0, \qquad \lim_{i\to\infty} \int_0^T \|y(t) - y_i(t)\|_X^q \, dt = 0.$$

Passing to a subsequence, if necessary, we may also assume that the sequence $(y_i)_{i\in\mathbb{N}}$ converges almost everywhere to y by part (i) of Theorem 7.5.6. Thus there is a Borel set $Z \subset I$ of measure zero such that

$$y(t) = \lim_{i\to\infty} AS(t)x_i \qquad \text{for all } t \in I \setminus Z.$$

Since A is closed and

$$\lim_{i\to\infty} S(t)x_i = S(t)x,$$

this implies

$$S(t)x \in \mathrm{dom}(A), \qquad AS(t)x = y(t) \qquad \text{for all } t \in I \setminus Z.$$

Since Z has measure zero, we obtain $S(t)x \in \mathrm{dom}(A)$ for all $t > 0$ and

$$\int_0^T \|AS(t)x\|_X^q \, dt = \int_0^T \|y(t)\|_X^q \, dt < \infty.$$

Hence $x \in X_{A,q}$ and

$$\begin{aligned}\lim_{i\to\infty} \|x - x_i\|_{A,q} &= \lim_{i\to\infty} \|x - x_i\|_X + \lim_{i\to\infty} \left(\int_0^T \|y(t) - AS(t)x_i\|_X^q \, dt \right)^{1/q} \\ &= 0.\end{aligned}$$

This shows that $X_{A,q}$ is a Banach space. That the obvious inclusions

$$\mathrm{dom}(A) \hookrightarrow X_{A,q}, \qquad X_{A,q} \hookrightarrow X$$

are continuous follows directly from the definition of the norms. That $X_{A,q}$ is dense in X follows from Lemma 7.1.16 and the fact that $\mathrm{dom}(A) \subset X_{A,q}$. To prove that $\mathrm{dom}(A)$ is dense in $X_{A,q}$, one can use part (ii), which is an easy exercise left to the reader, and observe that the domain of the infinitesimal generator of the restricted semigroup $S(t)|_{X_{A,q}}$ contains $\mathrm{dom}(A)$. This proves parts (i) and (ii). Part (iii) follows directly from the definitions.

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To prove part (iv), assume that S is maximal q-regular, and define the bounded linear operator

$$\mathscr{S} : L^q(I,X) \to \mathscr{W}_A^{1,q}(I,X)$$

by

$$(\mathscr{S}f)(t) := \int_0^t S(t-s)f(s)\,ds$$

for $f \in L^q(I,X)$ and $0 \le t \le T$. Composing $\mathscr{S}$ with the bounded linear operator

$$\mathscr{W}_A^{1,q}(I,X) \to L^q(I,X) : x \mapsto \dot{x} - Ax$$

we obtain a bounded linear operator

$$\mathscr{T} : \mathscr{W}_A^{1,q}(I,X) \to \mathscr{W}_A^{1,q}(I,X)$$

given by

$$(\mathscr{T}x)(t) := \int_0^t S(t-s)\big(\dot{x}(s) - Ax(s)\big)\,ds$$

for $x \in \mathscr{W}_A^{1,q}(I,X)$ and $0 \le t \le T$. For $x \in C^1(I,\mathrm{dom}(A))$ and $0 \le t \le T$ we obtain from Lemma 7.1.14 the equation

$$(\mathscr{T}x)(t) - x(t) = S(t)x(0).$$

This implies the inequality

$$\|x(0)\|_{A,q} \le \|S(\cdot)x(0)\|_{\mathscr{W}_A^{1,q}} \le \Big(1 + \|\mathscr{T}\|\Big)\,\|x\|_{\mathscr{W}_A^{1,q}}$$

for all $x \in C^1(I,\mathrm{dom}(A))$. Since $C^1(I,\mathrm{dom}(A))$ is dense in $\mathscr{W}_A^{1,q}(I,X)$, this inequality continues to hold for all $x \in \mathscr{W}_A^{1,q}(I,X)$. Similar estimates, with a constant independent of t, for all the evaluation maps

$$\mathscr{W}_A^{1,q}(I,X) \to X_{A,q} : x \mapsto x(t)$$

can be obtained by shortening the interval for $0 \le t \le T/2$ and in addition reversing time for $T/2 \le t \le T$. Here one must use the fact that in the definition of the norm (7.6.18) on the space $X_{A,q}$, the number T can be chosen arbitrarily. Different choices of T give rise to equivalent norms. This proves part (iv). Part (v) follows directly from part (iv) and this completes the proof of Lemma 7.6.10. □

The preceding discussion sets up a general abstract framework for suitable Banach spaces of initial conditions and solutions for linear Cauchy problems. Under the assumption of maximal q-regularity these spaces can be used to obtain well-posed Cauchy problems for PDEs with nonlinearities in the highest order terms (see Remark 7.6.14 below).

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7.6.5. Regularity in Hilbert Spaces. For self-adjoint semigroups on Hilbert spaces maximal q-regularity is easy to verify for $q = 2$.

Theorem 7.6.11. *Every self-adjoint semigroup on a Hilbert space is maximal 2-regular.*

Proof. Let H be a Hilbert space and let $S : [0, \infty) \to \mathcal{L}(H)$ be a strongly continuous semigroup of self-adjoint operators with infinitesimal generator $A : \mathrm{dom}(A) \to H$. Then, by Theorem 7.3.4, we have

$$\omega := \sup_{x \in \mathrm{dom}(A) \setminus \{0\}} \frac{\langle x, Ax \rangle_H}{\|x\|_H^2} < \infty. \tag{7.6.21}$$

Let $V \subset H$ be the completion of $\mathrm{dom}(A)$ with respect to the norm

$$\|x\|_V := \sqrt{\langle x, cx - Ax \rangle}, \qquad c := \omega + 1. \tag{7.6.22}$$

Now let $x_0 \in \mathrm{dom}(A)$, let $f : [0, T] \to H$ be a continuously differentiable function, and define the function $x : [0, T] \to H$ by

$$x(t) := S(t)x_0 + \int_0^t S(t - s)f(s)\, ds \qquad \text{for } 0 \le t \le T. \tag{7.6.23}$$

Then $x(t) \in \mathrm{dom}(A)$ for all t and the function $x : [0, T] \to H$ is continuously differentiable and satisfies $\dot{x}(t) = Ax(t) + f(t)$ for all t (Lemma 7.1.14). Thus the function $t \mapsto \frac{1}{2}\|x(t)\|_V^2$ is continuously differentiable and

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|x(t)\|_V^2 &= \langle \dot{x}(t), cx(t) - Ax(t) \rangle_H \\
&= \langle f(t) + Ax(t), cx(t) - Ax(t) \rangle_H \\
&\le \|f(t)\|_H \|cx(t)\|_H + \|Ax(t)\|_H \|cx(t)\|_H \\
&\quad + \|f(t)\|_H \|Ax(t)\|_H - \|Ax(t)\|_H^2 \\
&\le \frac{3}{2}\|f(t)\|_H^2 + \frac{3c^2}{2}\|x(t)\|_H^2 - \frac{1}{2}\|Ax(t)\|_H^2.
\end{aligned}$$

Integrate this inequality over the interval $[0, T]$ to obtain

$$\|x(T)\|_V^2 + \int_0^T \|Ax(t)\|_H^2\, dt \le \|x_0\|_V^2 + 3\int_0^T \left(\|f(t)\|_H^2 + c^2\|x(t)\|_H^2\right) dt.$$

Now take $x_0 = 0$ and define $c_T := (2\omega)^{-1}(e^{2\omega T} - 1)$ when $\omega \ne 0$ and $c_T := T$ when $\omega = 0$. Then $\int_0^T \|x(t)\|_H^2\, dt \le Tc_T \int_0^T \|f(t)\|_H^2\, dt$ and so

$$\int_0^T \|Ax(t)\|_H^2\, dt \le 3\left(1 + c^2 Tc_T\right) \int_0^T \|f(t)\|_H^2\, dt.$$

This proves Theorem 7.6.11. □

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Remark 7.6.12. Let A and ω be as in the proof of Theorem 7.6.11. Let $B : \mathrm{dom}(B) \to H$ be the unique self-adjoint operator with $\langle x, Bx\rangle \geq 0$ for all $x \in \mathrm{dom}(B)$ that satisfies $B^2 = \omega\mathbb{1} - A$ (see Exercise 6.5.8). Then the space V in (7.6.22) is the domain of B, equipped with the graph norm of B. Moreover, V agrees with the space $X_{A,2}$ in (7.6.17) and hence there is a canonical inclusion $\mathscr{W}_A^{1,2}(I,H) \hookrightarrow C(I,V)$ (see part (iv) of Lemma 7.6.10).

Remark 7.6.13. For parabolic (second order) equations in an L^p-space, the question of finding the space of initial conditions that give rise to solutions in $W^{1,q}(I,L^p) \cap L^q(I,W^{2,p})$ has been studied by many mathematicians (see [**11**, **12**, **32**, **46**, **63**, **64**, **84**, **85**]). For the heat equation a theorem of Grigor'yan–Liu [**32**], which is based on work of Triebel [**84**, **85**], asserts that the initial conditions in the Besov space

$$B_q^{s,p}(\mathbb{R}^n), \qquad s = 2 - \frac{2}{q},$$

give rise to solutions in the space

$$\mathscr{W}^{1,q,p} := W^{1,q}([0,T], L^p(\mathbb{R}^n)) \cap L^q([0,T], W^{2,p}(\mathbb{R}^n)).$$

For $p = q = 2$ the relevant Besov space is the Hilbert space $W^{1,2}(\mathbb{R}^n)$ and the proof reduces to the simple abstract argument in Theorem 7.6.11. For $p \neq 2$ the Grigor'yan–Liu Theorem is a deep result which goes far beyond the scope of the present book. Another exposition is given in [**76**].

Remark 7.6.14. One reason for the importance of such results is that one can reformulate the existence and uniqueness problem for nonlinear parabolic equations of the form

$$\partial_t u = \Delta u + f(u), \qquad u(0,\cdot) = u_0, \tag{7.6.24}$$

as a fixed point problem for the map $\mathscr{W}^{1,q,p} \to \mathscr{W}^{1,q,p} : u \mapsto \mathcal{F}(u)$ given by

$$(\mathcal{F}(u))(t) := S(t)u_0 + \int_0^t S(t-s) f(u(s))\, ds. \tag{7.6.25}$$

Here $S : [0,\infty) \to \mathcal{L}(L^p(\mathbb{R}^n))$ is the heat semigroup and f can be a map from $W^{2,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Thus one can deal with nonlinearities in the highest order terms. Moreover, by standard regularity arguments, one can replace the Laplace operator by a general second order elliptic operator. In this situation it is sometimes important to choose $p > n/2$ to obtain the relevant nonlinear estimates, and so the easy case $p = 2$ may not suffice. Many important geometric PDEs, such as the Ricci flow, the mean curvature flow, the Yang-Mills flow, the harmonic map flow, or the Donaldson geometric flow for symplectic four-manifolds [**51**, **52**] can be formulated in this manner.

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7.7. Problems

Exercise 7.7.1 (**Semigroups on Complex Banach Spaces).** Let X be a complex Banach space and let $A : \mathrm{dom}(A) \to X$ be the infinitesimal generator of a strongly continuous semigroup $S : [0, \infty) \to \mathcal{L}(X)$. Suppose that $\mathrm{dom}(A)$ is a complex subspace of X and that A is complex linear. Prove that $S(t) \in \mathcal{L}^c(X)$ for all $t \geq 0$. **Hint:** Define the operator $T(t) \in \mathcal{L}(X)$ by

$$T(t)x := -\mathbf{i}S(t)\mathbf{i}x$$

for $x \in X$ and $t \geq 0$. Show that T is a strongly continuous semigroup with infinitesimal generator A and use Corollary 7.2.3.

Exercise 7.7.2 (**Contraction Semigroups).** Let X be a complex Banach space and let $A : \mathrm{dom}(A) \to X$ be a complex linear operator with a dense domain $\mathrm{dom}(A) \subset X$. Consider the following conditions.

(i) A generates a contraction semigroup.

(ii) A has a closed graph and both A and A^* are dissipative.

Prove that (ii) implies (i). If X is reflexive prove that (i) is equivalent to (ii). Find an example of an operator on a nonreflexive Banach space that satisfies (i) but not (ii). **Hint:** Definition 7.2.10.

Exercise 7.7.3 (**Dual Semigroup).** Prove that the domain of the infinitesimal generator A of the group on $L^1(\mathbb{R})$ in Example 7.3.3 is the space of absolutely continuous real valued functions on $\mathbb{R}$ with integrable derivative. Prove that the domain of the dual operator A^* on $L^\infty(\mathbb{R})$ is the space of bounded Lipschitz continuous functions from $\mathbb{R}$ to itself. Prove that $\sigma(A) = \sigma(A^*) = \mathbf{i}\mathbb{R}$. Prove that the operator A^* does not satisfy the requirements of the Hille–Yosida–Phillips Theorem 7.2.5 because its domain is not dense.

Exercise 7.7.4 (**Infinitesimal Generators of Unitary Groups).** Let H be a complex Hilbert space and let $A : \mathrm{dom}(A) \to H$ be an unbounded complex linear operator with a dense domain $\mathrm{dom}(A) \subset H$. Prove that the following are equivalent.

(i) If $\lambda \in \mathbb{R} \setminus \{0\}$, then $\lambda\mathbb{1} - A$ is bijective and $\left\|(\lambda\mathbb{1} - A)^{-1}\right\| \leq |\lambda|^{-1}$.

(ii) If $\lambda \in \mathbb{C} \setminus \mathbf{i}\mathbb{R}$, then $\lambda\mathbb{1} - A$ is bijective and $\left\|(\lambda\mathbb{1} - A)^{-1}\right\| \leq |\mathrm{Re}\lambda|^{-1}$.

(iii) $\mathrm{dom}(A^*) = \mathrm{dom}(A)$ and $A^*x + Ax = 0$ for all $x \in \mathrm{dom}(A)$.

Hint: Each of these conditions is equivalent to the assertion that A generates a unitary group, by Theorem 7.2.11 and Theorem 7.3.6. The exercise is to establish their equivalence without using semigroup theory. Show that (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

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Exercise 7.7.5 (**The Sobolev Space** $W^{1,2}(\mathbb{R})$**).** Prove that the space of smooth functions $f : \mathbb{R} \to \mathbb{R}$ with compact support and mean value zero is dense in $L^2(\mathbb{R})$. Deduce that the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm $f \mapsto \|f'\|_{L^2}$ in Example 7.1.7 can be identified with the space of equivalence classes of absolutely continuous function $f : \mathbb{R} \to \mathbb{R}$ with $\frac{df}{dx} \in L^2(\mathbb{R})$ under the equivalence relation $f_1 \sim f_2$ iff $f_1 - f_2$ is constant.

Exercise 7.7.6 (**Maximal Regularity).** Let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup on a Banach space X with infinitesimal generator A and let $q > 1$. If the estimate (7.6.11) holds for some $T > 0$, prove that it holds for all $T > 0$ with a constant depending on T.

Exercise 7.7.7 (**The Banach Space** $L^p(I, X)$ **and its Dual).**

(a) Verify the assertions of Theorem 7.5.6 for $p = \infty$.

(b) Verify the assertions of Lemma 7.5.15 for $p = 1$ and $p = \infty$.

(c) Prove that the composition (7.5.27) in the proof of Corollary 7.5.17 is the canonical inclusion $\iota : L^p(I, X) \to L^p(I, X)^{**}$.

Exercise 7.7.8 (**The Radon–Nikodým Property).** Let $I := [0, 1]$ be the unit interval and $1 \le p \le \infty$. Define the function $f : [0, 1] \to L^p(I)$ by

$$(f(t))(s) := \begin{cases} 1, & \text{if } 0 \le s \le t, \\ 0, & \text{if } t < s \le 1, \end{cases} \qquad \text{for } s, t \in I.$$

When $p = \infty$, prove that f is everywhere discontinuous. When $1 < p < \infty$, prove that f is Hölder continuous. When $p = 1$, prove that f is Lipschitz continuous and nowhere differentiable. Deduce that $L^1(I)$ is not isomorphic to the dual space of any Banach space. **Hint:** Theorem 7.5.13.

Exercise 7.7.9 (**Lebesgue Differentiation).** Let X be a Banach space and let $f : I := [0, 1] \to X$ be a strongly measurable function such that $\int_0^1 \|f(t)\| \, dt < \infty$. Define the function $F : [0, 1] \to X$ by

$$F(t) := \int_0^t f(s) \, ds \qquad \text{for } 0 \le t \le 1.$$

Prove that F is absolutely continuous and that there is a Borel set $Z \subset I$ of Lebesgue measure zero such that F is differentiable on $I \setminus Z$ and

$$F'(t) = \lim_{h \to 0} \frac{F(t + h) - F(t)}{h} = f(t) \qquad \text{for every } t \in I \setminus Z.$$

Hint: The proof of the Lebesgue Differentiation Theorem in [**75**, Thm. 6.14] carries over verbatim to Banach space valued functions.

Exercise 7.7.10 (**Bounded Lipschitz Continuous Functions).** Prove that the closure of the space of bounded Lipschitz continuous functions in $L^\infty(\mathbb{R})$ is the space of bounded uniformly continuous functions on $\mathbb{R}$.

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Exercise 7.7.11 (**Weak and Strong Continuity**). Let X be a real Banach space and let $S : [0, \infty) \to \mathcal{L}(X)$ be a one-parameter semigroup. Prove that the following are equivalent.

(i) The function $[0, \infty) \to X : t \mapsto S(t)x$ is continuous for all $x \in X$.

(ii) The function $[0, \infty) \to \mathbb{R} : t \mapsto \langle x^*, S(t)x \rangle$ is continuous for all $x \in X$ and all $x^* \in X^*$.

Hint: To prove that (ii) implies (i), show first that

$$\sup_{0 \le t \le T} \|S(t)\| < \infty \qquad \text{for all } T > 0, \tag{7.7.1}$$

using the Uniform Boundedness Theorem 2.1.1. Second, use part (iii) of Theorem 7.5.2 and Lemma 7.5.5 to prove that

$$\lim_{\substack{h \to 0 \\ h > 0}} \int_0^{T-h} \|S(t+h)x - S(t)x\| \, dt = 0. \tag{7.7.2}$$

Third, fix a constant $\varepsilon > 0$, define

$$M := \sup_{0 \le s \le \varepsilon} \|S(s)\|,$$

prove the estimate

$$\|S(t+h)x - S(t)x\| \le \frac{M}{\varepsilon} \int_{t-\varepsilon}^{t} \|S(s+h)x - S(s)x\| \, ds \tag{7.7.3}$$

for $x \in X$ and $0 < |h| < \varepsilon < t/2$, and use this estimate to show that the function $[0, \infty) \to X : t \mapsto S(t)x$ is continuous for $t > 0$. Fourth, prove that

$$\lim_{t \to 0} \|S(t)x - x\| = 0 \tag{7.7.4}$$

for all $x \in X$, by observing that x belongs to the closure of the linear subspace

$$Z := \operatorname{span}\left\{ S(t)x \,\middle|\, 0 < t < 1 \right\}$$

and using $\lim_{t \to 0} \|S(t)z - z\| = 0$ for all $z \in Z$.

Exercise 7.7.12 (**Regularity of Weak Solutions**). Let X be a Banach space and let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous semigroup with infinitesimal generator $A : \operatorname{dom}(A) \to X$. Let $I := [0, T]$ and $f \in \mathcal{L}^1(I, X)$ and define $x(t) := \int_0^t S(t-s) f(s) \, ds$ for $0 \le t \le T$.

(a) If $f \in W^{1,1}(I, X)$, prove that $x \in W^{1,1}(I, X)$.

(b) If $f(t) \in \operatorname{dom}(A)$ for all t and $Af \in \mathcal{L}^1(I, X)$, prove that $x \in W^{1,1}(I, X)$.

Hint: For part (a) use Lemma 7.1.14 and approximation. For part (b) assume first that $Af : I \to X$ is continuous and then use approximation.

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Exercise 7.7.13 (**Semigroups and Compact Operators**). Let $I = [0,1]$ be the unit interval, let U, X, Y be real Banach spaces, and let $[0,\infty) \to \mathcal{L}(X) : t \mapsto S(t)$ be a strongly continuous semigroup.

(a) Let $I \to \mathcal{L}(X,Y) : t \mapsto K(t)$ be a strongly continuous family of operators. Prove that the operator

$$X \to C(I,Y) : x \mapsto K(\cdot)x \tag{7.7.5}$$

is compact if and only if the operator $K(t) \in \mathcal{L}(X,Y)$ is compact for every $t \in I$ and the map $K : I \to \mathcal{L}(X,Y)$ is continuous with respect to the operator norm on $\mathcal{L}(X,Y)$. **Hint:** Consider the set $\mathscr{F} \subset C(I,Y)$ whose elements are the functions $f_x := K(\cdot)x$ for all $x \in X$ with $\|x\| \le 1$. Prove that $\mathscr{F}$ is equi-continuous if and only if the map $K : I \to \mathcal{L}(X,Y)$ is continuous with respect to the operator norm. Use Theorem 1.1.11.

(b) For $t \in I$ let $K(t) \in \mathcal{L}(X,Y)$ be a compact operator and suppose that the function $K : I \to \mathcal{L}(X,Y)$ is continuous with respect to the norm topology. Prove that the operator

$$L^1(I,X) \to Y : f \mapsto \int_0^1 K(t)f(t)\,dt \tag{7.7.6}$$

is compact. **Hint:** Show first that the function $I \to Y : t \mapsto K(t)f(t)$ is strongly measurable whenever $f : I \to X$ is strongly measurable. Second, use part (a) to prove that the operator $Y^* \to C(I,X^*) : y^* \mapsto K^*(\cdot)y^*$ is compact. Third, show that the composition of this operator with the canonical isometric inclusion $C(I,X^*) \to L^1(I,X)^*$ (Lemma 7.5.15) is the dual operator of (7.7.6). Then use Theorem 4.2.10.

(c) Let $B \in \mathcal{L}(U,X)$ be a compact operator. Prove that the operator

$$L^1(I,U) \to X : f \mapsto \int_0^1 S(t)Bf(t)\,dt \tag{7.7.7}$$

is compact. **Hint:** Show that the map $I \to \mathcal{L}(U,X) : t \mapsto S(t)B$ is continuous in the norm topology and use part (b).

(d) Let $C \in \mathcal{L}(X,Y)$ be a compact operator. If X is reflexive, prove that the operator

$$X \to C(I,Y) : x \mapsto CS(\cdot)x \tag{7.7.8}$$

is compact. Find an example of a semigroup on a nonreflexive Banach space X and a compact operator $C : X \to Y$ such that the operator (7.7.8) is not compact. **Hint:** Consider the shift semigroup on $X = L^1([0,1])$ and let $C : X \to \mathbb{R}$ be the bounded linear functional $x \mapsto \int_0^1 x(t)\,dt$. Relate this to the fact that the inclusion of $W^{1,1}(I)$ into $C(I)$ is not a compact operator. (See Exercise 4.5.16.)

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Exercise 7.7.14 (Semigroups and Functional Calculus). Let H be a complex Hilbert space, let $A : \mathrm{dom}(A) \to H$ be an unbounded self-adjoint operator on H with spectrum

$$\Sigma := \sigma(A) \subset (-\infty, 0],$$

and let $\Psi_A : C_b(\Sigma) \to \mathcal{L}^c(H)$ be the functional calculus in Theorem 6.4.1. Let $\mathcal{B} \subset 2^\Sigma$ be the Borel σ-algebra and, for $x, y \in H$, define the signed Borel measure $\mu_{x,y} : \mathcal{B} \to \mathbb{R}$ by $\mu_{x,y}(\Omega) := \mathrm{Re}\langle x, \Psi_A(\chi_\Omega)y\rangle$ for all $\Omega \in \mathcal{B}$ as in Definition 6.4.3 and Theorem 6.4.4. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$ define the linear operator $S(z) \in \mathcal{L}^c(H)$ by

$$\mathrm{Re}\langle x, S(z)y\rangle := \int_\Sigma e^{z\lambda} d\mu_{x,y}(\lambda) \tag{7.7.9}$$

for $x, y \in H$ (see Theorem 5.6.2).

(a) Verify that $S(z) = \Psi_A(f_z)$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$, where the function $f_z : \Sigma \to \mathbb{C}$ is defined by $f_z(\lambda) := e^{\lambda z}$ for $\lambda \in \Sigma$.

(b) Verify the formulas $S(0) = \mathrm{id}$ and $S(z + w) = S(w)S(z)$ for all $w, z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$ and $\mathrm{Re}(w) \geq 0$.

(c) Show that the map $z \mapsto S(z)$ is continuous in the norm topology on the open right half-plane and is strongly continuous on the closed right half-plane.

(d) Show that the map $z \mapsto S(z)$ is the analytic semigroup generated by A (see Example 7.4.5 and Theorem 7.4.4).

(e) Show that the map $\mathbb{R} \to \mathcal{L}^c(H) : t \mapsto S(\mathbf{i}t)$ is the unitary group generated by $\mathbf{i}A$ (see Theorem 7.3.6).

(f) By considering the Laplace operator on $\mathbb{R}^n$, deduce from (e) that the heat equation in Example 7.1.6 and the Schrödinger equation in Example 7.3.8 (adapted to dimension n) fit into a single strongly continuous semigroup on the right half-plane.

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