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An Introduction to Their Analysis and Numerical Solution

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Preface

Convection-diffusion problems attract much attention in the research literature. For numerical analysts working in this area, a standard reference is the text by Roos, Stynes, and Tobiska [**RST96**, **RST08**]. This book contains a lot of useful information, but it is daunting for those beginners who have some familiarity with numerical methods and their analysis but who have not previously worked with convection-diffusion and other singularly perturbed differential equations. For many years I felt that an easier, more introductory book was needed to encourage new people to enter our fascinating research area. This belief was encouraged by the popularity of a survey article, "Steady-state convection-diffusion problems", that I wrote for *Acta Numerica* in 2005 [**Sty05**]. The present book is an extended and updated version of that 2005 article, and I have added exercises and other material to try to make it more attractive and more useful for the novice reader.

The feeling that a book of this type was desirable did not lead me to take any action until I was invited to present a course on this topic at the AARMS (Atlantic Association for Research in the Mathematical Sciences) Summer School at Dalhousie University in Halifax, Nova Scotia, Canada, during July 2015. The organisers encourage their lecturers to transform their lecture notes into books, and after much delay I have done this. I am very grateful to AARMS for their invitation to lecture and for the enjoyable month I spent in the delightful city of Halifax.

Here we list the prerequisites for the reader. In Chapters 1–3 some knowledge of two-point boundary value problems and their numerical solution by finite difference methods is enough for almost all of the material.

For Chapter 4 it is desirable to have some previous experience of partial differential equations. Chapter 5 uses only ideas from earlier chapters. Finite element methods (FEMs) appear for the first time in the long Chapter 6, and here I assume that the reader already has a general understanding of how FEMs are constructed and analysed. The Lebesgue spaces $L^p(\Omega)$ and the standard Sobolev spaces $H^k(\Omega)$ are used occasionally in the earlier chapters of the book and more heavily in Chapter 6; the reader should have some familiarity with these well-known concepts.

The book was written where I work, in the research paradise known as Beijing Computational Science Research Center. I owe a great debt to CSRC's director Hai-Qing Lin for the positive environment he has created at CSRC through his friendly yet no-nonsense approach to productive research. My work was supported by the 1000 Talents (Foreign Experts) Program of the People's Republic of China.

All comments on this book are welcome. No doubt it will (inevitably) contain some mistakes, so corrections are also welcome, though the fewer the better! My email address is `m.stynes@csrc.ac.cn`

Martin Stynes

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