GRADUATE STUDIES IN MATHEMATICS 198

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American Mathematical Society

10.1090/gsm/198

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2010 *Mathematics Subject Classification*. Primary 37P40;
Secondary 37P30, 37P50.

For additional information and updates on this book, visit
www.ams.org/bookpages/gsm-198

Library of Congress Cataloging-in-Publication Data

Names: Benedetto, Robert L., 1972– author.
Title: Dynamics in one non-archimedean variable / Robert L. Benedetto.
Description: Providence, Rhode Island : American Mathematical Society, [2019] | Series: Graduate studies in mathematics ; volume 198 | Includes bibliographical references and index.
Identifiers: LCCN 2018044194 | ISBN 9781470446888 (alk. paper)
Subjects: LCSH: Geometry, Analytic–Textbooks. | p-adic analysis–Textbooks. | Analytic spaces–Textbooks. | AMS: Dynamical systems and ergodic theory – Arithmetic and non-Archimedean dynamical systems – Non-Archimedean Fatou and Julia sets. msc | Dynamical systems and ergodic theory – Arithmetic and non-Archimedean dynamical systems – Height functions; Green functions; invariant measures. msc | Dynamical systems and ergodic theory – Arithmetic and non-Archimedean dynamical systems – Dynamical systems on Berkovich spaces. msc
Classification: LCC QA551 .B4255 2019 | DDC 515/.39—dc23
LC record available at https://lccn.loc.gov/2018044194

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10 9 8 7 6 5 4 3 2 1 24 23 22 21 20 19

For Danielle

Contents

List of Notation

$\mathbb{P}^1(K)$ .... projective line over field K; p. 11
ϕ^n .... nth iterate of ϕ; p. 11
$\mathrm{Orb}_\phi^+(x)$ .... forward orbit of x under ϕ; p. 11
$\mathrm{Orb}_\phi^-(x)$ .... backward orbit of x under ϕ; p. 11
$\mathrm{GO}_\phi(x)$ .... grand orbit of x under ϕ; p. 11
T_d .... dth Chebyshev polynomial; p. 14
$\deg(\phi)$ .... degree of the rational function ϕ; p. 15
E_ϕ .... exceptional set of ϕ; p. 21
$\mathrm{ord}_b(\phi)$ .... order of vanishing of ϕ at b; p. 30
Σ .... set of symbol sequences; p. 31
$C_{A,n}$ .... cylinder set p. 31
$|\cdot|$ .... absolute value on a field K; p. 33
$\mathcal{O}_K$ .... ring of integers of a non-archimedean field K; p. 34
$\mathcal{M}_K$ .... maximal ideal of a non-archimedean field K; p. 34
$\mathbb{F}((t))$ .... field of formal Laurent series over the field $\mathbb{F}$; p. 35
$\mathbb{C}_v$ .... a complete, algebraically closed non-archimedean field; p. 35
$\mathbb{Q}_p$ .... the field of p-adic rationals; p. 37
$\mathbb{Z}_p$ .... the ring of p-adic integers; p. 37
$\mathbb{C}_p$ .... the completion of an algebraic closure of $\mathbb{Q}_p$; p. 37
$D(a,r)$ .... open disk of radius r centered at $a \in \mathbb{C}_v$; p. 38
$\overline{D}(a,r)$ .... closed disk of radius r centered at $a \in \mathbb{C}_v$; p. 38

Preface

This book grew from my lecture notes for a short course I gave at the Arizona Winter School in March 2010. My aim has been to present an exposition accessible to students in their second or third year of graduate school, while also providing a one-stop reference for active researchers in arithmetic and non-archimedean dynamics. In particular, I do not assume the reader has any prior familiarity with the basics of dynamics, with non-archimedean analysis, or with Berkovich spaces. I *do* assume the reader has already seen some non-archimedean fields, usually the p-adic numbers and hopefully the complete, algebraically closed p-adic field $\mathbb{C}_p$, but even those topics are reviewed briefly in Chapter 2.

The theory of p-adic dynamics, and more generally, non-archimedean dynamics, is modeled on the theory of complex dynamics. Naturally, a student of complex dynamics is expected to have already completed a full course on complex analysis. For the sake of argument, though, one could probably learn quite a bit of complex dynamics with minimal prior complex analysis coursework if one were willing to accept certain analytic facts on faith. It would of course be unwise to try to approach complex dynamics that way, but my point is that it *could* be done.

The same is true of non-archimedean dynamics: mastery of the subject requires fluency in non-archimedean analysis, but getting started requires far less background. In practice, after all, most students of non-archimedean dynamics learn non-archimedean analysis along the way, not as a prerequisite. Or at least, that was how *I* learned it.

Thus, this book interleaves the basics of non-archimedean analysis and of elementary dynamics with the main topics of non-archimedean dynamics. For example, the more advanced theory relies heavily on the Berkovich projective line, discussed in Chapter 6. However, a lot can be done with less background, and therefore the fundamentals of non-archimedean dynamics appear earlier, in Chapter 4.

In addition, even in the exposition of non-archimedean analysis (Chapter 3) and Berkovich's theory (Chapters 6 and 7), I defer the more involved proofs to later chapters (specifically, Chapters 14–16). My aim is to acquaint the reader with the requisite analysis as quickly as possible before returning to our primary topic of dynamics. At the same time, a reader who wishes to see those proofs can easily find them by flipping to the back of the book.

As is the case for any project, there are many people who helped make this book possible. I would therefore like to thank the University of Arizona and the organizers of the 2010 Arizona Winter School: Matt Papanikolas, Rachel Pries, Fernando Rodriguez-Villegas, David Savitt, and Dinesh Thakur, as well as the staff of the University of Arizona. I am similarly grateful to Ina Mette, Marcia Almeida, and the rest of the publishing staff at the American Mathematical Society, especially Jennifer Wright Sharp for her careful reading of the manuscript. Many thanks also to Matt Baker, Xander Faber, Charles Favre, and Juan Rivera-Letelier, whose expert input was enormously valuable for organizing the material, improving certain proofs, and presenting important results and concepts clearly. Further thanks are due to Ryan Alvarado, John Benedetto, Amalia Culiuc, Jeff Diller, Fan Shilei, and the anonymous referees who provided mathematical expertise, found errors in early drafts, or otherwise made helpful suggestions. I am particularly grateful to Joe Silverman, who read a draft of the manuscript very closely and taught a course out of it; his countless excellent suggestions improved the presentation substantially. In addition, I gratefully acknowledge the support of NSF grants DMS-1201341 and DMS-1501766. Above all, I thank my wonderful wife Danielle for her constant love and support.

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