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The Distribution of Prime Numbers

Dimitris Koukoulopoulos

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To Jennifer

Contents

Preface

The main goal of this book is to introduce beginning graduate students to analytic number theory. In addition, large parts of it are suitable for advanced undergraduate students with a good grasp of analytic techniques.

Throughout, the emphasis has been put on exposing the main ideas rather than providing the most general results known. Any student wishing to do serious research in analytic number theory should broaden and deepen their knowledge by consulting some of the several excellent research-level books on the subject. Examples include: the books of Davenport [31] and of Montgomery-Vaughan [146] for classical multiplicative number theory; Tenenbaum's book [172] for probabilistic number theory and the saddle-point method; the book by Iwaniec-Kowalski [114] for the general theory of L-functions, of modular forms and of exponential sums; Montgomery's book [144] for the harmonic analytic aspects of analytic number theory; and the book of Friedlander-Iwaniec [59] for sieve methods.

Using the book

The book borrows the structure of Davenport's masterpiece *Multiplicative Number Theory* with several short- to medium-length chapters. Each chapter is accompanied by various exercises. Some of them aim to exemplify the concepts discussed, while others are used to guide the students to self-discover certain more advanced topics. A star next to an exercise indicates that its solution requires total mastery of the material.

The contents of the book are naturally divided into six parts as indicated in the table of contents. The first two parts study elementary and classical complex-analytic methods. They could thus serve as the manual for an introductory graduate course to analytic number theory. The last three parts of the book are devoted to the theory of sieves: Part 4 presents the basic elements of the theory of the small sieve, whereas Part 5 explores the method of bilinear sums and develops the large sieve. These techniques are then combined in Part 6 to study the spacing distribution of prime numbers and prove some of the recent spectacular results about small and large gaps between primes. Finally, Part 3 studies multiplicative functions and the anatomy of integers, and serves as a bridge between the complex-analytic techniques and the more elementary theory of sieves. Topics from it could be presented either in the end of an introductory course to analytic number theory (Chapter 13 most appropriately), or in the beginning of a more advanced course on sieves (the most relevant material is contained in Chapters 14 and 15, as well as in Theorem 16.1).

Certain portions of the book can be used as a reference for an undergraduate course. More precisely, Chapters 1–8 can serve as the core of such a course, followed by a selection of topics from Chapters 14, 15, 17 and 21.

A short guide to the main theorems of the book. Below is a list of the main results proven and of their prerequisites.

Chebyshev's and Mertens' estimates are presented in Chapters 2 and 3, respectively. Their proofs rest on the material contained in Part 1.

The landmark *Prime Number Theorem* is proven in Chapter 8. Understanding it requires a good grasp of all preceding chapters.

The *Siegel-Walfisz theorem*, which is a uniform version of the Prime Number Theorem for arithmetic progressions, is presented in Chapter 12. Its proof builds on all of the material preceding it.

The *Landau-Selberg-Delange method* is a key tool in the study of multiplicative functions. It is presented in Chapter 13. Appreciating its proof requires a firm understanding of Chapters 1–8 for the main analytic tools, as well as of Chapter 12 for dealing with uniformity issues.

The foundations of *probabilistic number theory* are explained in Chapters 15 and 16, where the *Erdős-Kac theorem* and the *Sathe-Selberg theorem* are proven. The main prerequisites can be found in Part 1 and in Chapter 14. In addition, Chapter 13 is needed for the Sathe-Selberg theorem.

The *Fundamental Lemma of Sieve Theory* is proven in Chapter 19. Its proof uses ideas and techniques from Part 1 and Chapters 14–17.

Vinogradov's method, one of the foundations of modern analytic number theory, is presented in Chapter 23. It builds on the material of Chapters 1–12 and 19.

The *Hardy-Littlewood circle method* is presented in Chapter 24. It is used to detect additive patterns among the primes and, more specifically, to count ternary arithmetic progressions all of whose members are primes.

The *Bombieri-Vinogradov theorem*, often called the "Generalized Riemann Hypothesis on average", is established in Chapter 26. Understanding its proof requires mastery of Vinogradov's method (Chapter 23) and of the *large sieve* (Chapter 25).

Linnik's theorem provides a very strong bound on the least prime in an arithmetic progression. It is proven in Chapter 27 and its prerequisites are Chapters 1–12, 17–20, 22–23 and 25.

The breakthrough of Zhang-Maynard-Tao about the existence of infinitely many *bounded gaps between primes* is presented in Chapter 28. Its proof requires a firm understanding of the Fundamental Lemma of Sieve Theory (Chapter 19), of Selberg's sieve (Chapter 21) and of the Bombieri-Vinogradov theorem (Chapter 26).

The recent developments about *large gaps between primes* of Ford-Green-Konyagin-Tao and Maynard are presented in Chapter 29. Understanding them necessitates knowledge of the same concepts as the proof of the existence of bounded gaps between primes, with the addition of the results on smooth numbers presented in Chapters 14 and 16.

Maier discovered in 1985 that the distribution of prime numbers has certain unexpected irregularities. His results are presented in Chapter 30 and they assume knowledge of Linnik's theorem (and of its prerequisites), as well as of Buchstab's function (see Chapter 14 and, more precisely, Theorem 14.4).

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Notation

Throughout the book, we make use of some standard and some less standard notation. We list here the most important conventions.

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers (we do not include zero in $\mathbb{N}$), integers, rational numbers, real numbers and complex numbers, respectively. Furthermore, given an integer $n \geqslant 1$, we write $\mathbb{Z}/n\mathbb{Z}$ for the set of residues mod n, as well as $(\mathbb{Z}/n\mathbb{Z})^*$ for the set of reduced residues mod n.

We write $\mathbb{P}$ to indicate a probability measure, and $\mathbb{E}[X]$ and $\mathbb{V}[X]$ for the expectation and the variance, respectively, of a random variable X.

Given a set of real numbers A and a parameter y, we write $A_{\leqslant y}$ for the set of numbers $a \in A$ that are $\leqslant y$; similarly for $A_{>y}$, $A_{\geqslant y}$, $A_{<y}$. We also write $|A|$ or $\#A$ for the cardinality of A, whichever is more convenient.

The letter p always denotes a prime, and the letter n always denotes an integer (usually, a natural number). We write $d|n$ to mean that d divides n, and that $p^k\|n$ to mean that p^k is the exact power of p dividing n. Lastly, $d|n^\infty$ means that all prime factors of d appear in the factorization of n too.

When we write (a, b), we might mean the open interval with endpoints a and b, the pair of a and b, or the greatest common divisor of the integers a and b. The meaning will always be clear from the context. Similarly, the symbol $[a, b]$ will sometimes denote the closed interval with endpoints a and b, and some other times the least common multiple of the integers a and b.

We write $P^+(n)$ and $P^-(n)$ to denote the largest and smallest prime factors of n, respectively, with the convention that $P^+(1) = 1$ and $P^-(1) = \infty$. Given a parameter y and an integer $n \geqslant 1$, we say that n is *y-smooth* if all its prime factors are $\leqslant y$ (i.e., if $P^+(n) \leqslant y$). The set of y-smooth numbers is denoted by $\mathcal{S}(y)$. Lastly, we say that n is *y-rough* if all its prime factors are $> y$ (i.e., if $P^-(n) > y$). Equivalently, $(n, P(y)) = 1$, where $P(y) := \prod_{p \leqslant y} p$.

The symbol log denotes the natural logarithm (base e). We also let $\mathrm{li}(x) = \int_2^x \mathrm{d}t/\log t$ denote the *logarithmic integral*.

Given $x \in \mathbb{R}$, we write $\lfloor x \rfloor$ for its integer part (defined to equal $\max \mathbb{Z}_{\leqslant x}$, and also called the "floor" of x), $\lceil x \rceil$ for the "ceiling" of x (defined to equal $\min \mathbb{Z}_{\geqslant x}$) and $\{x\}$ for the fractional part of x (defined to equal $x - \lfloor x \rfloor$).

Given $\alpha \in \mathbb{R}$, we write $\|\alpha\|$ to denote its distance from the nearest integer. On the other hand, if ψ is a bilinear form, then $\|\psi\|$ denotes its norm (see Chapter 25). Finally, if $\vec{v} \in \mathbb{C}^n$ or $f : \mathbb{N} \to \mathbb{C}$ is an arithmetic function, we write $\|\vec{v}\|_2$ and $\|f\|_2$ for their ℓ^2-norm.

The symbol $C^k(X)$, where $X \subseteq \mathbb{R}$ and $k \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, denotes the set of functions $f : X \to \mathbb{C}$ whose first k derivatives exist and are continuous.

We write 1_E to denote the indicator function of a set or of an event E. For example, $1_{[0,1]}$ denotes the indicator function of the interval $[0,1]$ and $1_{(n,10)=1}$ denotes the indicator function of the event that n is coprime to 10. In particular, 1_P will denote the indicator function of the set of primes.

The letter s will usually denote a complex number, in which case we denote its real part by σ and its imaginary part by t following Riemann's original notation that has now become standard. In addition, non-trivial zeroes of the Riemann zeta function and of Dirichlet L-functions will be denoted by $\rho = \beta + i\gamma$. Notice that we also use the letter γ for the Euler-Mascheroni constant, whereas $\rho(u)$ will also refer to the Dickman-de Bruijn function. The precise meaning of each letter will be clear from the context.

We employ frequently the usual asymptotic notation $f = O(g)$, $f \ll g$, $f \asymp g$, $f \sim g$ and $f = o(g)$, whose precise definition is given in Chapter 1.

Finally, we list below some other symbols and the page of their definition:

$1_P(n)$	xii	$\zeta(s)$	2	$\tau(n)$	33
$B(u)$	150	$\theta(x)$	13	$\tau_k(n)$ $(k \in \mathbb{N})$	33
$e(x)$	102	$\Lambda(n)$	37	$\tau_\kappa(n)$ $(\kappa \in \mathbb{C})$	131
$\mathcal{G}(\chi)$	103	$\Lambda^\sharp(n)$	237	$\varphi(n)$	4
$\mathrm{li}(x)$	1	$\Lambda^\flat(n)$	237	$\chi_0(n)$	97, 100
$L(s,\chi)$	97	$\Lambda^\sharp_{\mathrm{sieve}}(n)$	239	$\chi(n)$	96, 100
$P(y)$	xii	$\Lambda^\flat_{\mathrm{sieve}}(n)$	239	$\psi(x)$	22
$P^\pm(n)$	xi	$\mu(n)$	35	$\psi(x;q,a)$	98
$S(\mathcal{A},\mathcal{P})$	182	$\pi(x)$	1	$\psi(x,\chi)$	98
$\mathcal{S}(y)$	xii	$\pi(x;q,a)$	4	$\Psi(x,y)$	152
$\Gamma(s)$	17	$\rho(u)$	152	$\omega(n)$, $\Omega(n)$	29

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Prime numbers have fascinated mathematicians since the time of Euclid. This book presents some of our best tools to capture the properties of these fundamental objects, beginning with the most basic notions of asymptotic estimates and arriving at the forefront of mathematical research. Detailed proofs of the recent spectacular advances on small and large gaps between primes are made accessible for the first time in textbook form. Some other highlights include an introduction to probabilistic methods, a detailed study of sieves, and elements of the theory of pretentious multiplicative functions leading to a proof of Linnik's theorem.

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