

GRADUATE STUDIES  
IN MATHEMATICS **216**

**A Concise  
Introduction to  
Algebraic Varieties**

**Brian Osserman**



**AMS**

AMERICAN  
MATHEMATICAL  
SOCIETY

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Providence, Rhode Island

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To Fu, Lingkai, and Taokai



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# Contents

Preface	xi
Chapter 1. Introduction: An overview of algebraic geometry through the lens of plane curves	1
§1.1. Plane curves	1
§1.2. Elliptic curves	5
Chapter 2. Affine algebraic varieties	11
§2.1. Zero sets and the Zariski topology	11
§2.2. Zero sets and ideals	15
§2.3. Noetherian spaces	22
§2.4. Dimension	24
Chapter 3. Regular functions and morphisms	31
§3.1. Regular functions	31
§3.2. Morphisms	39
§3.3. Rational maps	48
§3.4. Chevalley's theorem	52
§3.A. Recovering geometry from categories	55
Chapter 4. Singularities	59
§4.1. Tangent lines and singularities	60
§4.2. Zariski cotangent spaces	62
§4.3. The Jacobian criterion	65
§4.4. Completions and power series	71



---

§4.5. Normality and normalization	74
§4.A. Local generation of ideals	77
Chapter 5. Abstract varieties via atlases	81
§5.1. Prevarieties	81
§5.2. Regular functions and morphisms	86
§5.3. Abstract varieties	90
§5.4. Normalization revisited	96
Chapter 6. Projective varieties	103
§6.1. Projective space	103
§6.2. Projective varieties and morphisms	108
§6.3. Blowup of subvarieties	114
§6.A. Homogeneous ideals and coordinate rings	122
Chapter 7. Nonsingular curves and complete varieties	125
§7.1. Curves, regular functions, and morphisms	125
§7.2. Quasiprojectivity	130
§7.3. Projective curves	132
§7.4. Completeness	135
§7.5. A limit-based criterion	137
§7.6. Irreducibility of polynomials in families	141
Chapter 8. Divisors on nonsingular curves	147
§8.1. Morphisms of curves	147
§8.2. Divisors on curves	150
§8.3. Linear equivalence and morphisms to projective space	153
§8.4. Embeddings of curves	160
§8.5. Secant varieties and curves in projective space	163
Chapter 9. Differential forms	169
§9.1. Differential forms	169
§9.2. Differential forms on curves	173
§9.3. Differential forms and ramification	176
§9.A. Field automorphisms and Frobenius	181
Chapter 10. An invitation to the theory of algebraic curves	187
§10.1. The Riemann-Roch theorem	187
§10.2. The Riemann-Hurwitz theorem	190

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§10.3. Brill-Noether theory and moduli spaces of curves	195
§10.A. Remarks on proofs of the Riemann-Roch theorem	199
Appendix A. Complex varieties and the analytic topology	207
§A.1. Quasiaffine complex varieties	207
§A.2. The analytic topology on prevarieties	209
§A.3. Fundamental results	211
§A.4. Nonsingularity and complex manifolds	212
§A.5. Connectedness	213
Appendix B. A roadmap through algebra	217
§B.1. Field theory	218
§B.2. Algebras	221
§B.3. Noetherian rings	222
§B.4. Rings of fractions	223
§B.5. Nakayama's lemma	225
§B.6. Unique factorization	228
§B.7. Integral extensions	229
§B.8. Integral closure	231
§B.9. The principal ideal theorem and regular local rings	232
§B.10. Noether normalization and first applications	236
§B.11. Dimension theory over fields	238
§B.12. Extensions of Dedekind domains	241
§B.13. Completion and power series	244
Bibliography	247
Index of Notation	251
Index	253



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# Preface

This book is an introduction to the theory of abstract algebraic varieties over algebraically closed fields, intended for a one-semester or one-quarter course. It provides a foundation in the basic properties of varieties, preparing the reader for either learning a more specialized topic such as toric varieties or moduli of curves, or moving on to the theory of schemes. Fundamental objects such as affine and projective varieties, regular functions and morphisms, and differential forms are introduced and interspersed with a mixture of local and global properties including nonsingularity, normality, and completeness. These ideas are combined to treat the theory of abstract nonsingular curves and linear series on them, concluding with an invitation to further directions in the form of a treatment of consequences of the Riemann-Roch theorem (a proof is not provided) and a brief discussion of Brill-Noether theory and moduli spaces of curves. The central object of the book is the abstract algebraic variety, which we define using atlases, as is commonly done for differentiable manifolds. Although this is, as far as I am aware, ahistorical,<sup>1</sup> it seems to strike a good balance between keeping the material down to earth while providing sufficient generality to serve as the foundation for learning about subjects such as toric varieties.

Our point of view is that abstract varieties are not motivated primarily by wanting to construct varieties that cannot be embedded in projective space, but rather because certain important constructions, such as normalization and toric varieties, are expressed most naturally in the context of abstract rather than projective varieties. Indeed, this philosophy goes back to the origins of the concept: Weil introduced abstract varieties in order to

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<sup>1</sup>As I understand it, abstract varieties were first introduced by Weil over arbitrary fields in [Wei46], requiring much more technical definitions.

construct Jacobian varieties of curves over arbitrary fields [Wei46], [Wei48], which he used to prove his conjectures on zeta functions in the case of curves. Only subsequently was it proved by Chow that Weil's Jacobian varieties were in fact projective [Cho54]. Of course, from a practical present-day point of view it is also important that abstract varieties provide a far stronger foundation for learning about schemes.

A guiding principle of the book is the demonstration of varied techniques and approaches to proving results. In terms of approaches, we prove a number of results via reduction to commutative algebra (and include an appendix on algebra definitions and results), but we also prove other results via more geometric approaches, as for instance the proof in Chapter 7 that morphisms from projective varieties have closed images, and that regular functions on projective varieties are constant: this proof emerges out of the theory of complete varieties and extensions of morphisms from punctured nonsingular curves. We have also included illustrations of various commonly used techniques in algebraic geometry. For instance, in §3.4 we argue via factorizations of morphisms and exploit the geometry of integral ring extensions. In §7.6 we use fibrations to induct on dimension, reduce to a universal case, and make a nonemptiness argument using extension of the base field. Finally, in studying the secant variety in §8.5, we demonstrate the utility when studying parameter spaces of introducing auxiliary spaces for which conditions on the existence of certain objects are replaced with choices of those objects.<sup>2</sup>

Later chapters emphasize the case of algebraic curves, and while this aspect may be viewed as providing elements of a course on curves, it also serves to introduce general concepts such as divisors in a simpler setting. Perhaps more importantly, we also use results involving curves to prove statements on general varieties, including the above-described proof that regular functions on projective varieties are constant, and the proof that complex varieties are connected in the analytic topology. From this point of view, our approach can be viewed as an invitation to the circle of techniques involving studying higher-dimensional varieties through the curves on them, as is described for instance in Kollár's survey of Mori's work on the structure of threefolds [Kol87], and his book on rational curves on varieties [Kol96].

Following Hartshorne [Har77], we have taken a regular-function-centered approach to defining morphisms of varieties, which thus leads naturally into scheme theory. More broadly, this book is heavily influenced by—and intended to work seamlessly as a substitute for—Chapter I of [Har77]. Many other aspects are adapted from Shafarevich [Sha94a], [Sha94b], with some influence also from Mumford [Mum99] and Harris [Har92].

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<sup>2</sup>In the case of the secant variety of a variety  $X$ , the condition that there exist a pair of points of  $X$  collinear with a given point is replaced by actually parametrizing such pairs of points.

**Prerequisites.** In any algebraic geometry course, students will benefit from having mastered as much background material as possible. In an ideal world, they would arrive having already taken an undergraduate course in algebraic geometry, a graduate course in algebra that includes a strong dose of commutative algebra, and maybe even a course in differential topology or geometry. However, it is my experience that this is rarely feasible, and this book is written around the reality that many students will arrive having learned far less. Even in a solid year-long graduate algebra course, which particular topics in commutative algebra are covered will vary drastically from course to course, so Appendix B on background in algebra is included to allow students to fill in any gaps in their own particular backgrounds as quickly and painlessly as possible. Ultimately, the only true prerequisites for this book are a strong undergraduate algebra course and a working familiarity with the basics of point-set topology, although graduate algebra is certainly strongly recommended.

**Organization.** Chapter 1 is an introduction to many of the ideas of algebraic geometry in the elementary context of plane curves and elliptic curves.

Chapters 2–4 constitute the local building blocks of the subject, with material on affine varieties, morphisms and rational maps, and singularities. Chapter 2 treats individual affine varieties, introducing the Zariski topology, the ideal-variety correspondence, and fundamental facts of dimension theory. Chapter 3 introduces regular functions, morphisms, and rational maps in the context of quasiaffine varieties, and proves the fundamental results describing regular functions and morphisms of affine varieties in terms of polynomial rings. A proof of Chevalley’s theorem on the image of morphisms is also provided. Chapter 4 then develops the concept of singularities of affine varieties, contrasting the more geometric notion of tangent space with the more algebraic notion of Zariski cotangent space, proving the Jacobian criterion, and ultimately surveying the notions of complete local rings and of normality.

Chapters 5–7 form an introduction to the global theory, starting with the construction of abstract varieties via atlases, and continuing with projective varieties and blowups, and the basics of nonsingular curves and complete varieties. Chapter 5 lays out the foundations of the atlas approach to prevarieties and their morphisms, and develops the analogue of the Hausdorff condition which is used to define varieties. It concludes with a treatment of normalization for prevarieties. Chapter 6 introduces projective space and projective varieties as examples of varieties, makes a preliminary study of their regular functions and morphisms, and develops the theory of blowups. Chapter 7 studies extensions of morphisms from punctured nonsingular curves to projective varieties, proving quasiprojectivity of nonsingular curves and the existence of nonsingular projective compactifications of nonsingular

curves. It then introduces the concept of complete varieties and proves the geometric analogue of the valuative criteria, stated in terms of extensions of morphisms from punctured nonsingular curves. Finally, these ideas are applied to the study of irreducibility of polynomials in families, leading to a proof of the fact that any two points in a variety can be connected by a chain of curves.

The main material concludes with Chapters 8–10, which develop the theory of nonsingular curves in more detail, as well as differentials on nonsingular varieties. Chapter 8 studies the role of divisors on curves in understanding morphisms to projective space, and also includes material on morphisms between nonsingular projective curves, and on testing for closed embeddings through injectivity on points and on tangent spaces. Chapter 9 develops the general theory of differential forms on nonsingular varieties, and then specializes to curves to study divisors associated to differential forms as well as the relationship between differential forms and ramification. Finally, Chapter 10 is an invitation to further topics; it initially takes the Riemann-Roch theorem as a black box and develops a variety of consequences, including the Riemann-Hurwitz theorem. It concludes with a brief sketch of Brill-Noether theory and moduli spaces of curves, although overviews of two proofs of the Riemann-Roch theorem are provided in an appendix.

Appendix A treats the analytic topology for complex varieties, relating the algebraic properties of being a variety, completeness, connectedness, and nonsingularity to the topological properties of Hausdorffness, compactness, connectedness, and being a (complex) manifold. The connectedness theorem is by far the deepest, pulling together ideas from several different parts of the book.

Finally, Appendix B is a guide to the algebra background used elsewhere in the book, collecting the definitions and statements, as well as selected proofs.

Given the goal of serving as a one-term textbook, many topics have been omitted. Obviously, sheaves, schemes, and cohomology have been left for a follow-up course. Other notable subjects which have been mostly or entirely omitted include aspects of projective geometry such as Hilbert functions and intersection theory, Bertini theorems, the theory of surfaces, Grassmannians, algebraic groups, vector bundles, and properties of morphisms. In addition, the material is focused entirely on theoretical algebraic geometry. Computational algebraic geometry has developed rapidly, and many explicit calculations are now possible and even routine. Even if one is ultimately interested in theoretical questions, a solid familiarity for what is computable can be valuable in working with examples. The development of computational techniques would take us too far afield, but the interested reader can

find approachable accounts in the books of Cox-Little-O'Shea [CLO07] or Hassett [Has07].

**How to use this book.** This book is intended to provide roughly the right amount of material for a one-quarter or one-semester course. My own experience is that the ten chapters making up the main body of the book constitute slightly too much material to fit into a ten-week quarter, and I describe which sections are good candidates for omission below. Exercises are interspersed throughout and are a mixture of working with specific examples, developing supplemental theoretical statements, and working out (generally comparatively routine) theoretical statements which are used in proofs. In some cases (such as hyperelliptic curves, a determinantal variety, and a nonnormal surface), examples recur across multiple exercises.

Many of the choices of which particular results to include were made with subsequent applications in mind, so there is a high degree of interconnectedness, and the book is intended to be taught for the most part systematically and in the order presented. However, there is a certain amount of flexibility in how to approach the material.

First, the contents of Chapter 1 are intended to serve as the basis for a single introductory lecture giving a broad overview of many ideas in algebraic geometry in an approachable context. However, the main content of the book starts in Chapter 2 and is written independently of Chapter 1, so the latter material may freely be skipped or replaced with an alternative introduction to the subject.

There are also a few topics which are not used elsewhere or only used very narrowly, and could easily be omitted entirely if desired. These include the following, the first three of which are not used anywhere else in the book:

- the discussion of categories and geometry in §3.A;
- the treatment of completion and analytic singularity type in §4.4;
- the treatment of blowups in §6.3;
- the treatment in §4.A of the ideal-theoretic formulation of local complete intersections is used only once, in the treatment of blowups;
- both the material in §7.6 on irreducibility of polynomials in families and path-connectedness of varieties and the material in §8.5 on secant varieties and curves in projective space are used only in Appendix A;
- the material on Frobenius morphisms in §9.A is used only in Corollary 10.2.9 to remove the separability hypothesis from the statement that a curve cannot cover a curve of higher genus.



Of course, if one wishes to place a greater emphasis on higher-dimensional varieties, another natural option is to teach most of the above-mentioned material, and simply truncate the curve-specific material at the end. In this context, it is worth remarking that the general material on differential forms in §9.1 can be covered at any point after the end of Chapter 5. A similar option is to replace the invitation to further topics offered in Chapter 10 with the material on the analytic topology from Appendix A, in which case the connectedness theorem can serve as a capstone result (note however that this does require the Riemann-Roch inequality proved in Proposition 10.A.3).

While the organization largely follows a natural order for teaching, there are a few sections which are natural candidates for delayed treatment. Specifically, Chevalley's theorem is proved in §3.4, but is used only at the end of Chapter 7 and in Appendix A. Similarly, normality and normalization are treated in the affine case in §4.5 and in general in §5.4, but are also not used until Chapter 7. Both of these topics require a certain degree of sophistication and could reasonably be taught immediately prior to using them.

**Acknowledgments.** The mathematical perspective expressed in this book was deeply influenced by the courses I took from Ravi Vakil, Johan de Jong, and Joe Harris. This book was greatly improved in turn as a result of the many questions and comments I received from students in my courses on algebraic geometry, among whom I would especially highlight the contributions of Federico Castillo and John Challenor. I would also like to thank the reviewers for their careful readings and detailed comments, David Eisenbud for his feedback on the commutative algebra appendix, and Eriko Hironaka for her feedback as the book's editor.

**Conventions.** All rings (including algebras over rings) are assumed to be commutative, with multiplicative identity. In particular, an  $R$ -algebra is equivalent to a ring  $S$ , together with a homomorphism  $R \rightarrow S$ . We work throughout over an algebraically closed base field  $k$  except where we explicitly state otherwise.

Compact topological spaces are not assumed Hausdorff.

Newly defined terms are in bold. Words and phrases appear in quotes to indicate that they are being used in a less formal manner, to paint an intuitive picture of a situation, or to informally introduce terminology which either will be defined formally later in the text, or is only being mentioned in passing.

Brian Osserman  
Davis, California  
March 2021

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# Index of Notation

- $j(E)$ ,  $j$ -invariant of  $E$ , 10  
 $\mathbb{A}_k^n$ , affine  $n$ -space, 12  
 $R_n$ , polynomial ring in  $n$  variables, 12  
 $Z(T)$ , zero set of  $T$ , 12  
 $I(Y)$ , ideal of polynomials vanishing on  $Y$ , 16  
 $A(Y)$ , affine coordinate ring of  $Y$ , 20  
 $\dim X$ , dimension of  $X$ , 24  
 $\dim R$ , dimension of  $R$ , 25  
 $\text{codim}_X Y$ , codimension of  $Y$  in  $X$ , 26  
 $\mathcal{O}(U)$ , ring of regular functions on  $U$ , 32  
 $\mathcal{O}_{P,Y}$ , local ring on  $Y$  at  $P$ , 35  
 $K(Y)$ , field of rational functions of  $Y$ , 35  
 $\dim_P X$ , dimension of  $X$  at  $P$ , 38  
 $\varphi^*$ , pullback of functions under  $\varphi$ , 40  
 $Y_f$ , complement of  $Z(f)$  in  $Y$ , 42  
 $T_P(X)$ , tangent space to  $X$  at  $P$ , 62  
 $T_P^*(X)$ , Zariski cotangent space of  $X$  at  $P$ , 63  
 $k[[x_1, \dots, x_r]]$ , ring of formal power series over  $k$ , 72  
 $\mathcal{O}(U)$ , ring of regular functions on  $U$ , 86  
 $X \times Y$ , product of  $X$  and  $Y$ , 91  
 $T_P(X)$ , tangent space to  $X$  at  $P$ , 94  
 $\mathbb{P}_k^n$ , projective  $n$ -space, 104  
 $Z_h(S)$ , projective zero set of  $S$ , 104  
 $h_i(f)$ , homogenization of  $f$ , 105  
 $d_i(F)$ , dehomogenization of  $F$ , 105  
 $\text{Bl}_Z X$ , blowup of  $X$  along  $Z$ , 114  
 $\text{ord}_P(f)$ , order of vanishing of  $f$  at  $P$ , 127  
 $\deg f$ , degree of  $f$ , 147  
 $\deg D$ , degree of  $D$ , 150  
 $\varphi^*(D)$ , pullback of  $D$  under  $\varphi$ , 150  
 $D(f)$ , divisor associated to  $f$ , 150  
 $\mathcal{L}(D)$ , linear series associated to  $D$ , 151  
 $\ell(D)$ , dimension of  $\mathcal{L}(D)$ , 152  
 $\varphi^*(H)$ , pullback divisor from hyperplane, 153  
 $\text{Sec}(X)$ , secant variety of  $X$ , 164  
 $\text{Tan}(X)$ , tangent variety of  $X$ , 167  
 $df$ , differential form associated to  $f$ , 170  
 $\Omega(U)$ , space of regular differential forms on  $U$ , 170  
 $D(\omega)$ , divisor associated to  $\omega$ , 173  
 $\Omega(D)$ , space of differential forms associated to  $D$ , 174  
 $\varphi^*(\omega)$ , pullback of  $\omega$  under  $\varphi$ , 176  
 $X^{(p)}$ , target of Frobenius map, 182  
 $\text{dord}_P \varphi$ , order of differential pullback, 191  
 $X_{\text{an}}$ , analytic topology on  $X$ , 208  
 $S^{-1}R$ , ring of fractions, 224  
 $R_f$ , fraction ring inverting  $f$  in  $R$ , 224  
 $R_{\mathfrak{p}}$ , localization of  $R$  at  $\mathfrak{p}$ , 224  
 $\dim R$ , dimension of  $R$ , 232



---

# Index

- absolute Frobenius map, **185**
- admissible atlas, **84**
- affine (algebraic) variety, **14**
  - morphisms, 43
- affine cone, **107**, 113
- affine coordinate ring, **20**
  - and affine varieties, 43, 44
  - and morphisms, 89
  - and regular functions, 36
  - injects into regular functions, 32
- affine open subset, **43**, **90**
- affine plane, **3**
- algebra over a ring, xvi, **221**
- algebraic set
  - affine, **12**, 17
  - corresponds to ideal, 17, 122
  - projective, **104**, 122
- algebraically independent, **218**
- analytic singularity type, **73**
- analytic topology, **207**, **209**
- ascending chain condition, **222**
- atlas, **82**
- automorphism group of  $\mathbb{P}_k^1$ , 113
  
- basepoint, **156**
- basepoint-free linear series, **156**, 188, 190
- Bertini's theorem, 167
- Bezout's theorem, **8**, **160**
- birational curves, 133
- birational map, **49**, **95**, 129
- birational varieties, **49**, 50, 51
- blowup, **114**, 114–122
  - independent of choices, 117
  - of a nodal curve, 120
  - of a nonsingular subvariety of a nonsingular variety, 118
  - universal property, 121
- branch point, **148**
- Brill-Noether theorem, **196**
  
- canonical embedding, **190**
- canonical linear series, **174**, 188
  - is basepoint-free, 190
- chain rule for differential forms, 170
- chart, **82**
- Chevalley's theorem, **52**, 96, 139, 211
- Chow's lemma, **141**
- closed embedding
  - criterion for, 160, 162, 188
  - of affine varieties, **45**
  - of prevarieties, **90**
- closed map, **53**
- closed subprevariety, **86**
- closed subvariety, **39**, **92**
- cocycle condition for gluing maps, **83**
- codimension, **26**, 28
- cofinite topology, **14**
- complete intersection, **69**
- complete linear series, **155**, 188, 190
- complete local ring of an algebraic set, **71**, 71–74
- complete variety, **135**, 135–137
  - and Nagata compactification, 136
  - criterion for, 138
  - global regular functions, 136



- image is closed, 135
- is compact in complex case, 212
- products thereof, 140
- completion of a local ring, **71**, **244**
  - properties preserved under, 246
- connectedness in analytic topology, 213
- coordinates of a point, **12**
- cotangent space, **94**
- curve
  - abstract, **94**
  - affine, **25**
  - birational theory, 133
  - closed embedding, 162
  - compactification in projective curve, 132
  - descriptions in low genus, 196
  - embedding in  $\mathbb{P}_k^3$ , 167
  - extending morphisms from, 128
  - is quasiprojective, 130
  - cusp of a curve, 6, 49, **74**
    - blowup, 121
    - nonregular function, 33
    - normalization, 77
- Dedekind domain, **241**
- degree of a divisor, **150**
  - preserved under linear equivalence, 154
  - principal, 151
  - under pullback, 150
- degree of a linear series, **155**
  - and plane curves, 160
- degree of a morphism of curves, **147**
  - and the preimage of a point, 148
- degree of a morphism to projective space, **154**
- degree-genus formula, 4, **193**
- depth, 235
- determinantal variety, 41
  - not a complete intersection, 29
- diagonal map
  - and analytic topology, 212
  - and Hausdorff condition, 91
  - and variety condition, 92
  - argument using, 95
  - is an embedding, 93
  - of a prevariety, **91**
  - of affine variety, **47**
- differential form, **169**, 169–181
  - regular, 170
  - representation with local coordinates, 170
  - zero set is closed, 172
- dimension, 24–30
  - local, **38**
  - of a linear series, **155**
  - of a ring, **25**, **232**
    - bounded in terms of Zariski cotangent space, 63, 234
  - of a topological space, **24**
- divisor, **150**
  - associated to rational differential form, **173**
  - associated to rational function, 150
  - exceptional, 116
  - families and linear series, 155, 156
  - of high degree, 188
  - on moduli space of curves, 199
  - pulled back from a general hyperplane, 167
  - pulled back from a hyperplane, 153
  - space of associated differential forms, 174
- divisors, 150–160
- domain of definition of a rational map, **95**
- dominant morphism, **46**
  - and injective ring map, 46
  - image contains open subset, 52
- dominant rational map, **48**
  - and function fields, 50
  - of curves, 133
- effective divisor, **150**
  - families and linear series, 155
  - on moduli space of curves, 199
  - pulled back from a hyperplane, 153
- elliptic curve, **5**, 5–10
  - associativity of group law, 175
  - differential forms, 175
  - point at infinity, 107
- embedding, **90**
  - curve in  $\mathbb{P}_k^3$ , 167
  - diagonal map is, 93
- enumerative geometry, **7**
- equivalent atlases, **87**
- equivalent linear series, **156**
- exceptional divisor of a blowup, **116**
  - is locally defined by one equation, 116
  - not necessarily connected, 120
- extremal epimorphism, **56**
- extremal monomorphism, **56**
- Fermat hypersurface, **107**, 195

- fiber, **46**
  - and families of varieties, 197
  - finiteness of, 53
  - of a morphism of curves, 148
- field of rational functions, **35**
- finite morphism, **134**, 160
- finitely generated algebra, **221**
- finitely generated module and integral ring extension, 229
- form, homogeneous, **108**
- formal power series ring, **72**, **244**
- Frobenius morphism
  - for curves, 185, 193
  - of a prevariety, **184**
  - of an affine variety, **182**
  - on  $\mathbb{A}_k^1$ , **51**
- function field, **35**, **94**
  - and cocycle condition, 84
  - and dominant rational map, 50
  - as fraction field, 36
  - degree of a morphism, 147
  - of a projective variety, 123
  - of curves, 133
  - separability, 176
  - transcendence degree, 37
- generators of an algebra, **221**
- genus of a curve, **4**, **174**
  - and degree in plane, 4, 193
  - and Frobenius morphism, 184
  - and linear series, 187–190, 196
  - and morphisms, 191, 193
  - complex case, 195
  - hyperelliptic, 194
  - realized as nodal in plane, 206
- gluing a prevariety from affine pieces, **83**, 97
- going-down theorem, **240**
- going-up theorem, 53, **230**, 238
- gonality of a curve, **196**
- graph of a morphism, **95**, 136
- graph of a rational map, **95**, 114
- height, **26**, 28, **233**, 238, 240
- Hilbert basis theorem, **15**, 23, **223**
- homogeneous coordinate ring, **123**
- homogeneous coordinates, 3, **104**
- homogeneous ideal, **122**
- homogeneous polynomial, **3**, 21, **104**
  - and morphisms of projective varieties, 110
  - and rational functions on  $\mathbb{P}_k^n$ , 109
  - associated to inhomogeneous polynomial, 105
  - closure of affine hypersurface, 106
- hyperelliptic curve, **190**, 194
  - affine plane model, 70, 194
  - genus, 194
  - genus 3, 194
- hyperplane, **111**
  - divisor pulled back from, 153
  - general, 167
- hypersurface, **19**, 28
  - and normality, 75
  - closure in projective space, 106
  - every variety is birational to, 51
  - Jacobian criterion for, 66
  - projective, 108
- ideal-variety correspondence, 17
  - for morphisms, 47
  - projective, 122
- inseparable extension, **219**
- inseparable morphism, **176**
  - and differential forms, 177
  - and ramification, 177
  - factorization, 185
- integral closure, **76**, **231**, 231–232
  - finiteness of, 237
  - rings of fractions, 97, 231
- integral element, **53**, **74**, **229**
- integral ring extension, **53**, **229**, 229–230, 236, 239
  - and dimension, 238
  - and finite module generation, 229
  - geometric consequences, 53
- integrally closed ring, **74**, 96, **231**, 241
  - local rings, 75
  - regular local ring is, 234
  - Serre’s criterion, 235
- intersection
  - codimension bound, 71, 108
- intersection multiplicity, 8, **159**
- intersection theory, **7**
- irreducibility in families, 144
- irreducible algebraic set
  - and nonsingularity, 69
  - corresponds to prime ideal, 17
  - examples of, 41
- irreducible closed subset
  - and dimension, 25
  - codimension of, 26
- irreducible components, **24**
- irreducible topological space, **13**, 82

- and fibers of open map, 46
- basic properties, 15
- criterion for, 41
- isomorphism
  - of elliptic curves, 10
  - of prevarieties, **88**, 96
  - of quasiprojective varieties, **39**
- $j$ -invariant, **10**, 182
- Jacobian criterion, 2, **66**, 70, 78, 119, 171, 183, 213, 236
  - and local coordinates, 68
  - for hypersurfaces, 66
- Jacobian matrix, **66**
- $k$ -algebra, **221**
  - and affine varieties, 43
  - and function fields, 50
  - and morphisms to an affine variety, 89
  - as affine coordinate ring, 20
  - dimension, 26, 240
  - importance of, 185
  - Noether normalization, 236
  - ring of regular functions, 32
- Krull dimension, **26**, 232
- Krull intersection theorem, **72**, **244**
- Krull's principal ideal theorem, **28**, 63, 66, **233**, 246
  - converse to, 233
- Leibniz rule, **170**
- limit
  - interpretation of completeness, variety condition, 137
  - version in algebraic geometry, 129
- linear projection, **112**, 118, 166, 167, 193
  - and tangent spaces, 159
- linear series, **155**, 155–159
  - and moduli spaces of curves, 198
  - associated to a divisor, **151**
  - criterion for closed embedding, 163
  - dimension bound, 152
  - of high degree, 188
  - on a general curve, 196
- linear span in  $\mathbb{P}_k^n$ , **111**
- linear subspace of  $\mathbb{P}_k^n$ , **106**
- linearly equivalent divisors, **154**
  - and linear series, 155
  - have same degree, 154
- local complete intersection, **70**, 118
  - and nonsingularity, 70, 71
  - and normality, 75, 235
  - ideal theoretic, 77, 79
- local coordinate at a point on a curve, **64**, 126
  - and inseparability, 177
- local coordinates
  - system of, **64**
- local ring, **35**, **94**, **224**
  - as ring of fractions, 36
  - dimension bound, 63, 234
  - injects into completion, 72, 244
  - of a projective variety, 123
  - properties, 37
  - test for isomorphism, 96
  - when an integral domain, 68
- localization, 36, **224**
  - and integral closure, 231
- locally closed subset, **39**
  - diagonal is, 93
  - graph is, 95
- $m$ -canonical embedding, **190**
- moduli space of curves, 198
- morphism
  - description of image, 52
  - factorization, 89
  - from curve to projective space, 157
  - from punctured curve extends, 128
  - graph, 95
  - in terms of atlas, 89
  - is local property, 88
  - of curves and ramification, 178, 191
  - of curves, fiber, 148
  - of curves, genus nonincreasing, 193
  - of prevarieties, **88**, 88–90
  - of projective varieties and tuples of homogeneous forms, 110
  - of quasiprojective varieties, **39**, 39–47
  - rigidity, 48, 94
  - to affine variety and affine coordinate ring, 43, 89
  - to projective variety and tuples of regular functions, 109
- Nagata compactification theorem, 136
- Nakayama's lemma, 63, **226**, 227, 233
- nodal plane curve, 205
- node of a curve, 6, **73**
  - blowup, 120
  - nonregular function, 128
  - odd example involving, 137
- Noether normalization theorem, **236**

- dimension consequences, 239, 240
- proof of finiteness of integral closure, 237
- proof of Nullstellensatz, 236
- Noetherian local ring
  - cotangent space, 63, 233
  - dimension bound, 63, 234
  - injects into completion, 72, 244
- Noetherian ring, **15**, **222**, 222–223
  - and polynomial rings, 15, 223
  - and rings of fractions, 225
  - height and principal ideals, 28, 233
- Noetherian topological space, **22**, 22–24
  - and algebraic sets, 23
  - and dimension, 25
  - irreducible components, 24
- nonalgebraically closed fields, 20, 29, 38
- nondegenerate morphism, **157**
- nonsingular (pre)variety
  - and blowup, 118, 121
  - is normal, 74, 96
- nonsingular affine algebraic set, **63**
- nonsingular curve
  - criterion for completeness involving, 138
  - divisors on, 150
  - extending morphisms from, 128
  - fiber of morphism, 148
  - is quasiprojective, 130
  - nonsingular projective
    - compactification, 132
  - order of vanishing at a point, 127
  - ramification of morphism, 148
- nonsingular plane curve, **2**
- nonsingular point, **63**, **94**
  - complete local ring, 73
  - contained in unique component, 69
  - Jacobian criterion, 66
  - local complete intersection, 70, 71, 77
- nonsingular variety
  - differential forms on, 169
- normal affine variety, **74**, 74–76
- normal prevariety, **96**
  - has singularities in codimension at least 2, 96
- normality
  - and nonsingularity, 74
  - determined locally, 75, 96
  - properties, 75
- normalization, **76**, 76–77, **97**, **98**, 96–101, 211, 214
  - has finite fibers, 77, 98
  - independent of atlas, 101
  - is closed, 77, 98
  - is surjective, 77, 98
  - of a nodal curve, 120
  - of projective curve is projective, 132
  - universal property, 100
- Nullstellensatz, Hilbert, **18**
  - algebraically closed base field, 20
  - equivalent form, 21
  - proof, 236
- open map, **46**
  - and irreducibility, 46
- open morphism, **46**, 54
  - from affine cone, 108
  - projection from product, 47
- open subprevariety, **86**
- open subvariety, **39**, **92**
- order of a differential form, **173**
  - and ramification, 178
- order of a function, **127**
  - and intersection multiplicity, 160
- plane curve, 1–5
  - formula for genus, 193
  - intersection formula, 160
  - local coordinates, 68
  - notions of degree agree, 160
- point, **12**
  - of a plane curve, **2**
- prevariety, **82**, 81–86
  - connected by chains of curves, 146
  - equivalent atlases, 87
  - gluing from affine varieties, 83
- primitive element theorem, 51, 146, **219**
- principal divisor, **150**
  - has degree 0, 151
- principal ideal
  - and exceptional divisors, 116
  - and height, 28, 233
  - in UFD, 28, 228
- product
  - in analytic topology, 210
  - not preserving homeomorphism, 137
  - not product topology, 14
  - of affine varieties, **46**
  - of complete varieties, 140
  - of prevarieties, **91**
  - of varieties, 140
  - role in adapting topological concepts, 90

- product morphism, **91**
- projection morphism, **47, 91**  
 is open, 47
- projective  $n$ -space, **104**, 104–106  
 automorphisms, 111  
 intersections in, 108  
 is a variety, 111  
 morphisms from curves, 157  
 rational functions, 109
- projective hypersurface, **108**  
 example, 107
- projective line, 83, 88, 93, 133, 188  
 automorphism group, 113  
 degree of morphisms to, 155  
 genus, 175  
 linear series on, 152
- projective plane, **3**  
 intersection of curves in, 8, 160
- projective variety, **106**  
 and homogeneous ideal, 122  
 compact in analytic topology, 210  
 global regular functions, 138  
 is complete, 138  
 local rings and function field, 123  
 morphisms between, 110  
 morphisms from a punctured curve  
 extend, 128  
 morphisms to, 109  
 surjecting onto complete variety, 141
- projectivization of a vector space, **118**
- pullback  
 of a differential form, **176**, 178  
 of a divisor, **150**  
 of a hyperplane, 153  
 of a hypersurface, 205  
 of a regular function, **40**
- purely inseparable extension, **219**, 237
- quadric surface, **112**
- quasi-affine variety, **39**  
 covered by affines, 42  
 morphism, 39  
 not necessarily affine, 44
- quasiprojective variety, **106**
- radical ideal, **16**  
 and projective algebraic sets, 122  
 corresponds to algebraic set, 17
- radical of an ideal, **16**
- ramification index, **148**, 150  
 and pullback of differential forms, 178  
 formula for genus and degree in terms  
 of, 191
- ramification point, **148**  
 finiteness for separable morphisms,  
 177  
 projection of plane curve, 159
- rational differential form, **172**  
 and inseparability, 177  
 associated divisor, **173**  
 basis in terms of local coordinates,  
 172  
 degree of divisor, 188
- rational function, **35, 94**  
 associated divisor, 150  
 on projective space, 109
- rational map, **48, 95**  
 and blow-up construction, 114  
 and function fields, 50  
 domain of definition, 95  
 graph, 95  
 resolved by blowups, 122
- rational normal curve, **163**
- rational variety, **51**  
 curve case, 133
- regular differential form, **170**  
 on an elliptic curve, 175  
 pullback, 177  
 rigidity, 172
- regular function, **32**, 31–38, **86**  
 analytic in complex case, 213  
 locally a restriction from ambient  
 variety, 88  
 nonextendable example, 33  
 not a restriction from ambient space,  
 45  
 on affine variety, 36  
 on complete variety, 136  
 on normal variety, 75  
 on projective variety, 138  
 on quasiprojective variety, 109  
 requiring more than one expression,  
 34, 38  
 rigidity, 33
- regular local ring, **63, 234**  
 completion, 245  
 is a unique factorization domain, 234  
 is an integral domain, 234  
 is integrally closed, 234
- relative Frobenius map, **185**
- residue of a rational differential form,  
 201

- Riemann-Hurwitz theorem, **191**  
 Riemann-Roch theorem, **187**, 187–190, 199–206  
 ring of fractions, 22, 36, 42, **224**  
   and integral closure, 97  
   integral closure, 231  
 secant variety, **164**  
 Segre embedding, **114**, 130  
 separable element, **219**  
 separable extension, **219**  
 separable morphism, **176**  
   formula for ramification, 191  
   has finitely many ramification points, 177  
 singularity, **2**, **63**  
   analytic type, 73  
 smooth, **2**  
 subadditivity of codimension for  
   intersections, 71, 108  
 subprevariety, **86**  
   characterization of regular functions, 88  
 subvariety, **39**, **92**  
 surface  
   abstract, **94**  
   affine, **25**  
   quadric, 112  
 surjectivity on local rings, 96  
   criterion for, 227  
 system of local coordinates, **64**  
   and differential forms, 170, 172  
   and Jacobian criterion, 68  
   complex case, 213  
 tamely ramified, **177**, 191  
 tangent line, **60**, **61**  
 tangent space, **62**  
   and blowup, 118  
   and closed embeddings, 160  
   and ramification, 149  
   dual to Zariski cotangent space, 65  
   in projective space, 166  
   to a prevariety, **94**  
 tangent variety, **167**  
 tangent vector, **61**  
 torsion points, 7  
 transcendence basis, **218**  
   and separability, 220  
 transcendence degree, **219**  
   and dimension, 26, 239  
 transition map, **82**  
   is analytic, 213  
 unique factorization domain (UFD), 28, 45, 74, **228**  
   is integrally closed, 232  
   polynomial ring over, 228  
 unirational variety, **197**  
 universally closed, **137**  
 unramified, **148**, 149  
 variety (abstract), **92**  
   criterion for, 93, 97, 138  
   Hausdorff in analytic topology, 212  
   normalization, 98  
   products thereof, 140  
   projective space, 111  
 Weierstrass form, **5**  
 wildly ramified, **177**  
 Zariski cotangent space, **62**, **63**, **233**  
   and dimension, 63, 234  
   dual to tangent space, 65  
 Zariski topology, **13**  
   coarser than analytic topology, 208  
   is Noetherian, 23  
   on product, 14  
   on projective space, 104  
   recoverable from category, 55  
 zero set, **12**  
   and radical ideals, 16  
   in projective space, 104  
   of a regular differential form, 172  
   polynomials vanishing on, 18



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