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Lectures on Differential Topology

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Dedico tutto questo ai miei nipoti Pietro(lino),
Martin(in)a, e Andrea bibo-chicco

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Preface

Over the years, I have taught several courses on differential topology in the master's degree program in mathematics at the University of Pisa. The class was usually attended by students who had accomplished (or were accomplishing) a first three-year degree in mathematics, together with a few peer physicists and a few beginner Ph.D. students. Considering the initial knowledge of these students, time after time, a collection of different topics, in different combinations, as well as a certain way of presenting them, has emerged. This textbook summarizes such teaching experiences; therefore it presents itself more as "lecture notes" than as a complete and systematic treatise. Sometimes, in a class, a "short cut" to an interesting application is chosen over broader generality. Similarly, in this text we will focus, for example, on compact manifolds (especially when we consider the sources of smooth maps), allowing simplifications in dealing, for instance, with function spaces or with certain "globalization procedures" of maps. There are already a lot of interesting facts concerning compact manifolds, so we will do it without remorse.

There are several classical well-known references (such as [**M1**], [**GP**], [**H**], [**M2**], [**M3**], [**Mu**], ...) which I used in preparing the courses and which have strongly influenced these pages. So, why another textbook on differential topology? An important motivation came to me from the students, looking at their notes and from their remark that they had "not been able to find some of the topics addressed in the course anywhere." It would be very hard to claim any "originality" in dealing with such a classical matter. However, that remark, at least in reference to textbooks addressed mainly to undergraduate readers, has some truth to it. Let's give an example. A theme of this text (similar, for example, to [**H**]) is the synergy between *bordism* and *transversality.* One of the limits imposed by the students' presumed initial knowledge, as mentioned above, is that we can't assume any familiarity with algebraic topology or homological algebra (besides, perhaps, the very basic facts about homotopy groups); on the other hand, it is very useful and meaningful to dispose of a (co)homology theory suited to support several differential topology constructions. We will show that (oriented or nonoriented) bordism provides instances of so-called (covariant) "generalized" homology theories for arbitrary pairs (X, A) of topological spaces, constructed via geometric means. Then, by specializing X to be a smooth compact manifold, and after a re-indexing of the bordism modules by the *codimension* (so that they are now called cobordism modules), transversality allows us to incorporate the bordism modules into a *contravariant cobordism functor* with the category of *graded rings* as the target; the product on cobordism modules is also defined by direct geometric means. This multiplicative structure is a substantial enhancement and it will lead to several important and often very classical applications. For example, it is the natural context for unavoidable topics such as degree theory or the Poincaré-Hopf index theorem. The verification that several constructions are well-defined is eventually reduced to the fact that the cobordism product is well-defined. Moreover, when possible, the "invariance up to bordism" is emphasized rather than the "invariance up to homotopy", compared to most of the established references. Not assuming any familiarity with algebraic topology, this presentation could also be useful as an intuitive, geometrically based introduction to some topics of that discipline. Overall, this book is a collection of themes, in some cases advanced and of historical importance and whose choice was certainly due in part to personal preferences, with the common characteristic that they can be treated with "bare hands", meaning by combining specific differential-topological cut-and-paste procedures and applications of transversality, mainly through the cobordism multiplicative structure. The trait of geometric construction sets the "tone" of this textbook, intended to be accessible and useful to motivated undergraduate students and Ph.D. students, but also to a more expert reader to recognize very basic reasons for some facts already known as the result of more advanced theories or technologies.

Riccardo Benedetti

Sassetta, February 2021

Introduction

These lecture notes were conceived with a typical class of rather good and motivated students in mind, who have accomplished (or are accomplishing) a first three-year degree in mathematics and whose mathematical background is likely limited. For example, besides very basic facts about homotopy groups, no familiarity with algebraic topology or homological algebra is assumed. Concerning general topology, some knowledge is assumed about compactness in Hausdorff second-countable topological spaces, but not about paracompactness.

In some sense, the most natural way to read this text is from the beginning to the end. Nonetheless, different reading paths and various combinations of subjects are also possible and meaningful. These pages have originated from teaching experiences. Whereas not a single course has covered the whole content of the book, parts of each chapter have been treated during some of the classes.

The text (as much as the lectures it derives from) intends to give accurate definitions, statements, and descriptions of the main constructions; it also aims to develop an articulated and coherent exposition. On the other hand, proofs are intentionally not uniformly detailed (and sometimes are even omitted). The text is addressed to an actively involved and motivated reader. The active participation of the reader is often required to complete some arguments or to check some statements, especially in the last two chapters. For this reason, we considered a list of exercises at the end of each chapter unnecessary.

The use of figures is limited; pictures containing substantial, not only allusive, information have been introduced. Drawing pictures following geometric reasoning is often useful, but this is left to the reader's initiative.

The bibliography is far from being exhaustive; besides a few classical references which have certainly influenced these pages (such as [**M1**], [**GP**], [**H**], [**M2**], [**M3**], and [**Mu**]), we just list the texts which have been cited.

The language of categories is moderately used, in the same way, for example, that it is used in textbooks of algebraic topology like [**Hatch**] and [**Mu2**]. The few necessary notions are collected in an appendix. Differential topology concerns the category of *smooth manifolds* and *smooth maps*; this includes the study of smooth manifolds considered up to *diffeomorphism*, that is, the equivalence in that category. A first necessary task is to define these objects and morphisms. We do it from scratch in Chapters 1, 2, and 4 by progressively extending the category from the category of open sets in Euclidean spaces and smooth maps, through the category of *embedded* smooth manifolds in some Euclidean space, and ending with the category of "abstract" smooth manifolds defined by the abstraction of some properties of embedded ones. Along with these generalizations, the notions of *submanifold*, manifold with *boundary*, and (orientable) *oriented* manifold with *oriented boundary* are developed. Basic notions such as immersion, submersion, embedding, smooth homotopy, isotopy or diffeotopy between smooth maps are also introduced.

In most applications, we will focus on *compact* manifolds, especially when we consider the sources of smooth maps. We will not present the most general version of many results; there are already a lot of interesting facts concerning compact manifolds, and the assumption of compactness simplifies many arguments in dealing, for example, with function space topology or with cut-and-paste constructions where one can use only *finite* partitions of unity, avoiding any reference to paracompactness.

Moreover, we will show that every compact manifold is diffeomorphic to an embedded one. Then several important facts, such as a tubular neighbourhood theory, can be developed by exploiting the embedding in some Euclidean space, but they eventually hold for arbitrary compact manifolds.

Let us describe the content of each chapter.

In Chapter 1, we assume the knowledge of basic differential calculus in several variables and we collect some facts concerning smooth maps between open sets of Euclidean spaces. Some of these facts (such as the *inverse map theorem* and its geometric applications to the local normal forms of immersions and submersions) should be familiar to the reader. Other facts are presumably less familiar, such as Morse's lemma, the linearization of diffeomorphisms of $\mathbb{R}^n$ up to isotopy, bump functions, and the smooth homogeneity of connected open sets (which later extends to arbitrary connected manifolds). Two characteristic features of differential topology already emerge. On one hand, there is a sort of "local rigidity": up to a local change of smooth coordinates, linear algebra provides the actual local models in many "generic and stable" smooth situations. But on the other hand, smooth maps are very "flexible", the existence of bump functions being a typical example. This will be the key for globalization procedures and cut-and-paste constructions. Flexibility is a quality expected from a topological theory, but this is moderated by that sort of local rigidity which allows for having a good geometric control; this moderate flexibility eliminates too "wild" phenomena that occur in general topology, even dealing with merely topological manifolds, or allows simple proofs of facts (such as the invariance of dimension up to diffeomorphism) whose topological counterparts hold as well but are more demanding. Moreover, the homogeneity property, in particular, indicates that the true questions in differential topology concern the *global* structure of manifolds.

In Chapter 2, we extend the notions of smooth maps and diffeomorphisms to *arbitrary* topological subspaces of some Euclidean spaces; then an embedded smooth manifold M of dimension m is defined as a topological subspace of some $\mathbb{R}^n$ which is locally diffeomorphic to open subsets of $\mathbb{R}^m$. Although not so demanding, this extension leads to many embedded manifolds beyond the open sets, including very familiar objects like the graphs of smooth maps between open sets, which ultimately are the local models for any embedded smooth manifold.

Chapter 3 is dedicated to a detailed presentation of two distinguished families of manifolds, that is, *Stiefel and Grassmann manifolds*, including projective spaces. Stiefel manifolds are naturally embedded; we provide embedded models also for the Grassmann manifolds. Besides the fact that they are nontrivial examples of (embedded) smooth manifolds, they shall be crucial in the study of vector (and frame) bundles on arbitrary manifolds. This chapter is essentially self-contained; it can be read independently and at a later stage when it is necessary.

Every embedded smooth manifold is naturally endowed with a maximal *atlas* of smooth *charts* (with corresponding smooth *local coordinates*) and smooth maps between embedded manifolds have natural *representations in local coordinates*. These notions are the key to the final abstraction made in Chapter 4.

After having stressed in Chapter 1 the functorial nature of the elementary *chain rule*, following the progressive generalizations of the concept of manifold, we build the fundamental covariant *tangent functor* which associates each manifold with its *tangent bundle* and each smooth map with its *tangent map*. This incorporates the notion of a tangent vector space at each point of a smooth manifold, of which we provide different interpretations.

The tangent functor is an important source of invariants of smooth manifolds. For embedded manifolds, tangent bundles and maps are constructed as a direct generalization of the elementary case of open sets in Euclidean spaces. For abstract manifolds, tangent bundles and maps must be somehow "invented", with the constraint that they must be compatible with what is already done in the embedded category. This is probably the most demanding extension, passing from the embedded to the abstract category. Eventually, this leads us, in Section 4.4, to the general notion of *principal bundle* with a given *structural group* G and *associated fibre bundles*, governed by a suitably defined G-valued *cocycle*, and we elaborate on different notions of fibred bundle equivalence. The principal *frame bundle* of a smooth manifold with the associated *tensor bundles* (including the tangent bundle) shall be a fundamental example (see Section 4.5).

Our typical student is probably already aware of the topology of the uniform convergence on compact sets of continuous maps between open sets of Euclidean spaces. This directly extends to $\mathcal{C}^r$-maps, $r \geq 0$, in terms of the uniform convergence on compact sets of the maps and their partial derivatives up to the order r. These topologies restrict to the set of smooth maps, for which we can also consider the union topology over $r \in \mathbb{N}$. Using the representation in local coordinates, the definition of these function spaces extends to smooth maps $f : M \to N$ between smooth manifolds, giving us the spaces $\mathcal{E}^r(M, N)$ endowed with the so-called $\mathcal{C}^r$ *weak topology* and $\mathcal{E}(M, N)$ endowed with the union topology. The adjective "weak" alludes to further function space topologies, the so-called *strong topologies*. These coincide with the weak ones if the source manifold is compact; otherwise, they are much finer and aimed at having a control "at infinity". We will not deal with the strong topology because in the relevant applications considered in this text, the source manifold M will be *compact*. For example, in Section 4.11.2 we show that if M is compact, $f : M \to N$ is an embedding if and only if it is an injective immersion and that immersions, submersions, and embeddings, respectively, form (possibly empty) *open sets* in $\mathcal{E}(M, N)$.

At the end of Chapter 4, we show that every abstract compact smooth manifold can be embedded in some $\mathbb{R}^n$. Then, considered up to diffeomorphism, it is not restrictive to assume that compact manifolds are embedded. As we are mainly concerned with compact manifolds, the abstraction of Chapter 4 might sound a bit superfluous. However, we will point out natural constructions to build new abstract compact manifolds, starting from given ones, even embedded. It would be artificial to force these constructions in the embedded setting. It is more convenient to use the embedding result a posteriori, to exploit the facts that we will establish for compact embedded manifolds.

In Chapter 5, we introduce the *pull-back* construction on fibred bundles; then we apply it to the so-called *tautological (vector or frame) bundles* over the Grassmann manifolds. This construction can be compared to a powerful machine that produces vector bundles (and the associated frame bundles) over smooth manifolds and naturally incorporates the tangent bundles of embedded manifolds and their tensorial relatives. We show that, up to equivalence, every vector bundle over a compact manifold arises in this way. After having constructed, via a suitable limit procedure, the *infinite Grassmannian* $\mathfrak{G}_{\infty,k}$ of k planes in $\mathbb{R}^\infty$, with its limit tautological bundles, an important result of the chapter is the classification of these vector bundles over a *compact* manifold M, partitioned by the rank k, up to "strict" equivalence. The classifying space is $[M, \mathfrak{G}_{\infty,k}]$: the set of *homotopy classes* of smooth maps from M to $\mathfrak{G}_{\infty,k}$. A typical way to get algebraic topological invariants is to construct functors from some subcategory of topological spaces to some category of algebraic structures (groups, rings, vector spaces, etc.). At the end of Chapter 5, we present a nontrivial implementation of this idea based on this family of vector bundles. By augmenting the strict equivalence to a suitable *stable equivalence*, we realize that the quotient set $\mathbf{K}_0(M)$ of the whole collection of vector bundles (all ranks confused) carries a natural *ring structure*; combined with the pull-back construction, this eventually builds a *contravariant functor* from the (sub)category of compact manifolds to the category of Abelian rings which satisfies the *homotopy invariance property*.

In Chapter 6, we focus on embedded *compact* manifolds, that is, following the above considerations, on compact manifolds exploiting the existence of an embedding in some Euclidean space. We develop a theory of *tubular neighbourhoods* of submanifolds and of *collars* for the boundary of a manifold with boundary. We present some applications of this technology. For simplicity, let us consider here boundaryless manifolds. If M and N are both compact, then we prove that smooth maps are *dense* in $\mathcal{C}^r(M, N)$ for every $r \geq 0$. Primary topological invariants, as the fundamental group or higher homotopy groups, are defined in general in terms of homotopy classes of *continuous* maps defined on spheres. As an application of the density theorem, we see that they can be equivalently defined in terms of smooth homotopy between smooth maps $f : S^n \to N$. We use this fact to classify vector bundles on spheres. Another important application, for every $r \geq 1$, is the approximation of every compact $\mathcal{C}^r$-manifold $M \subset \mathbb{R}^h$ by smooth embedded manifolds and the existence and uniqueness up to diffeomorphism of a smooth structure on each such $\mathcal{C}^r$-manifold. We state the *Sard-Brown* theorem, which is the base of *transversality* that shall be more systematically developed in Chapter 8; here, we anticipate some manifestation. By using the restriction to $M \subset \mathbb{R}^h$ of "generic" linear projections of $\mathbb{R}^h$ to lines, we show that *Morse functions* form an *open and dense* subset of $\mathcal{E}(M, \mathbb{R})$. We also study some instances of generic linear projections to hyperplanes and, eventually, prove the *"easy" Whitney immersion/embedding theorem*: every m-dimensional compact smooth manifold M can be immersed in $\mathbb{R}^{2m}$ and embedded in $\mathbb{R}^{2m+1}$. In the last section of the chapter, we discuss a huge refinement of the approximation theorem by smooth manifolds. Exploiting the fact that Grassmann manifolds are not only embedded smooth manifolds but actually *regular real algebraic sets* and that the tautological bundles are also real algebraic, we outline Nash's celebrated result that every embedded smooth manifold $M \subset \mathbb{R}^h$ can be approximated by a regular sheet of a real algebraic set of $\mathbb{R}^h$ (shortly, by a *Nash manifold*) and that every compact embedded smooth manifold admits a Nash manifold structure, unique up to Nash diffeomorphism. We also discuss a version of the Sard-Brown theorem in the category of Nash manifolds. In the general setting, the result is expressed in measure-theoretic terms, while in the Nash case it is purely a geometric statement, as well as its proof.

Chapters 1 to 6, with the exceptions of the end of Chapter 5 about the rings $\mathbf{K}_0(*)$ and the digression on Nash's manifolds, form the strict foundation part of this text. The following chapters articulate a more advanced discourse.

In Chapter 7, we collect several constructions that produce new compact manifolds by modifying given ones. At first, we prove the so-called *Thom lemma* about the extension of any isotopy defined on a compact source manifold to an ambient diffeotopy; this is the main tool to prove that such constructions are well-defined up to diffeomorphism. Among cut-and-paste procedures, we recall *gluing along diffeomorphic boundary components*, *connected sum* with a discussion about the related notion of *twisted spheres*, and *attaching a p-handle* (i.e., a standard handle $D^p \times D^{m-p}$ of index p to an m-manifold M via an embedding in ∂M of the *attaching tube* $S^{p-1} \times D^{m-p}$). In many cases, the immediate result is a *smooth manifold with corners*. Corners also arise by taking the product of two manifolds with nonempty boundary. We discuss a standard procedure of *smoothing the corners* that produces ordinary smooth manifolds well-defined up to diffeomorphism. We also discuss the *strong Whitney immersion/embedding theorem* of any m-dimensional compact manifold M in $\mathbb{R}^{2m-1}$ and in $\mathbb{R}^{2m}$, respectively. The main difference compared with the "easy" Whitney theorems is that the strong ones are not entirely based on "generic position arguments" (i.e., transversality). The strong embedding is achieved by performing a robust alteration of a "generic" immersions in $\mathbb{R}^{2m}$, the strong immersion by modifying certain "generic" maps of M in $\mathbb{R}^{2m-1}$. The proof of the strong embedding theorem introduces the so-called *Whitney trick* to eliminate pairs of self-intersection points in the image of a generic immersion in $\mathbb{R}^{2m}$; this "trick"'will be considered again in Chapter 18 and in Chapter 20. By elaborating on the strong immersion theorem, we present *Rohlin's embedding theorem in* $\mathbb{R}^{2m-1}$ *up to surgery.* This shows that for every orientable manifold M as above, there is M' such that the disjoint union $M \amalg M'$ is the boundary of a compact orientable $(m+1)$-manifold W and M' can be *embedded* in $\mathbb{R}^{2m-1}$. In the last section of the chapter, we describe the modification obtained by *blowing up a manifold* M *along a smooth centre* $X \subset M$; this replaces X with its *projectivized normal bundle* in M.

In Chapter 8, we develop the *transversality* concept in a more systematic way. As usual, the source manifold M is compact, possibly with a nonempty boundary, and for simplicity we assume here that the target manifold N is also compact and boundaryless; Z is a boundaryless compact submanifold of N. There are two kinds of *basic transversality theorems.* The first kind concerns a certain geometric tameness under the transversality hypothesis: if $f : M \to N$ is transverse to Z, then $(Y, \partial Y) := (f^{-1}(Z), (\partial f)^{-1}(Z))$ is a "proper submanifold" of $(M, \partial M)$ of the same *codimension* of Z in N. There is also a specialization within the category of oriented manifolds. The second kind of basic transversality theorem states that transverse maps are generic and stable; that is, they form an *open and dense* set in $\mathcal{E}(M, N)$. There is also a relative version, concerning maps which coincide on ∂M, provided that this restriction is already transverse to Z by itself. The bridge between the two kinds of theorems is represented by the so-called *parametric transversality*, whose proof is substantially based on the Sard-Brown theorem. These basic transversality theorems suffice for most applications later in the text. However, transversality (i.e., "general position" reasoning) is a profound, potent, and pervasive paradigm beyond such basic results. Without any pretension of completeness, in the second part of the chapter we collect a few examples of further applications (including the notion of "generic immersion", already employed while discussing Whitney's strong embedding theorem).

In Chapter 9, we formalize the notion of a *smooth triad* (M, V_0, V_1), where M is a compact smooth m-manifold and V_0 and V_1 are unions of connected components of ∂M in such a way that the boundary is the disjoint union $\partial M = V_0 \amalg V_1$; M might be boundaryless, so that the triad $(M, \emptyset, \emptyset)$ is allowed. In some way, a triad realizes a "transition" from V_0 to V_1. We define generic Morse functions $f : M \to [0, 1]$ on a triad, meaning that $f^{-1}(j) = V_j$, $j = 0, 1$, f has only nondegenerate critical points placed outside a neighbourhood of ∂M, and they have distinct critical values. The density and stability of these functions are assured by the results of Chapter 8. An important achievement of Chapter 9 is that every Morse function carries a *handle decomposition of the triad*, that is, a way to reconstruct the triad (up to diffeomorphism) from a collar of V_0 in M, by attaching successively a handle of index p for every nondegenerate critical point of index p of f. Associated with every decomposition of a triad (M, V_0, V_1), there is a *dual decomposition* of the triad (M, V_1, V_0) where every p-handle is converted into an $(m-p)$-handle and these are attached backward starting from a collar of V_1 in M. If the initial decomposition is carried by a Morse function f, then the dual decomposition is carried by the function $1-f$. In a sense, Morse functions are used as a tool to prove the *existence* of handle decompositions. Then, handle decompositions are used as they are and are eventually modified, not addressing the issue of whether the new decompositions are carried by a Morse function. We point out two *basic moves* which modify a given decomposition without changing the triad (up to diffeomorphism): the so-called *sliding handles* (which is nothing other than the possibility of modifying any attaching map up to isotopy, already treated in Chapter 7) and the *elimination/insertion of pairs of complementary handles.* We show some elementary specialization ("reordering") or simplification ("elimination of 0- and m-handles") of handle decompositions obtained by using the basic moves. As a simple but important application, we get the classification up to diffeomorphism of compact 1-dimensional manifolds, confirming the intuition: a connected compact 1-manifold is diffeomorphic either to S^1 or to the 1-disk $[-1, 1]$.

In Chapter 10, we develop *bordism.* There is an unoriented version and an oriented one. Two (unoriented) compact boundaryless m-manifolds M_0 and M_1 are bordant manifolds if $M_0 \amalg M_1$ is the boundary of a compact $(m+1)$-manifold W. In the oriented case, the manifolds M_0, M_1, and W are oriented and $M_0 \amalg -M_1$ is the oriented boundary of W. Case by case, the quotient set of the relation generated by "being bordant" and (oriented) diffeomorphisms is denoted by Ω_m in the oriented case and is a $\mathbb{Z}$-module, while it is denoted by η_m in the nonoriented case and is a $\mathbb{Z}/2\mathbb{Z}$-vector space. The operation is induced by the *disjoint union.* If X is any topological space, a continuous map $f : M \to X$ is called a *singular* smooth m-manifold in X and we can extend the definition of bordism to such singular manifolds and, consequently, define the modules $\Omega_m(X)$ or $\eta_m(X)$, sometimes denoted by $\mathcal{B}_m(X; R)$, $R = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$. When X consists of a single point, we recover the earlier modules because the maps f are immaterial in this case. We can also define relative versions $\mathcal{B}_m(X, A; R)$ for topological pairs (X, A) (X being, as usual, identified with $(X, \emptyset)$). We prove that in this way we define a *covariant functor* from the category of topological pairs to the category of R-modules, which turns out to be a *generalized homology theory.* This means that all *Eilenberg-Steenrod axioms* are satisfied with the possible exception of "dimension"; its failure depends on the nontriviality of $\mathcal{B}_m(X; R)$, $m \geq 1$, when X consists of a single point. This issue will be considered throughout the rest of the text. We discuss some relationships between bordism and homotopy group functors.

In Chapter 11, we specialize bordism assuming that X is a compact boundaryless smooth manifold. Just like the homotopy groups, thanks to the approximation theorems of Chapter 6, it is not restrictive to deal only with smooth maps $f : M \to X$. The bordism modules $\mathcal{B}_m(X; \mathbb{Z}/2\mathbb{Z})$ are indexed over $\mathbb{Z}$, by postulating that they are the trivial module 0 if $m < 0$. We formally re-index them by the *codimension*, by setting $\mathcal{B}^r(X; \mathbb{Z}/2\mathbb{Z}) = \mathcal{B}_m(X; \mathbb{Z}/2\mathbb{Z})$, $r = \dim X - m$, so that they are trivial if $r > \dim X$ and they are now called *cobordism modules*. The key point is that by combining a slight extension of the basic transversality theorems of Chapter 8 with variations on the *pull-back construction*, we incorporate $X \Rightarrow \bigoplus_r \mathcal{B}^r(X; \mathbb{Z}/2\mathbb{Z})$ into a *contravariant functor* from the (sub)category of compact boundaryless smooth manifolds to the category of *graded rings*; this means that $\bigoplus_r \mathcal{B}^r(X; \mathbb{Z}/2\mathbb{Z})$ is endowed with a multiplicative structure which distributes itself in a family of $\mathbb{Z}/2\mathbb{Z}$-bilinear maps

$$\bigsqcup : \mathcal{B}^r(X; \mathbb{Z}/2\mathbb{Z}) \times \mathcal{B}^s(X; \mathbb{Z}/2\mathbb{Z}) \to \mathcal{B}^{r+s}(X; \mathbb{Z}/2\mathbb{Z})$$

defined geometrically via transversality and implementation of the pull-back construction. If X is oriented, we can perform all the construction within the oriented category, using the $\mathbb{Z}$-modules $\mathcal{B}^r(X; \mathbb{Z})$. If $\alpha = [M_1]$ and $\beta = [M_2]$ are represented by submanifolds of X, then $\alpha \sqcup \beta$ is represented by any transverse intersection $M_1' \cap M_2'$ where M_j' is a suitable small perturbation of M_j, $j = 1, 2$. Over both $R = \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}$, the product satisfies the relation

$$\alpha \sqcup \beta = (-1)^{rs} \beta \sqcup \alpha$$

which can be checked in a geometric way. If X consists of a single point, then the product reduces to $[M] \sqcup [N] = [M \times N]$. If $r + s = \dim X = n$ and X is connected (possibly oriented), then $\mathcal{B}^n(X; R) = R$ and the product $\sqcup$ induces a linear map $\phi^r : \mathcal{B}^r(X; R) \to \mathrm{Hom}(\mathcal{B}_r(X; R), R)$; in many situations it is convenient to consider the quotient module

$$\mathcal{H}^r(X; \mathbb{R}) := \mathcal{B}^r(X; R) / \ker(\phi^r)$$

with the induced linear injection

$$\hat{\phi}^r : \mathcal{H}^r(X; R) \to \mathrm{Hom}(\mathcal{H}_r(X; R), R) \ .$$

In particular, if X is oriented, then $\mathcal{H}^r(X; \mathbb{Z})$ is torsion free. If X is connected (possibly oriented) and $\dim X = 2m$, then we have the *intersection form*

$$\bigsqcup : \mathcal{H}^m(X; R) \times \mathcal{H}^m(X; R) \to R$$

which is symmetric if either $R = \mathbb{Z}/2\mathbb{Z}$ or $R = \mathbb{Z}$ and m is even; it is antisymmetric otherwise. Sometimes it is expressed as

$$\bullet : \mathcal{H}_m(X;R) \times \mathcal{H}_m(X;R) \to R \ .$$

The cobordism multiplicative structure is a substantial enhancement of the theory. In Chapter 12, we collect a few classical applications: the *fundamental class* $[X] \in \mathcal{H}^0(X;R)$ when X is connected and possibly oriented; *Brouwer's fixed point theorem* for continuous maps $f : D^n \to D^n$, $n \geq 1$; a *separation theorem* for hypersurfaces in S^n, $n > 1$; *intersection and linking numbers*; the *R-degree* of continuous maps $f : M \to N$ between (possibly oriented) compact connected boundaryless smooth manifolds of the same dimension; a proof of the *fundamental theorem of algebra*; the *Borsuk-Ulam theorem*. We also define the *Euler class* $\omega(\xi) \in \mathcal{B}^k(X;R)$ of a rank-k vector bundle ξ over X (everything possibly suitably oriented), defined by the transverse self-intersection of the zero section of ξ in its total space. A non-vanishing Euler class is a primary obstruction to the existence of a nowhere vanishing section of ξ.

In Chapter 13, we focus on line (i.e., rank-1) bundles on X, on oriented rank-2 vector bundles (provided that also X is oriented), and on their Euler classes in $\mathcal{B}^1(X;\mathbb{Z}/2\mathbb{Z})$, $\mathcal{B}^1(X;\mathbb{Z})$, or $\mathcal{B}^2(X;\mathbb{Z})$. A key point here is that $\mathbf{P}^\infty(\mathbb{R})$ is a $\mathbf{K}(1,\mathbb{Z}/2\mathbb{Z})$-space, S^1 is a $\mathbf{K}(1,\mathbb{Z})$-space, and $\mathbf{P}^\infty(\mathbb{C})$ is a $\mathbf{K}(2,\mathbb{Z})$-space. This eventually gives precise information, case by case, about $\mathcal{H}^1(X;R)$ and $\mathcal{H}^2(X;\mathbb{Z})$. For example, every class in $\mathcal{H}^1(X;R)$ is the Euler class of a line bundle over X (oriented if $R = \mathbb{Z}$). It can be represented by an embedded hypersurface S of X (oriented if $R = \mathbb{Z}$); moreover, $[S_0] = [S_1]$ (in the appropriate bordism module) is equivalent to the fact that the associated bundles are strictly equivalent, and it is also equivalent to the fact that a bordism between S_0 and S_1 is realized through a triad (W, S_0, S_1) properly embedded in $X \times [0,1]$ (all manifolds being oriented if $R = \mathbb{Z}$), similarly for $\mathcal{B}^2(X;\mathbb{Z})$.

In Chapter 14, we focus on the Euler class in $\mathcal{B}^m(M;\mathbb{Z}) = \mathbb{Z}$ of the tangent bundle of a compact oriented connected boundaryless smooth m-manifold M. This integer is denoted by $\chi(M)$ and is called the *Euler-Poincaré (E-P) characteristic* of M. Essentially by definition, it can be computed using any section of $T(M)$ transverse to the zero section, that is, using any tangent vector field on M with only nondegenerate zeros. This can be extended to any tangent vector field on M with only isolated (not necessarily nondegenerate) zeros. This is the content of the *index theorem*; the key point is the reformulation of the sign of a nondegenerate zero in terms of the $\mathbb{Z}$-degree of a suitable map $f : S^{m-1} \to S^{m-1}$, defined locally at the zero using the vector field; this reformulation by the degree makes sense also for any isolated zero and well-defines its *index*. Then $\chi(M)$ is eventually equal to the sum of such indices. Invariance of the degree up to bordism plays a crucial role in this achievement. The Euler-Poicaré characteristic is multiplicative with regards to the product of compact boundaryless manifolds. The value of $\chi(X)$ does not depend on the choice of the orientation of X; eventually $\chi(M) := \frac{1}{2}\chi(\tilde{M})$ is well-defined also if M is not orientable, $\tilde{M} \to M$ being the orientation 2-to-1 covering map. We extend the index formula to define the *relative characteristic* $\chi(M, V_0)$ of a triad (M, V_0, V_1) by using suitable tangent vector fields on M, transverse to the boundary and with only isolated zeros. The characteristic has certain homotopy invariance properties so that, for example, if B is the total space of a disk bundle over a boundaryless M, then $\chi(M) = \chi(B, \emptyset, \partial B)$. In the special case when M is embedded in $\mathbb{R}^h$ and B is a tubular neighbourhood of M in $\mathbb{R}^h$, this leads to the classical fact that $\chi(M)$ coincides with the degree of the *Gauss map* $\partial B \to S^{h-1}$. The extended characteristic also has remarkable additive properties concerning the composition of triads. Moreover, $\chi(M, V_0)$ can be computed using any gradient vector field of any Morse function $f : M \to [0, 1]$ on the triad. By combining these facts, we obtain, for example, that if M is boundaryless and *odd*-dimensional, then $\chi(M) = 0$ (use both f and $1 - f$ to compute $\chi(M)$ in two ways); if V is even-dimensional and it is the boundary of some M, then $\chi(V) \equiv 0 \bmod(2)$. It follows that for every even m, η_m is nontrivial because $\chi(\mathbf{P}^m(\mathbb{R})) = 1$. At the end of the chapter, we briefly discuss other ways (combinatorial or algebraic/topological) to recover the Euler-Poincaré characteristic.

In Chapter 15, we apply several tools developed in the previous chapters to classify compact surfaces (i.e., compact smooth 2-manifolds) up to diffeomorphism and also to determine both bordism moduli η_2 and Ω_2. If M is a connected boundaryless compact surface, we show that $\eta_1(M)$ is a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space and that the symmetric intersection form $\bullet : \eta_1(M) \times \eta_1(M) \to \mathbb{Z}/2\mathbb{Z}$ is nondegenerate. We focus on its isometry class as the main invariant up to diffeomorphism. After having established the abstract algebraic classification, up to isometry, of nondegenerate symmetric bilinear forms on finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-spaces, we show, step by step, that there is a perfect 2D topological counterpart. Finally, every isometry class can be realized as the intersection form of some surface M, and two surfaces are diffeomorphic if and only if they have isometric intersection forms. In particular, from that isometry class, we can derive whether M is orientable or not and the value of $\chi(M)$. If M is orientable, then it is the connected sum of S^2 with g copies of $S^1 \times S^1$, where $\chi(M) = 2 - 2g$; if M is not orientable, then M is a connected sum of copies of $\mathbf{P}^2(\mathbb{R})$ whose number is determined by $\chi(M)$. As for the bordism, $\Omega_2 = 0$, while $\eta_2 = \mathbb{Z}/2\mathbb{Z}$ generated by $[\mathbf{P}^2(\mathbb{R})]$. We also discuss some aspect of the *stable equivalence* generated by diffeomorphisms and the elementary stabilization that consists of performing the connected sum with $\mathbf{P}^2(\mathbb{R})$; in particular, we refer to the relationship with the so-called *Nash rational model question* in dimension 2. The theme of the *quadratic enhancements* of the intersection form associated with the immersion of a surface in a higher-dimensional manifold will emerge later in the text. At the end of Chapter 15, we develop the abstract theory of these quadratic enhancements, including the introduction of the *Arf* and the *Arf-Brown* invariants.

The Euler-Poincaré characteristic mod(2) is a first example of characteristic number for the nonoriented bordism modules η_m; that is, for every $m \geq 0$, it defines a homomorphism $\chi_{(2)} : \eta_m \to \mathbb{Z}/2\mathbb{Z}$ which is surjective for each even m. Pontryagin remarked that a huge family of characteristic numbers can be produced using the cohomology ring with $\mathbb{Z}/2\mathbb{Z}$-coefficients of the infinite Grassmann manifolds $\mathfrak{G}_{\infty,\bullet}$ and the classifying map $M \to \mathfrak{G}_{\infty,m+1}$ of the stable tangent bundle $T(M) \oplus \epsilon^1$ of each compact boundaryless m-manifold M. These are called *SW-characteristic numbers* as they are incorporated into the theory of (cohomological) Stiefel-Whitney *characteristic classes*. We do not dispose of cohomology, but in Chapter 16, it is easy to reformulate the definition using the cobordism rings, which we have defined from scratch, instead of the cohomology rings. We call (stable) *η-characteristic numbers* the ones obtained in this way. In [**T**], Thom determined the ring $\eta^\bullet$, using the Pontryagin-Thom construction (that we treat in Chapter 17) and combining geometric tools and homotopy theory. A byproduct of Thom's work is the *completeness* of the SW-characteristic numbers: *$\beta \in \eta_m$ is equal to zero if and only if every SW-characteristic number vanishes on β*. Later, the authors obtained in [**BH**] a nice geometric proof of this remarkable result, ultimately based on transversality and simple cohomological computations. In Chapter 16, we show that this proof can be entirely performed using the cobordism rings, and we eventually get the completeness of the η-characteristic numbers. At the end of the chapter, we extend η to Ω-characteristic numbers and we briefly discuss why they are not sufficient to completely detect the oriented boundaries. Cohomology cannot be avoided in dealing with the oriented bordism. However, any characteristic number for $\Omega_\bullet$, however it is defined, should vanish on $[M]$ if the m-manifold M is *parallelizable*. At least in this special case we prove that $[M] = 0 \in \Omega_m$ for any choice of the orientation of M, using similar geometric tools.

Chapter 17 is dedicated to the *Pontryagin-Thom* construction. The original Pontryagin construction was invented to rephrase the homotopy groups of spheres $\pi_{n+k}(S^n)$, $k \geq 0$, $n > 1$, in terms of a certain more geometric (therefore presumably more accessible at that time, around 1938) codimension-n embedded oriented bordism theory with target S^{n+k}. This so-called *framed bordism* makes sense for arbitrary compact target M of dimension $n+k$ and recovers $[M, S^n]$. Vice versa, Thom's extension of Pontryagin construction was mainly intended as a way to rephrase the cobordism rings $\eta^\bullet$ or $\Omega^\bullet$ in terms of the homotopy groups (more accessible at that time, around 1954, after the impressive progress in homotopy theory since Serre's thesis [**Se**]) of the so-called Thom spaces. Concerning the determination of $\pi_{n+k}(S^n)$, Pontryagin succeeded for $k \leq 2$; in Chapter 17, we outline these results. For $k = 0$, the $\mathbb{Z}$-degree establishes an isomorphism between $\pi_n(S^n)$ and $\mathbb{Z}$. As a corollary, we show that a compact connected boundaryless manifold M is *combable* (i.e., it admits a nowhere vanishing tangent vector field) if and only if $\chi(M) = 0$; in particular, every odd-dimensional M is combable. The difficulty increases by k. For $k = 2$, a key ingredient is the Arf invariant of the quadratic enhancement of the intersection form of every framed orientable surface in S^6. The hardest application of this geometric method is for $k = 3$ and is due to Rohlin. We limit ourselves to stating the result. This is of major importance for its consequences in the theory of 4-manifolds and will be considered again in Chapter 20.

In differential topology, there is a precise distinction between "high" (6 or greater) dimensions and "low" (4 or fewer) dimensions, 5 being on the border. The main reason is that for $d \geq 6$, Smale's (simply connected) *h-cobordism theorem* holds; moreover there is a "stable proof" that works uniformly for all high dimensions. This is an important application of handle decomposition theory. This proof does not apply to lower dimensions. In some cases the theorem fails, and in some cases it is still an open question. In Chapter 18, we briefly discuss this issue. We do not prove the whole stable h-cobordism theorem; rather we focus on an important step (the cancellation of *algebraically* complementary handles) where the high dimension assumption is crucial. This is related to the possibility of applying Whitney's trick (first introduced for the strong embedding theorem) to eliminate pairs of intersection points of opposite sign between transverse submanifolds of complementary dimension of a simply connected manifold of dimension greater than or equal to 5.

For "very low" dimensions $0 \leq d \leq 2$, we have achieved a complete classification of compact manifolds up to diffeomorphism. This is essentially "hopeless" for $d > 2$, even for $d = 3, 4$. In Chapters 19 and 20, we address some aspects of these low-dimensional theories. We stress that, in both cases, we do not touch the mainstream themes of the last decades (the *geometrization conjecture* — now a theorem — of 3-manifolds or the use of powerful *gauge theories* applied to the study of 4-manifolds). We limit ourselves to developing a few primary differential topological results, combining several tools established in the previous chapters.

In Chapter 19, we present a few elementary and self-contained proofs that compact orientable boundaryless 3-manifolds are parallelizable and we study combing and framing. A significant amount of the chapter is devoted to several proofs of "$\Omega_3 = 0$" and of the equivalent *Lickorish-Wallace theorem* about 3-manifolds considered up to "longitudinal" Dehn surgery equivalence. Each proof will enlighten different facets of the subject. The last two sections of the chapter are more advanced. We determine the bordism semigroup (which turns out to be a group) of immersions of surfaces in a given compact connected boundaryless 3-manifold M. If M is orientable, a key ingredient will be the Arf-Brown invariant of the quadratic enhancement of the intersection form associated with every immersion of a surface in M (endowed with an auxiliary framing). We also classify compact boundaryless 3-manifolds up to certain equivalence relations generated by diffeomorphisms and blow-up-down along smooth centres (a notion introduced in Chapter 7). The subtler so-called "tear" equivalence, in the nonorientable case, also involves instances of quadratic enhancement of the intersection form of characteristic surfaces (i.e., representing the Euler class of the determinant bundle of the ambient 3-manifold). We also discuss an application to a solution of the so-called *Nash rational model question* in dimension 3.

In Chapter 20, in analogy to the case of surfaces, we focus on the isometry class of the intersection form $\bigsqcup_M : \mathcal{H}^2(M;\mathbb{Z}) \times \mathcal{H}^2(M;\mathbb{Z}) \to \mathbb{Z}$ as the main invariant of every compact connected oriented boundaryless 4-manifold M. It is a symmetric unimodular $\mathbb{Z}$-bilinear form on the finite rank-free $\mathbb{Z}$-module $\mathcal{H}^2(M;\mathbb{Z})$. We prove Rohlin's theorem that the signature σ of the intersection form determines an isomorphism $\sigma : \Omega_4 \to \mathbb{Z}$, so that Ω_4 is generated by $[\mathbf{P}^2(\mathbb{C})]$. We follow his original geometric proof. Trying to pursue the analogy with surfaces, we address the abstract arithmetic classification of such symmetric unimodular forms. An important difference is that it is complete only in the *indefinite* case. We try to develop, as much as possible, a parallel 4D counterpart, at least in the indefinite case, by restricting, in fact, to *simply connected* 4-manifolds. We establish a classification up to *odd stabilization*, the elementary ones being the connected sum with $\pm\mathbf{P}^2(\mathbb{C})$. We outline a more subtle classification up to *even stabilization* (i.e., up to connected sum with $S^2 \times S^2$). Arithmetic tells us that there are *characteristic elements* $\beta \in \mathcal{H}^2(M;\mathbb{Z})$ such that for every $\alpha \in \mathcal{H}^2(M;\mathbb{Z})$, $\alpha \sqcup \alpha = \beta \sqcup \alpha$ mod(2) and that $\sigma = \beta \sqcup \beta$ mod(8). Every β can be represented by an oriented surface F embedded in M, called a *characteristic surface*. We prove the congruence

$$\sigma - \beta \sqcup \beta = 8\alpha(F) \bmod(16)\,,$$

where $\alpha(F) \in \mathbb{Z}/2\mathbb{Z}$ is the Arf invariant of a quadratic enhancement of the intersection form of F which represents an obstruction to surgery F

within M to an embedded 2-sphere. If the intersection form is even, we can take $F = \emptyset$, so that we recover the original celebrated Rohlin congruence $\sigma = 0 \bmod(16)$ (originally obtained as a corollary of the fact that $\pi_{n+3}(S^n) = \mathbb{Z}/24\mathbb{Z}$ for n big enough). This implies, in particular, that there are unimodular symmetric forms which cannot be realized as the intersection form of any simply connected 4-manifold. We propose an elementary proof of the congruence due to [**Mat**] and based on the classification up to odd stabilization. Following [**A3**] and [**Roh**], we illustrate an application of these 4-dimensional congruences mod(16) to Hilbert's 16th problem. We end the chapter with an informative and discursive section about more recent achievements in the realm of 4-manifolds.

In conclusion, this book is a collection of themes, in some cases advanced and of historical importance, with the common characteristic that they can be treated with "bare hands", meaning by combining specific differential topological cut-and-paste procedures and applications of transversality, mainly through the cobordism multiplicative structure. Of course, the choice of the topics was due in part to personal preferences. The book is intended to be accessible and useful to motivated undergraduate students and to Ph.D. students, but also to a more expert reader to recognize very basic reasons for some facts already known as the result of more advanced theories or technologies.

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