GRADUATE STUDIES IN MATHEMATICS 219

Essentials of Tropical Combinatorics

Michael Joswig

American Mathematical Society

10.1090/gsm/219

Essentials of Tropical Combinatorics

GRADUATE STUDIES IN MATHEMATICS 219

Essentials of Tropical Combinatorics

Michael Joswig

AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island

Support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH$^+$ (EXC-2046/1, project ID 390685689); "Symbolic Tools in Mathematics and their Application" (TRR 195, project-ID 286237555); "Facets of Complexity" (GRK 2434, project-ID 385256563) gratefully acknowledged.

For additional information and updates on this book, visit
www.ams.org/bookpages/gsm-219

Library of Congress Cataloging-in-Publication Data

Names: Joswig, Michael, 1965- author.
Title: Essentials of tropical combinatorics / Michael Joswig.
Description: Providence, Rhode Island : American Mathematical Society, [2021] | Series: Graduate studies in mathematics, 1065-7339 ; 219 | Includes bibliographical references and index.
Identifiers: LCCN 2021023313 | ISBN 9781470466534 (hardback) | ISBN 9781470467418 (paperback) | ISBN 9781470467401 (v. 219 ; ebook)
Subjects: LCSH: Tropical geometry. | Combinatorial geometry. | Linear programming. | AMS: Algebraic geometry – Tropical geometry – Combinatorial aspects of tropical varieties. | Algebraic geometry – Tropical geometry – Applications of tropical geometry. | Convex and discrete geometry – Polytopes and polyhedra – Computational aspects related to convexity. | Operations research, mathematical programming – Mathematical programming – Linear programming. | Operations research, mathematical programming – Mathematical programming – Tropical optimization (e.g., max-plus optimization).
Classification: LCC QA582 .J67 2021 | DDC 516.3/5–dc23
LC record available at https://lccn.loc.gov/2021023313

10 9 8 7 6 5 4 3 2 1 26 25 24 23 22 21

To Claudia

Contents

Preface

The goal of this book is to explain, at the graduate student level, connections between tropical geometry and optimization. Tropical geometry is the geometry over the semiring $(\mathbb{R}, \min, +)$. Its key objects, tropical hypersurfaces and more general tropical varieties, admit polyhedral descriptions, i.e., in terms of finitely many ordinary linear inequalities and equations. This allows us to approach tropical geometry through geometric combinatorics, and this is what we call tropical combinatorics.

Building bridges between optimization and tropical geometry is fruitful in two ways. Through tropical geometry, optimization algorithms become applicable to questions in algebraic geometry. Conversely, looking at topics in optimization through the tropical geometry lens adds an additional layer of structure to known optimization problems. The main prerequisite for the reader is a working knowledge in polytope theory as contained in the first few chapters in the books by Grünbaum [**Grü03**] or Ziegler [**Zie95**]. For convenience, a terse summary is given in Appendix A. Due to the tremendous growth of tropical geometry it is impossible to cover all aspects. The strategy chosen here is to stay close to polyhedral geometry; this entails that some proofs of algebraic results are replaced by references.

Despite its rather young age, tropical geometry already has a rich history, as it unifies techniques from several directions. Early contributions, which established tropical geometry as a field, include work of Einsiedler, Kapranov, and Lind [**EKL06**] on non-Archimedean amoebas and Mikhalkin's enumerative results on plane algebraic curves [**Mik05**]. A main source of inspiration for this book, however, is the approach to algebraic geometry via polyhedral geometry as expressed in the books by Gel′fand, Kapranov, and Zelevinsky [**GKZ08**] and Sturmfels [**Stu96**]. The first reference to tropical geometry from this perspective is Sturmfels [**Stu02**, Chapter 9], quickly followed by the landmark paper of Speyer and Sturmfels on tropical Grassmannians [**SS04**]. Popular introductions to the subject include the book by Itenberg, Mikhalkin, and Shustin [**IMS09**] and the survey by Richter-Gebert, Sturmfels, and Theobald [**RGST05**], which access the subject from two different angles. The monograph by Maclagan and Sturmfels [**MS15**] is now a standard reference.

Specializing tropical geometry to the tropicalization of linear algebra leads to the subject of max-plus linear algebra. This field, rooted in optimization, is much older than tropical geometry. It is linked with signal processing, functional analysis, and other areas. A text containing the early history is the monograph by Baccelli, Cohen, Olsder, and Quadrat [**BCOQ92**]. Further influential contributions in this line of research include work of Litvinov, Maslov, and Shpiz [**LMS01**], Cohen, Gaubert, and Quadrat [**CGQ04**], and Butkovič [**But10**]. For a recent survey see Akian, Bapat, and Gaubert [**ABG14**].

As its inhabitants know well, the world of tropical geometry is divided between the "min" and "max" conventions. Here we follow a somewhat mixed approach, which needs a word of justification. First of all, sometimes it is beneficial to look into "min" and "max" *together*. Perhaps the most prominent example is that min-tropical convexity is most naturally described in terms of an arrangement of max-tropical hyperplanes, or conversely. But the reader will also find other examples in this book. Another aspect is that the various roots of tropical geometry naturally lead to opposite conventions: For tropical algebraic geometry non-Archimedean valuations on fields play a key role, with Puiseux series as the prime example. In their classical form Puiseux series depend on a formal parameter which may be seen as an infinitesimal, and this leads to the min-convention. Yet this has the drawback that the ordering gets reversed when we look at the ordered field of real Puiseux series. This is one reason why many connections between tropical geometry and optimization are naturally studied via the max-convention. I chose to be guided by the paradigm of tropical convexity, which relates $(\min, +)$-tropical polytopes with $(\max, +)$-tropical hyperplane arrangements. Consequently, in this book tropical point configurations, tropical polynomials, tropical hypersurfaces, and such are usually studied via "min". When it comes to describing tropical hyperplane arrangements and tropical linear optimization we switch to "max" as the default.

Using this book for teaching. This book grew out of courses given at TU Berlin and elsewhere. The chapters are organized into sections which are meant to be read sequentially. Yet some material may also be skipped on first reading or in a course which is too short to cover everything. For instance, a course of two hours a week for 15 weeks, with a focus on the relationship to optimization, could follow the path: Chapter 1 (omitting Section 1.6), Sections 2.1 and 2.2, Chapter 3, Chapter 5, Sections 6.1 through 6.3, Chapter 7, and Chapter 8.

Organization of the book. Each chapter ends with two special sections containing problems and remarks. The problems come in three flavors. The first type is comprised of standard exercises intended to help the reader get acquainted with the subject. The second type comes with hints in Appendix D; these problems are marked with "○". They are selected because often they are related to proofs of lemmas or particularly relevant examples that are used later in the text. The third kind, marked with "●", are research problems and open questions. References to the literature or historical explanations are usually collected as remarks at the very end of each chapter. Occasionally, we skip a proof of a result which we consider outside the scope of this book, despite its relevance for tropical geometry. In those cases, a textbook reference is given instantly; an example is the Newton–Puiseux Theorem, Theorem 2.1. Some specific examples are copied from the literature, always with an exact reference, usually in the remarks following each chapter. This is meant to ease further reading of the original texts, which often study additional aspects beyond the scope of this book. For instance, Example 4.34 originates from an article of Tran and Yu [**TY19**, Example 3].

There are four appendices. The first two, A on "Geometric Combinatorics" and B for "Computational Complexity", collect material that is more or less taken for granted throughout the book. While they cannot replace textbooks on their respective subjects, some readers may find the reminders convenient. Appendix C is very different. It comprises explicit computations with the software system `polymake` [**GJ00**, **HJ17**], covering large portions of the book. Looking it up frequently and experimenting with the software may be an additional help. The final appendix, Appendix D, contains hints to the problems marked "○".

Acknowledgements. I am very grateful to the many readers of early drafts of this book, in its various stages, for their comments, suggestions, and corrections. This includes, in particular, Xavier Allamigeon, Benjamin Assarf, Matthias Beck, Pascal Benchimol, Paul Breiding, Taylor Brysiewicz, Andrei Comăneci, Francisco Criado, Jules Depersin, Maria Dostert, Holger Eble, Stéphane Gaubert, Kevin Guo, Louis Hainaut, Girtrude Hamm, Martin Henk, Katrin Herr, Johannes Hofscheier, Aryaman Jal, Katharina Jochemko, Florian Kohl, Anton Leykin, Niels Lindner, Georg Loho, Ina Mette, Sybille Meyer, Henryk Nagel, Daniel Oberländer, Jorge Alberto

Olarte, Marta Panizzut, Manuel Radons, Janek Reichardt, Felix Rydell, Francisco Santos, Moritz Schmitt, Benjamin Schröter, Sylvain Spitz, Bernd Sturmfels, Paul Vater, André Wagner, and five anonymous referees. Special thanks to Robert Löwe, who helped me to get this book done, finally. In addition to commenting on the text, his support involved copyediting, discussing notation, and contributions to several illustrations.

Berlin, May 2021 Michael Joswig

Notation

The symbols are organized by topic.

Sets.

$\#M$	cardinality of the set M
$[n]$	set of first n positive integers $1, 2, \ldots, n$
$M + N$	disjoint union of the sets M and N
$M - N$	set difference of M and N where N is a subset of M
$M \triangle N$	symmetric difference of sets
$\binom{M}{k}$	set of all k-element subsets of the set M
2^M	power set, i.e., set of all subsets of M

Tropical Arithmetic.

$\mathbb{T}$	the set $\mathbb{R} \cup \{\infty\}$ which underlies the tropical semiring (with respect to min)
$\oplus$	tropical addition, usually: min
$\odot$	tropical multiplication of scalars, always: $+$
$\odot$	also: tropical matrix multiplication; if it is necessary to explicitly name the relevant tropical addition, we use $\odot^{\min}$ and $\odot^{\max}$
$\mathbb{R}^d/\mathbb{R}\mathbb{1}$	tropical $(d-1)$-torus $\mathbb{R}^d/\mathbb{R}\mathbb{1}$
$\mathbb{TP}^{d-1}$	tropical projective $(d-1)$-space

Vectors and Matrices.

e_i	ith standard basis vector of $\mathbb{R}^d$
e_σ	sum of standard basis vectors $\sum_{i\in\sigma} e_i$

e_i^{trop} ith tropical standard basis vector of $\mathbb{TP}^{d-1}$

$\zeta(p)$ canonical coordinates of a point $p \in \mathbb{TP}^{d-1}$

M_i ith row of matrix M

M_I submatrix of $m \times n$-matrix M formed from rows indexed by $I \subseteq [m]$

M_{-i} short for submatrix $M_{[m]-i}$ of M obtained by deleting ith row

$M^{(k)}$ kth column of M

$M^{(-k)}$ submatrix of M obtained by deleting the kth column

$A^\top$ transpose of the matrix A

$A^\#$ residuated operator defined by matrix A

Valued Fields.

$\mathbb{F}[\![t]\!]$ power series with coefficients in $\mathbb{F}$ and indeterminate t

$\mathbb{F}(\!(t)\!)$ Laurent series

$\mathbb{F}\{\!\{t\}\!\}$ standard Puiseux series

$\mathbb{F}\{\!\{t\}\!\}^*$ dual standard Puiseux series

$\mathbb{F}\{t\}$ Puiseux fractions

$\mathbb{F}\{\!\{t^{\mathbb{R}}\}\!\}$ generalized Puiseux series

$\mathbb{F}\{\!\{t^{\mathbb{R}}\}\!\}^*$ dual generalized Puiseux series

$\operatorname{ord}(\gamma)$ valuation of a Puiseux series

$\operatorname{sord}(\gamma)$ signed valuation of a real Puiseux series

$V(I)$ algebraic variety of the ideal I, affine or projective

$\mathrm{P}^m(\mathbb{F})$ m-dimensional projective space over $\mathbb{F}$

Tropical Polynomials and Polyhedral Geometry.

$\operatorname{pos}(M)$ cone generated by a set M

$\operatorname{conv}(M)$ convex hull of a set M

$\mathcal{T}(F)$ tropical hypersurface defined by tropical polynomial F

$\mathcal{D}(F)$ dome of the tropical polynomial F

$\mathrm{NC}(F)$ normal complex of the tropical polynomial F

$\operatorname{supp}(F)$ support of tropical polynomial F

$\operatorname{localsupp}_F(p)$ local support of point p with respect to tropical polynomial F

$\operatorname{ct}_F(p)$ coarse type of a point p with respect to F

$\operatorname{ft}_{F,G}(p)$ fine type of a point p with respect to F and G

$[p, q]$ ordinary line segment $\operatorname{conv}\{p, q\}$

Δ_{d-1}	ordinary $(d-1)$-simplex $\operatorname{conv}\{e_1, e_2, \ldots, e_d\}$
$\Delta(d,n)$	hypersimplex $\operatorname{conv}\{e_\sigma \mid \sigma \in \binom{[n]}{d}\}$
$U(A,\omega)$	unbounded polyhedron defined by the point configuration A and the height function ω
$\Sigma(A,\omega)$	regular subdivision of A induced by ω
$\mathcal{N}(f)$	Newton polytope of (tropical or ordinary) polynomial f
$\widetilde{\mathcal{N}}(f)$	extended Newton polyhedron of polynomial f
$\mathcal{S}(f)$	regular subdivision of $\mathcal{N}(f)$ dual to $\mathcal{T}(F)$
$\mathcal{F}(P)$	face poset of a not necessarily bounded polyhedron P
$\mathfrak{P}_d$	polytope algebra in $\mathbb{R}^d$
$\mathcal{C}(A,B)$	Cayley embedding of the point configurations A and B
$R(G)$	root polytope of the bipartite graph $G \subseteq K_{m,n}$
$\mathcal{S}(L)$	mixed subdivision dual to tropical hyperplane arrangement defined by the matrix L
$\mathcal{C}(L)$	Cayley embedding corresponding to tropical hyperplane arrangement defined by L
$C(L)$	tropical Cramer vector
$\mathrm{LP}(A,b,c)$	linear program: minimize $\langle c,x\rangle$ subject to $Ax+b \geq 0$
$\mathcal{P}(A,b)$	feasible set of an ordinary or tropical linear program
$\mathrm{face}_P(w)$	face of the polyhedron P maximizing the linear function w
$\mathrm{vol}(P)$	Euclidean volume of a polytope P

Graphs, Partially Ordered Sets, and Polytopes Derived.

$\operatorname{tdet} M$	tropical determinant of the matrix M
$\delta^{\mathrm{in}}(v)$	(multi)set of in-arcs of the node v in a directed graph
$\delta^{\mathrm{out}}(v)$	(multi)set of out-arcs of the node v in a directed graph
$B(A)$	bipartite graph recording the finite entries of the matrix A
$Q(W)$	weighted digraph polyhedron associated with the square matrix W
$\mathfrak{P}(\Gamma)$	partially ordered set arising from the digraph Γ
$\mathrm{Ord}(\mathfrak{P})$	order polytope of the poset $\mathfrak{P}$

Tropical Convexity.

$\mathrm{tpos}(M)$	tropical cone generated by M
$\mathrm{tconv}(M)$	tropical convex hull, in $\mathbb{R}^d/\mathbb{R}\mathbb{1}$ or $\mathbb{TP}^{d-1}$
$\mathrm{tconv}(V)$	tropical convex hull of the columns of $V \in \mathbb{T}^{d\times n}$

$\operatorname{tvol}(V)$	tropical volume of $\operatorname{tconv}(V)$, where $V \in \mathbb{T}^{d\times d}$
$[p,q]_{\text{trop}}$	tropical line segment $\operatorname{tconv}\{p,q\}$
$\mathcal{E}(V)$	envelope of the (columns of) the matrix V
S_k	closed sector of the tropical hyperplane with apex 0
S_k°	open sector of the tropical hyperplane with apex 0
$\operatorname{CovDec}(V)$	covector decomposition induced by (the columns of) V
$\operatorname{CovGraph}(C)$	covector graph of the cell C
$\operatorname{ext} K$	set of extreme points of the tropically convex set K
pr_i	projection $\mathbb{R}^d \to \mathbb{R}^{d-1}$ by omitting the ith coordinate
$\overline{\mathrm{S}_p}\sigma$	closed tropical half-space (in $\mathbb{R}^d/\mathbb{R}\mathbb{1}$) with apex p and closed sectors $\sigma \subset [d]$
$\mathbb{T}_\pm$	signed tropical numbers $\mathbb{T}_+ \cup \mathbb{T}_-$
$\ominus a$	negative tropical numbers, where $a \in \mathbb{T}$
$\operatorname{stdet} M$	signed tropical determinant of the matrix M
$\mathcal{H}(a', a'')$	signed tropical hyperplane defined by $a' \odot x = a'' \odot x$
$\operatorname{TLP}(A,b,c)$	tropical linear program, usually with respect to $\oplus = \max$

Games.

$\Gamma_\infty(A,B)$	mean payoff game for a pair of matrices
$\Gamma^\circ(A,B,s_0)$	finite game, starting at s_0; ends if a node is revisited
$\Gamma'(A,B,s_0;z)$	finite game, starting at s_0; restarting at z
$G(A,B)$	digraph on which the games above are played
$G_\beta(A,B)$	reduced digraph for (positional) strategy β

Matroids and Tropical Linear Spaces.

$U_{d,n}$	uniform matroid of rank d on n elements
$P(M)$	matroid polytope of the matroid M
$\mathcal{L}_\pi$	tropical linear space defined by tropical Plücker vector π
$\operatorname{Gr}(d,n)$	ordinary Grassmannian of d-planes in n-space
$\operatorname{TGr}(d,n)$	tropical Grassmannian
$\operatorname{Dr}(d,n)$	Dressian
M_x	initial matroid defined by weight vector w
$\widetilde{\mathcal{B}}(M)$	Bergman fan of the matroid M
$\mathcal{B}(M)$	Bergman complex of the matroid M

Bibliography

[AB07] Federico Ardila and Sara Billey, *Flag arrangements and triangulations of products of simplices*, Adv. Math. **214** (2007), no. 2, 495–524, DOI 10.1016/j.aim.2007.02.014. MR2349710 ↑124

[AB08] Peter Abramenko and Kenneth S. Brown, *Buildings*, Theory and applications, Graduate Texts in Mathematics, vol. 248, Springer, New York, 2008, DOI 10.1007/978-0-387-78835-7. MR2439729 ↑323, 324

[ABBN03] John W. Anthony, Richard A. Bideaux, Kenneth W. Bladh, and Monte C. Nichols, *Handbook of mineralogy*, vol. II, Mineral Data Publishing, 2003, reprinted version. 190

[ABG14] Marianne Akian, Ravindra Bapat, and Stéphane Gaubert, *Handbook of linear algebra*, 2nd ed., Discrete Mathematics and its Applications (Boca Raton), vol. 39, ch. 35, pp. 1–22, CRC Press, Boca Raton, FL, 2014. MR3013937 xiv

[ABGJ14] Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, and Michael Joswig, *Combinatorial simplex algorithms can solve mean payoff games*, SIAM J. Optim. **24** (2014), no. 4, 2096–2117, DOI 10.1137/140953800. MR3504692 ↑250, 279

[ABGJ15] Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, and Michael Joswig, *Tropicalizing the simplex algorithm*, SIAM J. Discrete Math. **29** (2015), no. 2, 751–795, DOI 10.1137/130936464. MR3336300 ↑213, 239, 250

[ABGJ18] Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, and Michael Joswig, *Log-barrier interior point methods are not strongly polynomial*, SIAM J. Appl. Algebra Geom. **2** (2018), no. 1, 140–178, DOI 10.1137/17M1142132. MR3772004 ↑248, 249, 250, 251, 359

[ABGJ21] Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, and Michael Joswig, *What tropical geometry tells us about the complexity of linear programming*, SIAM Rev. **63** (2021), no. 1, 123–164, DOI 10.1137/20M1380211. MR4209657 ↑250

[Abh90] Shreeram S. Abhyankar, *Algebraic geometry for scientists and engineers*, Mathematical Surveys and Monographs, vol. 35, American Mathematical Society, Providence, RI, 1990, DOI 10.1090/surv/035. MR1075991 ↑29

[ABS97] David Avis, David Bremner, and Raimund Seidel, *How good are convex hull algorithms?*, 11th ACM Symposium on Computational Geometry (Vancouver, BC, 1995), Comput. Geom. **7** (1997), no. 5-6, 265–301, DOI 10.1016/S0925-7721(96)00023-5. MR1447243 ↑340

[ACP15] Dan Abramovich, Lucia Caporaso, and Sam Payne, *The tropicalization of the moduli space of curves* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 4, 765–809, DOI 10.24033/asens.2258. MR3377065 ↑25

[AD09] Federico Ardila and Mike Develin, *Tropical hyperplane arrangements and oriented matroids*, Math. Z. **262** (2009), no. 4, 795–816, DOI 10.1007/s00209-008-0400-z. MR2511751 ↑124, 190, 250, 331

[AGG10] Xavier Allamigeon, Stéphane Gaubert, and Éric Goubault, *The tropical double description method*, STACS 2010: 27th International Symposium on Theoretical Aspects of Computer Science, LIPIcs. Leibniz Int. Proc. Inform., vol. 5, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2010, pp. 47–58. MR2853909 ↑212

[AGG12] Marianne Akian, Stéphane Gaubert, and Alexander Guterman, *Tropical polyhedra are equivalent to mean payoff games*, Internat. J. Algebra Comput. **22** (2012), no. 1, 1250001, 43, DOI 10.1142/S0218196711006674. MR2900854 ↑279

[AGG13] Xavier Allamigeon, Stéphane Gaubert, and Éric Goubault, *Computing the vertices of tropical polyhedra using directed hypergraphs*, Discrete Comput. Geom. **49** (2013), no. 2, 247–279, DOI 10.1007/s00454-012-9469-6. MR3017909 ↑213

[AGG14a] Assalé Adjé, Stéphane Gaubert, and Eric Goubault, *Computing the smallest fixed point of order-preserving nonexpansive mappings arising in positive stochastic games and static analysis of programs*, J. Math. Anal. Appl. **410** (2014), no. 1, 227–240, DOI 10.1016/j.jmaa.2013.07.076. MR3109834 ↑280

[AGG14b] Marianne Akian, Stéphane Gaubert, and Alexander Guterman, *Tropical Cramer determinants revisited*, Tropical and idempotent mathematics and applications, Contemp. Math., vol. 616, Amer. Math. Soc., Providence, RI, 2014, pp. 1–45, DOI 10.1090/conm/616/12324. MR3221324 ↑95, 124

[AGH+17] Benjamin Assarf, Ewgenij Gawrilow, Katrin Herr, Michael Joswig, Benjamin Lorenz, Andreas Paffenholz, and Thomas Rehn, *Computing convex hulls and counting integer points with* `polymake`, Math. Program. Comput. **9** (2017), no. 1, 1–38, DOI 10.1007/s12532-016-0104-z. MR3613012 ↑340, 347

[AGK11a] Xavier Allamigeon, Stéphane Gaubert, and Ricardo D. Katz, *The number of extreme points of tropical polyhedra*, J. Combin. Theory Ser. A **118** (2011), no. 1, 162–189, DOI 10.1016/j.jcta.2010.04.003. MR2737191 ↑250, 251, 369

[AGK11b] Xavier Allamigeon, Stéphane Gaubert, and Ricardo D. Katz, *Tropical polar cones, hypergraph transversals, and mean payoff games*, Linear Algebra Appl. **435** (2011), no. 7, 1549–1574, DOI 10.1016/j.laa.2011.02.004. MR2810655 ↑157

[AGNS11] Marianne Akian, Stéphane Gaubert, Viorel Niţică, and Ivan Singer, *Best approximation in max-plus semimodules*, Linear Algebra Appl. **435** (2011), no. 12, 3261–3296, DOI 10.1016/j.laa.2011.06.009. MR2831610 ↑156

[AGS18] Xavier Allamigeon, Stéphane Gaubert, and Mateusz Skomra, *Solving generic nonarchimedean semidefinite programs using stochastic game algorithms*, J. Symbolic Comput. **85** (2018), 25–54, DOI 10.1016/j.jsc.2017.07.002. MR3707850 ↑250

[AGS20] Xavier Allamigeon, Stéphane Gaubert, and Mateusz Skomra, *Tropical spectrahedra*, Discrete Comput. Geom. **63** (2020), no. 3, 507–548, DOI 10.1007/s00454-020-00176-1. MR4074332 ↑250

[AGT16] Fuensanta Aroca, Cristhian Garay, and Zeinab Toghani, *The fundamental theorem of tropical differential algebraic geometry*, Pacific J. Math. **283** (2016), no. 2, 257–270, DOI 10.2140/pjm.2016.283.257. MR3519102 ↑56

[AHLS21] Nima Arkani-Hamed, Thomas Lam, and Marcus Spradlin, *Positive Configuration Space*, Comm. Math. Phys. **384** (2021), no. 2, 909–954, DOI 10.1007/s00220-021-04041-x. MR4259378 ↑331

[AK06] Federico Ardila and Caroline J. Klivans, *The Bergman complex of a matroid and phylogenetic trees*, J. Combin. Theory Ser. B **96** (2006), no. 1, 38–49, DOI 10.1016/j.jctb.2005.06.004. MR2185977 ↑331

[AK17] Xavier Allamigeon and Ricardo D. Katz, *Tropicalization of facets of polytopes*, Linear Algebra Appl. **523** (2017), 79–101, DOI 10.1016/j.laa.2017.02.011. MR3624667 ↑213

[AKS04] Manindra Agrawal, Neeraj Kayal, and Nitin Saxena, *PRIMES is in P*, Ann. of Math. (2) **160** (2004), no. 2, 781–793, DOI 10.4007/annals.2004.160.781. MR2123939 ↑345

[AKS19] Manindra Agrawal, Neeraj Kayal, and Nitin Saxena, *Errata: PRIMES is in P [MR2123939]*, Ann. of Math. (2) **189** (2019), no. 1, 317–318, DOI 10.4007/annals.2019.189.1.6. MR3898710 ↑345

[Ale13] Daniele Alessandrini, *Logarithmic limit sets of real semi-algebraic sets*, Adv. Geom. **13** (2013), no. 1, 155–190, DOI 10.1515/advgeom-2012-0020. MR3011539 ↑57

[Ard04] Federico Ardila, *Subdominant matroid ultrametrics*, Ann. Comb. **8** (2004), no. 4, 379–389, DOI 10.1007/s00026-004-0227-1. MR2112691 ↑332, 371

[ARW05] Federico Ardila, Victor Reiner, and Lauren Williams, *Bergman complexes, Coxeter arrangements, and graph associahedra*, Sém. Lothar. Combin. **54A** (2005/07), Art. B54Aj, 25. MR2317682 ↑331

[AW21] Josh Alman and Virginia Vassilevska Williams, *A refined laser method and faster matrix multiplication*, Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA), 2021, pp. 522–539. 95

[AZ99] Nina Amenta and Günter M. Ziegler, *Deformed products and maximal shadows of polytopes*, Advances in discrete and computational geometry (South Hadley, MA, 1996), Contemp. Math., vol. 223, Amer. Math. Soc., Providence, RI, 1999, pp. 57–90, DOI 10.1090/conm/223/03132. MR1661377 ↑251

[AZ18] Martin Aigner and Günter M. Ziegler, *Proofs from The Book*, 6th ed.; see corrected reprint of the 1998 original [MR1723092]; including illustrations by Karl H. Hofmann, Springer, Berlin, 2018, DOI 10.1007/978-3-662-57265-8. MR3823190 ↑1, 23

[BA09] P. Butkovic and A. Aminu, *Introduction to max-linear programming*, IMA J. Manag. Math. **20** (2009), no. 3, 233–249, DOI 10.1093/imaman/dpn029. MR2511497 ↑250

[BB19] Matthew Baker and Nathan Bowler, *Matroids over partial hyperstructures*, Adv. Math. **343** (2019), 821–863, DOI 10.1016/j.aim.2018.12.004. MR3891757 ↑54, 57, 371

[BBRS20] Dominik Bendle, Janko Boehm, Yue Ren, and Benjamin Schröter, *Parallel computation of tropical varieties, their positive part, and tropical Grassmannians*, 2020, Preprint `arXiv:2003.13752`. 329

[BCOQ92] François Louis Baccelli, Guy Cohen, Geert Jan Olsder, and Jean-Pierre Quadrat, *Synchronization and linearity*, An algebra for discrete event systems, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Ltd., Chichester, 1992. MR1204266 ↑xiv, 94, 156

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy, *Real algebraic geometry*, translated from the 1987 French original, revised by the authors, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 36, Springer-Verlag, Berlin, 1998, DOI 10.1007/978-3-662-03718-8. MR1659509 ↑50, 57

[BD86] Hans-Jürgen Bandelt and Andreas Dress, *Reconstructing the shape of a tree from observed dissimilarity data*, Adv. in Appl. Math. **7** (1986), no. 3, 309–343, DOI 10.1016/0196-8858(86)90038-2. MR858908 ↑332

[BDM09] Rainer Burkard, Mauro Dell'Amico, and Silvano Martello, *Assignment problems*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009, DOI 10.1137/1.9780898717754. MR2488749 ↑94, 156

[Bel58] Richard Bellman, *On a routing problem*, Quart. Appl. Math. **16** (1958), 87–90, DOI 10.1090/qam/102435. MR102435 ↑94

[Ben14] Pascal Benchimol, *Tropical aspects of linear programming*, Theses, École Polytechnique, December 2014. 251

[Ber71] George M. Bergman, *The logarithmic limit-set of an algebraic variety*, Trans. Amer. Math. Soc. **157** (1971), 459–469, DOI 10.2307/1995858. MR280489 ↑56

[BG84] Robert Bieri and J. R. J. Groves, *The geometry of the set of characters induced by valuations*, J. Reine Angew. Math. **347** (1984), 168–195. MR733052 ↑56

[BGS17] Anna Lena Birkmeyer, Andreas Gathmann, and Kirsten Schmitz, *The realizability of curves in a tropical plane*, Discrete Comput. Geom. **57** (2017), no. 1, 12–55, DOI 10.1007/s00454-016-9816-0. MR3589056 ↑25

[BGW03] Alexandre V. Borovik, I. M. Gelfand, and Neil White, *Coxeter matroids*, Progress in Mathematics, vol. 216, Birkhäuser Boston, Inc., Boston, MA, 2003, DOI 10.1007/978-1-4612-2066-4. MR1989953 ↑330

[BH04] Walter Briec and Charles Horvath, *$\mathbb{B}$-convexity*, Optimization **53** (2004), no. 2, 103–127, DOI 10.1080/02331930410001695283. MR2058292 ↑156

[BH74] James R. Bunch and John E. Hopcroft, *Triangular factorization and inversion by fast matrix multiplication*, Math. Comp. **28** (1974), 231–236, DOI 10.2307/2005828. MR331751 ↑95

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999, DOI 10.1007/978-3-662-12494-9. MR1744486 ↑156

[BHV01] Louis J. Billera, Susan P. Holmes, and Karen Vogtmann, *Geometry of the space of phylogenetic trees*, Adv. in Appl. Math. **27** (2001), no. 4, 733–767, DOI 10.1006/aama.2001.0759. MR1867931 ↑332

[BJMS15] Sarah Brodsky, Michael Joswig, Ralph Morrison, and Bernd Sturmfels, *Moduli of tropical plane curves*, Res. Math. Sci. **2** (2015), Art. 4, 31, DOI 10.1186/s40687-014-0018-1. MR3333700 ↑25

[BK19] Elizabeth Baldwin and Paul Klemperer, *Understanding preferences: "demand types", and the existence of equilibrium with indivisibilities*, Econometrica **87** (2019), no. 3, 867–932, DOI 10.3982/ECTA13693. MR3957334 ↑122, 125

[BL21] Matthew Baker and Oliver Lorscheid, *Descartes' rule of signs, Newton polygons, and polynomials over hyperfields*, J. Algebra **569** (2021), 416–441, DOI 10.1016/j.jalgebra.2020.10.024. MR4187242 ↑57

[Bou03] Nicolas Bourbaki, *Algebra II, Chapters 4–7*, Elements of Mathematics (Berlin), translated from the 1981 French edition by P. M. Cohn and J. Howie, reprint of the 1990 English edition, Springer-Verlag, Berlin, 2003, MR1994218 28

[BPR06] Saugata Basu, Richard Pollack, and Marie-Françoise Roy, *Algorithms in real algebraic geometry*, 2nd ed., Algorithms and Computation in Mathematics, vol. 10, Springer-Verlag, Berlin, 2006. MR2248869 ↑56, 57

[BPR16] Matthew Baker, Sam Payne, and Joseph Rabinoff, *Nonarchimedean geometry, tropicalization, and metrics on curves*, Algebr. Geom. **3** (2016), no. 1, 63–105, DOI 10.14231/AG-2016-004. MR3455421 ↑57

[BR15] Matthias Beck and Sinai Robins, *Computing the continuous discretely,* Integer-point enumeration in polyhedra; with illustrations by David Austin, 2nd ed., Undergraduate Texts in Mathematics, Springer, New York, 2015, DOI 10.1007/978-1-4939-2969-6. MR3410115 ↑333

[Bre97] Glen E. Bredon, *Topology and geometry*, corrected third printing of the 1993 original, Graduate Texts in Mathematics, vol. 139, Springer-Verlag, New York, 1997. MR1700700 ↑124

[BS83] Walter Baur and Volker Strassen, *The complexity of partial derivatives*, Theoret. Comput. Sci. **22** (1983), no. 3, 317–330, DOI 10.1016/0304-3975(83)90110-X. MR693063 ↑95

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local* (French), Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251. MR327923 ↑323

[But95] P. Butkovič, *Regularity of matrices in min-algebra and its time-complexity*, Combinatorial optimization 1992 (CO92) (Oxford), Discrete Appl. Math. **57** (1995), no. 2-3, 121–132, DOI 10.1016/0166-218X(94)00099-Y. MR1327771 ↑95

[But10] Peter Butkovič, *Max-linear systems: theory and algorithms*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2010, DOI 10.1007/978-1-84996-299-5. MR2681232 ↑xiv, 70, 94, 213

[BY06] Florian Block and Josephine Yu, *Tropical convexity via cellular resolutions*, J. Algebraic Combin. **24** (2006), no. 1, 103–114, DOI 10.1007/s10801-006-9104-9. MR2245783 ↑190

[CDJ⁺20] Desmond Coles, Neelav Dutta, Sifan Jiang, Ralph Morrison, and Andrew Scharf, *Tropically planar graphs*, 2020, Preprint `arXiv:1908.04320`. 25

[CE20] Freddy Cachazo and Nick Early, *Minimal kinematics: An all k and n peek into* $\mathrm{Trop}^+ \mathrm{G}(k,n)$, 2020, Preprint `arXiv:2003.07958`. 331

[CEGM19] Freddy Cachazo, Nick Early, Alfredo Guevara, and Sebastian Mizera, *Scattering equations: from projective spaces to tropical Grassmannians*, J. High Energy Phys. **6** (2019), 039, 32, DOI 10.1007/jhep06(2019)039. MR3982543 ↑331

[CG79] Raymond Cuninghame-Green, *Minimax algebra*, Lecture Notes in Economics and Mathematical Systems, vol. 166, Springer-Verlag, Berlin-New York, 1979. MR580321 ↑24, 94, 156

[CGQ04] Guy Cohen, Stéphane Gaubert, and Jean-Pierre Quadrat, *Duality and separation theorems in idempotent semimodules*, Tenth Conference of the International Linear Algebra Society, Linear Algebra Appl. **379** (2004), 395–422, DOI 10.1016/j.laa.2003.08.010. MR2039751 ↑xiv, 156, 212

[CGQS05] Guy Cohen, Stéphane Gaubert, Jean-Pierre Quadrat, and Ivan Singer, *Max-plus convex sets and functions*, Idempotent mathematics and mathematical physics, Contemp. Math., vol. 377, Amer. Math. Soc., Providence, RI, 2005, pp. 105–129, DOI 10.1090/conm/377/06987. MR2149000 ↑156

[Cha96] T. M. Chan, *Optimal output-sensitive convex hull algorithms in two and three dimensions*, Eleventh Annual Symposium on Computational Geometry (Vancouver, BC, 1995), Discrete Comput. Geom. **16** (1996), no. 4, 361–368, DOI 10.1007/BF02712873. MR1414961 ↑212

[cJM⁺21] Türkü Özlüm Çelik, Asgar Jamneshan, Guido Montúfar, Bernd Sturmfels, and Lorenzo Venturello, *Wasserstein distance to independence models*, J. Symbolic Comput. **104** (2021), 855–873, DOI 10.1016/j.jsc.2020.10.005. MR4180150 ↑190

[CJS19] Francisco Criado, Michael Joswig, and Francisco Santos, *Tropical bisectors and Voronoi diagrams*, 2019, Preprint `arXiv:1906.10950`. 190

[CLO05] David A. Cox, John Little, and Donal O'Shea, *Using algebraic geometry*, 2nd ed., Graduate Texts in Mathematics, vol. 185, Springer, New York, 2005. MR2122859 ↑104

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011, DOI 10.1090/gsm/124. MR2810322 ↑333

[CLY20] Marcel Celaya, Georg Loho, and Chi Ho Yuen, *Oriented matroids from triangulations of products of simplices*, 2020, Preprint `arXiv:2005.01787`. 332

[CM03] Nathalie Caspard and Bernard Monjardet, *The lattices of closure systems, closure operators, and implicational systems on a finite set: a survey*, The 1998 Conference on Ordinal and Symbolic Data Analysis (OSDA '98) (Amherst, MA), Discrete Appl. Math. **127** (2003), no. 2, 241–269, DOI 10.1016/S0166-218X(02)00209-3. MR1984087 ↑308

[Coh89] P. M. Cohn, *Algebra. Vol. 2*, 2nd ed., John Wiley & Sons, Ltd., Chichester, 1989. MR1006872 ↑55

[Col75] George E. Collins, *Quantifier elimination for real closed fields by cylindrical algebraic decomposition*, Automata theory and formal languages (Second GI Conf., Kaiserslautern, 1975), Lecture Notes in Comput. Sci., Vol. 33, Springer, Berlin, 1975, pp. 134–183. MR0403962 ↑57

[Con92] Anne Condon, *The complexity of stochastic games*, Inform. and Comput. **96** (1992), no. 2, 203–224, DOI 10.1016/0890-5401(92)90048-K. MR1147987 ↑279

[Cos09] Arnaud Costinot, *An elementary theory of comparative advantage*, Econometrica **77** (2009), no. 4, 1165–1192, DOI 10.3982/ECTA7636. MR2547072 ↑274

[Cro19] Robert Alexander Crowell, *The tropical division problem and the minkowski factorization of generalized permutahedra*, 2019, Preprint `arXiv:1908.00241`. 124

[Dev05] Mike Develin, *The moduli space of n tropically collinear points in $\mathbb{R}^d$*, Collect. Math. **56** (2005), no. 1, 1–19. MR2131129 ↑331

[DGJ17] Jules Depersin, Stéphane Gaubert, and Michael Joswig, *A tropical isoperimetric inequality*, Sém. Lothar. Combin. – FPSAC 2017 – **78B** (2017), Art. 27, 12. MR3678609 143, 156, 187, 190

[DHK+12] Andreas Dress, Katharina T. Huber, Jacobus Koolen, Vincent Moulton, and Andreas Spillner, *Basic phylogenetic combinatorics*, Cambridge University Press, Cambridge, 2012. MR2893879 ↑332

[Dij59] E. W. Dijkstra, *A note on two problems in connexion with graphs*, Numer. Math. **1** (1959), 269–271, DOI 10.1007/BF01386390. MR107609 ↑94

[DJS12] Anton Dochtermann, Michael Joswig, and Raman Sanyal, *Tropical types and associated cellular resolutions*, J. Algebra **356** (2012), 304–324, DOI 10.1016/j.jalgebra.2011.12.028. MR2891135 ↑123, 178, 190

[DK04] Vladimir I. Danilov and Gleb A. Koshevoy, *Discrete convexity and unimodularity. I*, Adv. Math. **189** (2004), no. 2, 301–324, DOI 10.1016/j.aim.2003.11.010. MR2101223 ↑122, 125

[DKM01] Vladimir Danilov, Gleb Koshevoy, and Kazuo Murota, *Discrete convexity and equilibria in economies with indivisible goods and money*, Math. Social Sci. **41** (2001), no. 3, 251–273, DOI 10.1016/S0165-4896(00)00071-8. MR1818477 ↑125

[dlP13] María Jesús de la Puente, *On tropical Kleene star matrices and alcoved polytopes*, Kybernetika (Prague) **49** (2013), no. 6, 897–910. MR3182647 ↑190

[dlP20] María Jesús de la Puente, *Quasi-Euclidean classification of alcoved convex polyhedra*, Linear Multilinear Algebra **68** (2020), no. 10, 2110–2142, DOI 10.1080/03081087.2019.1572065. MR4160431 ↑190

[DLRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos, *Triangulations*, Structures for algorithms and applications, Algorithms and Computation in Mathematics, vol. 25, Springer-Verlag, Berlin, 2010, DOI 10.1007/978-3-642-12971-1. MR2743368 ↑24, 25, 57, 104, 116, 117, 123, 124, 163, 333, 337, 339, 367

[Dre84] Andreas W. M. Dress, *Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces*, Adv. in Math. **53** (1984), no. 3, 321–402, DOI 10.1016/0001-8708(84)90029-X. MR753872 ↑320, 332

[DS04] Mike Develin and Bernd Sturmfels, *Tropical convexity*, Doc. Math. **9** (2004), 1–27 (electronic), correction: ibid., pp. 205–206. MR2054977 94, 123, 156, 157, 178, 189, 190, 212, 369

[DSS05] Mike Develin, Francisco Santos, and Bernd Sturmfels, *On the rank of a tropical matrix*, Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., vol. 52, Cambridge Univ. Press, Cambridge, 2005, pp. 213–242. MR2178322 ↑190

[DTZ09] Antoine Deza, Tamás Terlaky, and Yuriy Zinchenko, *A continuous d-step conjecture for polytopes*, Discrete Comput. Geom. **41** (2009), no. 2, 318–327, DOI 10.1007/s00454-008-9096-4. MR2471877 ↑251

[DW92] Andreas W. M. Dress and Walter Wenzel, *Valuated matroids*, Adv. Math. **93** (1992), no. 2, 214–250, DOI 10.1016/0001-8708(92)90028-J. MR1164708 ↑331

[DY07] Mike Develin and Josephine Yu, *Tropical polytopes and cellular resolutions*, Experiment. Math. **16** (2007), no. 3, 277–291. MR2367318 ↑156, 190, 213

[Edm65] Jack Edmonds, *Minimum partition of a matroid into independent subsets*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 67–72. MR190025 ↑345

[Edm70] Jack Edmonds, *Submodular functions, matroids, and certain polyhedra*, Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969), Gordon and Breach, New York, 1970, pp. 69–87. MR0270945 ↑330

[EKL06] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind, *Non-Archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157, DOI 10.1515/CRELLE.2006.097. MR2289207 ↑xiii, 56

[EM79] A. Ehrenfeucht and J. Mycielski, *Positional strategies for mean payoff games*, Internat. J. Game Theory **8** (1979), no. 2, 109–113, DOI 10.1007/BF01768705. MR550954 ↑278, 279, 370

[ES52] Samuel Eilenberg and Norman Steenrod, *Foundations of algebraic topology*, Princeton University Press, Princeton, New Jersey, 1952. MR0050886 ↑124

[Ewa96] Günter Ewald, *Combinatorial convexity and algebraic geometry*, Graduate Texts in Mathematics, vol. 168, Springer-Verlag, New York, 1996, DOI 10.1007/978-1-4612-4044-0. MR1418400 ↑333

[FK11] Stefan Felsner and Kolja Knauer, *Distributive lattices, polyhedra, and generalized flows*, European J. Combin. **32** (2011), no. 1, 45–59, DOI 10.1016/j.ejc.2010.07.011. MR2727459 ↑95, 156

[FLM67] B. A. Farbey, A. H. Land, and J. D. Murchland, *The cascade algorithm for finding all shortest distances in a directed graph*, Management Sci. **14** (1967), 19–28, DOI 10.1287/mnsc.14.1.19. MR241185 ↑94

[Flo62] Robert W. Floyd, *Algorithm* 97 *shortest path*, Comm. Assoc. Comput. Mach. **5** (1962), 345. 70, 94

[FM16] Alex Fink and Luca Moci, *Matroids over a ring*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 4, 681–731, DOI 10.4171/JEMS/600. MR3474454 ↑331

[FO20] Alex Fink and Jorge Alberto Olarte, *Presentations of transversal valuated matroids*, 2020, Preprint `arXiv:1903.08288`. 332

[For56] Lester R. Ford, Jr., *Network flow theory*, Tech. report, The RAND Corporation, Paper P-923, 1956. 94

[FR15] Alex Fink and Felipe Rincón, *Stiefel tropical linear spaces*, J. Combin. Theory Ser. A **135** (2015), 291–331, DOI 10.1016/j.jcta.2015.06.001. MR3366480 ↑189, 190, 296, 332

[FS05] Eva Maria Feichtner and Bernd Sturmfels, *Matroid polytopes, nested sets and Bergman fans*, Port. Math. (N.S.) **62** (2005), no. 4, 437–468. MR2191630 ↑287, 315, 330, 331, 371

[Fuj84] Satoru Fujishige, *A characterization of faces of the base polyhedron associated with a submodular system* (English, with Japanese summary), J. Oper. Res. Soc. Japan **27** (1984), no. 2, 112–129, DOI 10.15807/jorsj.27.112. MR755473 ↑330

[Gal58] Tibor Gallai, *Maximum-minimum Sätze über Graphen* (German), Acta Math. Acad. Sci. Hungar. **9** (1958), 395–434, DOI 10.1007/BF02020271. MR124238 ↑94, 190

[Gan87] Bernhard Ganter, *Algorithmen zur formalen Begriffsanalyse* (German), Beiträge zur Begriffsanalyse (Darmstadt, 1986), Bibliographisches Inst., Mannheim, 1987, pp. 241–254. MR949457 ↑308

[Gau92] Stéphane Gaubert, *Theory of linear systems over dioids*, Ph.D. thesis, Ecole Nationale Supérieure des Mines de Paris, 1992. 56, 94

[GG98] Stéphane Gaubert and Jeremy Gunawardena, *The duality theorem for min-max functions* (English, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 1, 43–48, DOI 10.1016/S0764-4442(97)82710-3. MR1649473 ↑279

[GGMS87] I. M. Gel′fand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova, *Combinatorial geometries, convex polyhedra, and Schubert cells*, Adv. in Math. **63** (1987), no. 3, 301–316, DOI 10.1016/0001-8708(87)90059-4. MR877789 ↑330

[GGP97] Israel M. Gelfand, Mark I. Graev, and Alexander Postnikov, *Combinatorics of hypergeometric functions associated with positive roots*, The Arnold-Gelfand mathematical seminars, Birkhäuser Boston, Boston, MA, 1997, pp. 205–221, DOI 10.1007/978-1-4612-4122-5_10. MR1429893 ↑190

[Ghy17] Étienne Ghys, *A singular mathematical promenade*, ENS Éditions, Lyon, 2017. MR3702027 ↑57

[GJ79] Michael R. Garey and David S. Johnson, *Computers and intractability*, A guide to the theory of NP-completeness, A Series of Books in the Mathematical Sciences, W. H. Freeman and Co., San Francisco, Calif., 1979. MR519066 ↑122, 344

[GJ00] Ewgenij Gawrilow and Michael Joswig, *polymake: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73. MR1785292 ↑xv, 347

[GJ06] Bernd Gärtner and Martin Jaggi, *Tropical support vector machines*, ACS Technical Report No.: ACS-TR-362502-01, 2006. 332

[GK11] Stéphane Gaubert and Ricardo D. Katz, *Minimal half-spaces and external representation of tropical polyhedra*, J. Algebraic Combin. **33** (2011), no. 3, 325–348, DOI 10.1007/s10801-010-0246-4. MR2772536 ↑157

[GKK88] V. A. Gurvich, A. V. Karzanov, and L. G. Khachiyan, *Cyclic games and finding minimax mean cycles in digraphs* (Russian), Zh. Vychisl. Mat. i Mat. Fiz. **28** (1988), no. 9, 1407–1417, 1439, DOI 10.1016/0041-5553(88)90012-2; English transl., U.S.S.R. Comput. Math. and Math. Phys. **28** (1988), no. 5, 85–91 (1990). MR967535 ↑279

[GKM09] Andreas Gathmann, Michael Kerber, and Hannah Markwig, *Tropical fans and the moduli spaces of tropical curves*, Compos. Math. **145** (2009), no. 1, 173–195, DOI 10.1112/S0010437X08003837. MR2480499 ↑25

[GKS12] Stéphane Gaubert, Ricardo D. Katz, and Sergeĭ Sergeev, *Tropical linear-fractional programming and parametric mean payoff games*, J. Symbolic Comput. **47** (2012), no. 12, 1447–1478, DOI 10.1016/j.jsc.2011.12.049. MR2929038 ↑250

[GKZ08] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, reprint of the 1994 edition, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008. MR2394437 ↑xiv, 24

[GL01] S. Gao and A. G. B. Lauder, *Decomposition of polytopes and polynomials*, Discrete Comput. Geom. **26** (2001), no. 1, 89–104, DOI 10.1007/s00454-001-0024-0. MR1832732 ↑124

[GLS93] Martin Grötschel, László Lovász, and Alexander Schrijver, *Geometric algorithms and combinatorial optimization*, 2nd ed., Algorithms and Combinatorics, vol. 2, Springer-Verlag, Berlin, 1993, DOI 10.1007/978-3-642-78240-4. MR1261419 ↑212, 218, 246, 253, 279, 341, 345, 346

[GM07] Andreas Gathmann and Hannah Markwig, *The Caporaso-Harris formula and plane relative Gromov-Witten invariants in tropical geometry*, Math. Ann. **338** (2007), no. 4, 845–868, DOI 10.1007/s00208-007-0092-4. MR2317753 ↑21, 25

[GM08] Michel Gondran and Michel Minoux, *Graphs, dioids and semirings*, New models and algorithms, Operations Research/Computer Science Interfaces Series, vol. 41, Springer, New York, 2008. MR2389137 ↑94

[GM10] Stéphane Gaubert and Frédéric Meunier, *Carathéodory, Helly and the others in the max-plus world*, Discrete Comput. Geom. **43** (2010), no. 3, 648–662, DOI 10.1007/s00454-009-9207-x. MR2587842 ↑157

[GM19] Stéphane Gaubert and Marie MacCaig, *Approximating the volume of tropical polytopes is difficult*, Internat. J. Algebra Comput. **29** (2019), no. 2, 357–389, DOI 10.1142/S0218196719500061. MR3934791 ↑156

[GO16] Bernhard Ganter and Sergei Obiedkov, *Conceptual exploration*, Springer-Verlag, Berlin, 2016, DOI 10.1007/978-3-662-49291-8. MR3495406 ↑331

[GP15] Dima Grigoriev and Vladimir V. Podolskii, *Complexity of tropical and min-plus linear prevarieties*, Comput. Complexity **24** (2015), no. 1, 31–64, DOI 10.1007/s00037-013-0077-5. MR3320301 ↑279

[GP18] Dima Grigoriev and Vladimir V. Podolskii, *Tropical effective primary and dual Nullstellensätze*, Discrete Comput. Geom. **59** (2018), no. 3, 507–552, DOI 10.1007/s00454-018-9966-3. MR3770201 ↑368

[Gri13] Dima Grigoriev, *Complexity of solving tropical linear systems*, Comput. Complexity **22** (2013), no. 1, 71–88, DOI 10.1007/s00037-012-0053-5. MR3034020 ↑279

[Grü03] Branko Grünbaum, *Convex polytopes*, 2nd ed., prepared and with a preface by Volker Kaibel, Victor Klee, and Günter M. Ziegler, Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003, DOI 10.1007/978-1-4613-0019-9. MR1976856 ↑xiii, 156, 212, 333, 334

[GS16] Alexander Guterman and Yaroslav Shitov, *Rank functions of tropical matrices*, Linear Algebra Appl. **498** (2016), 326–348, DOI 10.1016/j.laa.2015.07.004. MR3478565 ↑190

[Gub07] Walter Gubler, *Tropical varieties for non-Archimedean analytic spaces*, Invent. Math. **169** (2007), no. 2, 321–376, DOI 10.1007/s00222-007-0048-z. MR2318559 ↑57

[Gub13] Walter Gubler, *A guide to tropicalizations*, Algebraic and combinatorial aspects of tropical geometry, Contemp. Math., vol. 589, Amer. Math. Soc., Providence, RI, 2013, pp. 125–189, DOI 10.1090/conm/589/11745. MR3088913 ↑57

[GV21] Dima Grigoriev and Nicolai Vorobjov, *Complexity of deciding whether a tropical linear prevariety is a tropical variety*, Appl. Algebra Engrg. Comm. Comput. **32** (2021), no. 2, 157–174, DOI 10.1007/s00200-019-00407-w. MR4216369 ↑57

[GZK90] I. M. Gel′fand, A. V. Zelevinskiĭ, and M. M. Kapranov, *Discriminants of polynomials in several variables and triangulations of Newton polyhedra* (Russian), Algebra i Analiz **2** (1990), no. 3, 1–62; English transl., Leningrad Math. J. **2** (1991), no. 3, 449–505. MR1073208 ↑25

[Har10] John Harris (ed.), *Lexicon technicum: or, a universal English dictionary of arts and sciences: explaining not only the terms of art, but the arts themselves*, vol. 2, London, 1710. 55

[HB] Daniel H. Huson and David Bryant, *SplitsTree4*, `https://github.com/husonlab/splitstree4`, retrieved 03 March 2021. 321

[HB06] Daniel H. Huson and David Bryant, *Application of phylogenetic networks in evolutionary studies*, Mol. Biol. Evol. **23** (2006), no. 2, 254–267. 321

[Hel88] Siegfried Helbig, *On Carathéodory's and Kreĭn-Milman's theorems in fully ordered groups*, Comment. Math. Univ. Carolin. **29** (1988), no. 1, 157–167. MR937558 ↑157

[Hir06] Hiroshi Hirai, *A geometric study of the split decomposition*, Discrete Comput. Geom. **36** (2006), no. 2, 331–361, DOI 10.1007/s00454-006-1243-1. MR2252108 ↑95, 332

[HJ08] Sven Herrmann and Michael Joswig, *Splitting polytopes*, Münster J. Math. **1** (2008), 109–141. MR2502496 ↑95, 332

[HJ10] Sven Herrmann and Michael Joswig, *Totally splittable polytopes*, Discrete Comput. Geom. **44** (2010), no. 1, 149–166, DOI 10.1007/s00454-009-9217-8. MR2639822 ↑124

[HJ17] Simon Hampe and Michael Joswig, *Tropical computations in `polymake`*, Algorithmic and experimental methods in algebra, geometry, and number theory (Gebhard Böckle, Wolfram Decker, and Gunter Malle, eds.), Springer, Cham, 2017, pp. 361–385. MR3792732 xv, 347

[HJJS09] Sven Herrmann, Anders Jensen, Michael Joswig, and Bernd Sturmfels, *How to draw tropical planes*, Special volume in honor of Anders Björner, Electron. J. Combin. **16** (2009), no. 2, Research Paper 6, 26. MR2515769 ↑310, 329

[HJP13] Sven Herrmann, Michael Joswig, and Marc E. Pfetsch, *Computing the bounded subcomplex of an unbounded polyhedron*, Comput. Geom. **46** (2013), no. 5, 541–551, DOI 10.1016/j.comgeo.2011.11.002. MR3015940 ↑331

[HJS14] Sven Herrmann, Michael Joswig, and David E. Speyer, *Dressians, tropical Grassmannians, and their rays*, Forum Math. **26** (2014), no. 6, 1853–1881, DOI 10.1515/forum-2012-0030. MR3334049 ↑318, 329, 332

[HJS19] Simon Hampe, Michael Joswig, and Benjamin Schröter, *Algorithms for tight spans and tropical linear spaces*, J. Symbolic Comput. **91** (2019), 116–128, DOI 10.1016/j.jsc.2018.06.016. MR3860887 ↑308, 329, 331, 371

[HK12] Hiroshi Hirai and Shungo Koichi, *On tight spans for directed distances*, Ann. Comb. **16** (2012), no. 3, 543–569, DOI 10.1007/s00026-012-0146-5. MR2960019 ↑332

[HLS19] Cvetelina Hill, Sara Lamboglia, and Faye Pasley Simon, *Tropical convex hulls of infinite sets*, 2019, Preprint `arXiv:1912.01253`. 188

[HM12] Sven Herrmann and Vincent Moulton, *Trees, tight-spans and point configurations*, Discrete Math. **312** (2012), no. 16, 2506–2521, DOI 10.1016/j.disc.2012.05.003. MR2927167 ↑332

[Hor12] Silke Horn, *A topological representation theorem for tropical oriented matroids* (English, with English and French summaries), 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012), Discrete Math. Theor. Comput. Sci. Proc., AR, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2012, pp. 135–146. MR2957992 ↑124, 250, 331

[HRS00] Birkett Huber, Jörg Rambau, and Francisco Santos, *The Cayley trick, lifting subdivisions and the Bohne-Dress theorem on zonotopal tilings*, J. Eur. Math. Soc. (JEMS) **2** (2000), no. 2, 179–198, DOI 10.1007/s100970050003. MR1763304 ↑123

[HS07] Serkan Hoşten and Bernd Sturmfels, *Computing the integer programming gap*, Combinatorica **27** (2007), no. 3, 367–382, DOI 10.1007/s00493-007-2057-3. MR2345814 ↑56

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990, DOI 10.1017/CBO9780511623646. MR1066460 ↑124

[IM65] Nagayoshi Iwahori and Hideya Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of* $\mathfrak{p}$*-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 5–48. MR185016 ↑323

[IMS09] Ilia Itenberg, Grigory Mikhalkin, and Eugenii Shustin, *Tropical algebraic geometry*, 2nd ed., Oberwolfach Seminars, vol. 35, Birkhäuser Verlag, Basel, 2009, DOI 10.1007/978-3-0346-0048-4. MR2508011 ↑xiv, 21, 25, 57

[Inc20] OEIS Foundation Inc., *The on-line encyclopedia of integer sequences*, `http://oeis.org/A001147`, 2020. 304

[IR09] Zur Izhakian and Louis Rowen, *The tropical rank of a tropical matrix*, Comm. Algebra **37** (2009), no. 11, 3912–3927, DOI 10.1080/00927870902828793. MR2573227 ↑190

[IS20] Hiroshi Iriyeh and Masataka Shibata, *Symmetric Mahler's conjecture for the volume product in the* 3*-dimensional case*, Duke Math. J. **169** (2020), no. 6, 1077–1134, DOI 10.1215/00127094-2019-0072. MR4085078 ↑251

[Isb64] John R. Isbell, *Six theorems about injective metric spaces*, Comment. Math. Helv. **39** (1964), 65–76, DOI 10.1007/BF02566944. MR182949 ↑332

[IV96] Ilia Itenberg and Oleg Viro, *Patchworking algebraic curves disproves the Ragsdale conjecture*, Math. Intelligencer **18** (1996), no. 4, 19–28, DOI 10.1007/BF03026748. MR1413249 ↑57

[Jac89] Nathan Jacobson, *Basic algebra. II*, 2nd ed., W. H. Freeman and Company, New York, 1989. MR1009787 ↑41, 55

[JdlP12] Andrea Jiménez and María Jesús de la Puente, *Six combinatorial classes of maximal convex tropical polyhedra*, 2012, Preprint `arXiv:1205.4162`. 187

[Jen10] Anders Nedergaard Jensen, *Traversing symmetric polyhedral fans*, Mathematical software—ICMS 2010, Lecture Notes in Comput. Sci., vol. 6327, Springer, Berlin, 2010, pp. 282–294. MR3663204 ↑340

[Jen17] Anders Nedergaard Jensen, *Gfan, a software system for Gröbner fans and tropical varieties, version 0.6.2*, Available at `http://home.imf.au.dk/jensen/software/gfan/gfan.html`, 2017. 369

[Jer73] Robert G. Jeroslow, *Asymptotic linear programming*, Operations Research **21** (1973), no. 5, 1128–1141. 56, 251

[JJK18] Charles Jordan, Michael Joswig, and Lars Kastner, *Parallel enumeration of triangulations*, Electron. J. Combin. **25** (2018), no. 3, Paper No. 3.6, 27, DOI 10.37236/7318. MR3829292 ↑114, 340, 369

[JJK20] Charles Jordan, Michael Joswig, and Lars Kastner, *mptopcom, version 1.2*, 2020, Open source software for the parallel enumeration of triangulations, `http://www.polymake.org/mptopcom`. 369

[JK10] Michael Joswig and Katja Kulas, *Tropical and ordinary convexity combined*, Adv. Geom. **10** (2010), no. 2, 333–352, DOI 10.1515/ADVGEOM.2010.012. MR2629819 ↑187

[JKM08] Colin N. Jones, Eric C. Kerrigan, and Jan M. Maciejowski, *On polyhedral projection and parametric programming*, J. Optim. Theory Appl. **138** (2008), no. 2, 207–220, DOI 10.1007/s10957-008-9384-4. MR2414994 ↑244

[JL16] Michael Joswig and Georg Loho, *Weighted digraphs and tropical cones*, Linear Algebra Appl. **501** (2016), 304–343, DOI 10.1016/j.laa.2016.02.027. MR3485070 ↑94, 95, 189, 190, 213

[JL20] Michael Joswig and Georg Loho, *Monomial tropical cones for multicriteria optimization*, SIAM J. Discrete Math. **34** (2020), no. 2, 1172–1191. MR4101364 155, 157, 251

[JLLS16] Michael Joswig, Georg Loho, Benjamin Lorenz, and Benjamin Schröter, *Linear programs and convex hulls over fields of Puiseux fractions*, Mathematical aspects of computer and information sciences, Lecture Notes in Comput. Sci., vol. 9582, Springer, [Cham], 2016, pp. 429–445, DOI 10.1007/978-3-319-32859-1_37. MR3517052 ↑251, 340

[Jos05] Michael Joswig, *Tropical halfspaces*, Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., vol. 52, Cambridge Univ. Press, Cambridge, 2005, pp. 409–431. MR2178330 ↑212

[Jos17] Michael Joswig, *The Cayley trick for tropical hypersurfaces with a view toward Ricardian economics*, Homological and computational methods in commutative algebra, Springer INdAM Ser., vol. 20, Springer, Cham, 2017, pp. 107–128. MR3751882 ↑123, 156, 190, 280

[JS17] Michael Joswig and Benjamin Schröter, *Matroids from hypersimplex splits*, J. Combin. Theory Ser. A **151** (2017), 254–284, DOI 10.1016/j.jcta.2017.05.001. MR3663497 ↑329

[JS20] Michael Joswig and Benjamin Schröter, *Parametric shortest-path algorithms via tropical geometry*, 2020, Preprint `arXiv:1904.01082v3`. 95

[JSY07] Michael Joswig, Bernd Sturmfels, and Josephine Yu, *Affine buildings and tropical convexity*, Albanian J. Math. **1** (2007), no. 4, 187–211. MR2367213 ↑156, 326, 332

[JSY18] Philipp Jell, Claus Scheiderer, and Josephine Yu, *Real tropicalization and analytification of semialgebraic sets*, 2018, Preprint `arXiv:1810.05132`. 57

[JT13] Michael Joswig and Thorsten Theobald, *Polyhedral and algebraic methods in computational geometry*, revised and updated translation of the 2008 German original, Universitext, Springer, London, 2013, DOI 10.1007/978-1-4471-4817-3. MR2905853 ↑24, 56, 218, 309, 333, 336, 341, 369

[JT21] Michael Joswig and Ayush Kumar Tewari, *Forbidden patterns in tropical plane curves*, Beitr. Algebra Geom. **62** (2021), no. 1, 65–81, DOI 10.1007/s13366-020-00523-6. MR4249854 ↑25

[Kap93] M. M. Kapranov, *Chow quotients of Grassmannians. I*, I. M. Gel′fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 29–110. MR1237834 ↑124, 331, 332

[Kar78] Richard M. Karp, *A characterization of the minimum cycle mean in a digraph*, Discrete Math. **23** (1978), no. 3, 309–311, DOI 10.1016/0012-365X(78)90011-0. MR523080 ↑84, 94

[Kar84] Narendra K. Karmarkar, *A new polynomial-time algorithm for linear programming*, Combinatorica **4** (1984), no. 4, 373–395, DOI 10.1007/BF02579150. MR779900 ↑246

[Kha79] Leonid G. Khachiyan, *A polynomial algorithm in linear programming*, Dokl. Akad. Nauk SSSR **244** (1979), no. 5, 1093–1096. MR522052 246, 346

[KK92] Gil Kalai and Daniel J. Kleitman, *A quasi-polynomial bound for the diameter of graphs of polyhedra*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), no. 2, 315–316, DOI 10.1090/S0273-0979-1992-00285-9. MR1130448 ↑247

[KL93] Alexander V. Karzanov and Vasilij N. Lebedev, *Cyclical games with prohibitions*, Math. Programming **60** (1993), no. 3, Ser. A, 277–293, DOI 10.1007/BF01580616. MR1234877 ↑279

[KM40] M. Krein and D. Milman, *On extreme points of regular convex sets* (English, with Ukrainian summary), Studia Math. **9** (1940), 133–138, DOI 10.4064/sm-9-1-133-138. MR4990 ↑157

[KM72] Victor Klee and George J. Minty, *How good is the simplex algorithm?*, Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin), Academic Press, New York, 1972, pp. 159–175. MR0332165 ↑245, 247

[Kna06] Anthony W. Knapp, *Basic algebra*, along with a companion volume *Advanced algebra*, Cornerstones, Birkhäuser Boston, Inc., Boston, MA, 2006, DOI 10.1007/978-0-8176-4529-8. MR2257570 ↑34

[Kon92] Maxim Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23. MR1171758 ↑25

[Koz08] Dmitry Kozlov, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics, vol. 21, Springer, Berlin, 2008, DOI 10.1007/978-3-540-71962-5. MR2361455 ↑124

[KP02] Volker Kaibel and Marc E. Pfetsch, *Computing the face lattice of a polytope from its vertex-facet incidences*, Comput. Geom. **23** (2002), no. 3, 281–290, DOI 10.1016/S0925-7721(02)00103-7. MR1927137 ↑331

[KP03] Kaibel Volker and Marc E. Pfetsch, *Some algorithmic problems in polytope theory*, Algebra, geometry, and software systems, Springer, Berlin, 2003, pp. 23–47. MR2011752 340

[KR06] Ki H. Kim and Fred W. Roush, *Kapranov rank vs. tropical rank*, Proc. Amer. Math. Soc. **134** (2006), no. 9, 2487–2494, DOI 10.1090/S0002-9939-06-08426-7. MR2213725 ↑190

[Kra57] Marc Krasner, *Approximation des corps valués complets de caractéristique $p \neq 0$ par ceux de caractéristique* 0 (French), Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, Centre Belge de Recherches Mathématiques, Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris, 1957, pp. 129–206. MR0106218 ↑57

[KS10] Edward D. Kim and Francisco Santos, *An update on the Hirsch conjecture*, Jahresber. Dtsch. Math.-Ver. **112** (2010), no. 2, 73–98, DOI 10.1365/s13291-010-0001-8. MR2681516 ↑251

[KT06] Sean Keel and Jenia Tevelev, *Geometry of Chow quotients of Grassmannians*, Duke Math. J. **134** (2006), no. 2, 259–311, DOI 10.1215/S0012-7094-06-13422-1. MR2248832 ↑324, 326

[KY94] J. J. Kosowsky and Alan L. Yuille, *The invisible hand algorithm: Solving the assignment problem with statistical physics*, Neural Networks **7** (1994), no. 3, 477–490. 156

[LMS01] Grigory L. Litvinov, Viktor P. Maslov, and Grigory B. Shpiz, *Idempotent functional analysis. An algebraic approach* (Russian, with Russian summary), Mat. Zametki **69** (2001), no. 5, 758–797, DOI 10.1023/A:1010266012029; English transl., Math. Notes **69** (2001), no. 5-6, 696–729. MR1846814 ↑xiv, 94, 156

[Loh17] Georg Loho, *Combinatorics of tropical linear programming*, Ph.D. thesis, TU Berlin, 2017. 124

[Loh20] Georg Loho, *Abstract tropical linear programming*, Electron. J. Combin. **27** (2020), no. 2, Paper No. 2.51, 68, DOI 10.37236/7718. MR4245106 ↑250

[LP98] Harry R. Lewis and Christos H. Papadimitriou, *Elements of the theory of computation*, 2nd ed., Prentice Hall, 1998. 247

[LP07] Thomas Lam and Alexander Postnikov, *Alcoved polytopes. I*, Discrete Comput. Geom. **38** (2007), no. 3, 453–478, DOI 10.1007/s00454-006-1294-3. MR2352704 ↑95, 190

[LS20a] Georg Loho and Matthias Schymura, *Tropical Ehrhart theory and tropical volume*, Res. Math. Sci. **7** (2020), no. 4, Paper No. 30, 34, DOI 10.1007/s40687-020-00228-1. MR4155409 ↑156

[LS20b] Georg Loho and Ben Smith, *Matching fields and lattice points of simplices*, Adv. Math. **370** (2020), 107232, 42, DOI 10.1016/j.aim.2020.107232. MR4104819 ↑124

[LSTY17] Bo Lin, Bernd Sturmfels, Xiaoxian Tang, and Ruriko Yoshida, *Convexity in tree spaces*, SIAM J. Discrete Math. **31** (2017), no. 3, 2015–2038, DOI 10.1137/16M1079841. MR3693600 ↑332

[LT19] Bo Lin and Ngoc Mai Tran, *Two-player incentive compatible outcome functions are affine maximizers*, Linear Algebra Appl. **578** (2019), 133–152, DOI 10.1016/j.laa.2019.04.027. MR3950503 ↑125

[Lun52] A. G. Lunc, *Algebraic methods of analysis and synthesis of relay contact networks* (Russian), Izvestiya Akad. Nauk SSSR. Ser. Mat. **16** (1952), 405–436. MR0052324 ↑24

[LV19] Georg Loho and László A. Végh, *Signed tropical convexity*, 2019, Preprint `arXiv:1906.06686`. 213

[LY18] Bo Lin and Ruriko Yoshida, *Tropical Fermat-Weber points*, SIAM J. Discrete Math. **32** (2018), no. 2, 1229–1245, DOI 10.1137/16M1071122. MR3810501 ↑332

[Mah38] Kurt Mahler, *Ein Minimalproblem für konvexe Polygone* (German), Mathematica B, Zutphen **7** (1938), 118–127. 251

[Mah39] Kurt Mahler, *Ein Übertragungsprinzip für konvexe Körper* (German), Časopis Pěst. Mat. Fys. **68** (1939), 93–102. MR0001242 ↑251

[Mar10] T. Markwig, *A field of generalised Puiseux series for tropical geometry*, Rend. Semin. Mat. Univ. Politec. Torino **68** (2010), no. 1, 79–92. MR2759691 ↑43, 44, 56

[Mas86] V. P. Maslov, *On a new superposition principle for optimization problem*, Séminaire sur les équations aux dérivées partielles, 1985–1986, École Polytech., Palaiseau, 1986, pp. Exp. No. XXIV, 14. MR874583 ↑49

[MC13] Bassel Mannaa and Thierry Coquand, *Dynamic Newton-Puiseux theorem*, J. Log. Anal. **5** (2013), Paper 5, 22, DOI 10.4115/jla.2013.5.5. MR3066398 ↑57

[McM70] P. McMullen, *The maximum numbers of faces of a convex polytope*, Mathematika **17** (1970), 179–184, DOI 10.1112/S0025579300002850. MR283691 ↑223

[McM89] Peter McMullen, *The polytope algebra*, Adv. Math. **78** (1989), no. 1, 76–130, DOI 10.1016/0001-8708(89)90029-7. MR1021549 ↑56

[Meg89] Nimrod Megiddo, *On the complexity of linear programming*, Advances in economic theory (Cambridge, MA, 1985), Econom. Soc. Monogr., vol. 12, Cambridge Univ. Press, Cambridge, 1989, pp. 225–268. MR1117042 ↑218

[Mik05] Grigory Mikhalkin, *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005), no. 2, 313–377, DOI 10.1090/S0894-0347-05-00477-7. MR2137980 ↑xiii, 24, 25

[Mik06] Grigory Mikhalkin, *Tropical geometry and its applications*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 827–852. MR2275625 ↑25

[Moi60] Grigore C. Moisil, *Sur une représentation des graphes qui interviennent dans l'économie des transports* (Romanian, with French and Russian summaries), Com. Acad. R. P. Romîne **10** (1960), 647–652. MR136445 ↑24

[MRTT53] Theodore S. Motzkin, Howard Raiffa, Gerald L. Thompson, and Robert M. Thrall, *The double description method*, Contributions to the theory of games, vol. 2, Annals of Mathematics Studies, no. 28, Princeton University Press, Princeton, N. J., 1953, pp. 51–73. MR0060202 ↑212

[MRZ21] Guido Montúfar, Yue Ren, and Leon Zhang, *Sharp bounds for the number of regions of maxout networks and vertices of Minkowski sums*, 2021, Preprint `arXiv:2104.08135`. 332

[MS15] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015, DOI 10.1090/gsm/161. MR3287221 ↑xiv, 21, 25, 51, 52, 94, 104, 156, 190, 369

[MSS04] Rolf H. Möhring, Martin Skutella, and Frederik Stork, *Scheduling with AND/OR precedence constraints*, SIAM J. Comput. **33** (2004), no. 2, 393–415, DOI 10.1137/S009753970037727X. MR2048448 ↑280

[MT01] Kazuo Murota and Akihisa Tamura, *On circuit valuation of matroids*, Adv. in Appl. Math. **26** (2001), no. 3, 192–225, DOI 10.1006/aama.2000.0716. MR1818743 ↑371

[Mur03] Kazuo Murota, *Discrete convex analysis*, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003, DOI 10.1137/1.9780898718508. MR1997998 ↑122, 125, 331

[Nel18] Peter Nelson, *Almost all matroids are nonrepresentable*, Bull. Lond. Math. Soc. **50** (2018), no. 2, 245–248, DOI 10.1112/blms.12141. MR3830117 ↑329

[NPRZ10] Fedor Nazarov, Fedor Petrov, Dmitry Ryabogin, and Artem Zvavitch, *A remark on the Mahler conjecture: local minimality of the unit cube*, Duke Math. J. **154** (2010), no. 3, 419–430, DOI 10.1215/00127094-2010-042. MR2730574 ↑251

[OPS19] Jorge Alberto Olarte, Marta Panizzut, and Benjamin Schröter, *On local Dressians of matroids*, Algebraic and geometric combinatorics on lattice polytopes, World Sci. Publ., Hackensack, NJ, 2019, pp. 309–329. MR3971702 ↑331

[OR88] G. J. Olsder and C. Roos, *Cramér and Cayley-Hamilton in the max algebra*, Linear Algebra Appl. **101** (1988), 87–108, DOI 10.1016/0024-3795(88)90145-0. MR941298 ↑124

[OT92] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992, DOI 10.1007/978-3-662-02772-1. MR1217488 ↑107

[Oxl11] James Oxley, *Matroid theory*, 2nd ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011, DOI 10.1093/acprof:oso/9780198566946.001.0001. MR2849819 ↑282, 288, 290, 329, 330, 370, 371

[Pap94] Christos H. Papadimitriou, *Computational complexity*, Addison-Wesley Publishing Company, Reading, MA, 1994. MR1251285 ↑247, 341, 342, 344, 345

[Pay09] Sam Payne, *Analytification is the limit of all tropicalizations*, Math. Res. Lett. **16** (2009), no. 3, 543–556, DOI 10.4310/MRL.2009.v16.n3.a13. MR2511632 ↑57

[Pos09] Alexander Postnikov, *Permutohedra, associahedra, and beyond*, Int. Math. Res. Not. IMRN **6** (2009), 1026–1106, DOI 10.1093/imrn/rnn153. MR2487491 ↑114, 124, 330

[PR03] Julian Pfeifle and Jörg Rambau, *Computing triangulations using oriented matroids*, Algebra, geometry, and software systems, Springer, Berlin, 2003, pp. 49–75. MR2011753 ↑340

[PRW08] Alex Postnikov, Victor Reiner, and Lauren Williams, *Faces of generalized permutohedra*, Doc. Math. **13** (2008), 207–273. MR2520477 ↑95

[PS82] Christos H. Papadimitriou and Kenneth Steiglitz, *Combinatorial optimization: algorithms and complexity*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1982. MR663728 178, 345

[PS05] Lior Pachter and Bernd Sturmfels (eds.), *Algebraic statistics for computational biology*, Cambridge University Press, New York, 2005. MR2205865 56, 332, 371

[Ram02] Jörg Rambau, *TOPCOM: triangulations of point configurations and oriented matroids*, Mathematical software (Beijing, 2002), World Sci. Publ., River Edge, NJ, 2002, pp. 330–340. MR1932619 ↑369

[Ram19] Jörg Rambau, *TOPCOM, version 0.17.8*, available at `http://www.rambau.wm.uni-bayreuth.de/TOPCOM/`, 2019. 369

[Ren88] James Renegar, *A polynomial-time algorithm, based on Newton's method, for linear programming*, Math. Programming **40** (1988), no. 1, (Ser. A), 59–93, DOI 10.1007/BF01580724. MR923697 ↑246

[Ren01] James Renegar, *A mathematical view of interior-point methods in convex optimization*, MPS/SIAM Series on Optimization, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Programming Society (MPS), Philadelphia, PA, 2001, DOI 10.1137/1.9780898718812. MR1857706 ↑246, 248, 341, 346

[RGST05] Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald, *First steps in tropical geometry*, Idempotent mathematics and mathematical physics, Contemp. Math., vol. 377, Amer. Math. Soc., Providence, RI, 2005, pp. 289–317, DOI 10.1090/conm/377/06998. MR2149011 ↑xiv, 25, 104, 124

[Ric17] David Ricardo, *On the principles of political economy and taxation*, London, John Murray, 1817. 274, 278

[Ric15] David Ricardo, *On the principles of political economy and taxation*, Cambridge Library Collection – British and Irish History, 19th Century, Cambridge: Cambridge University Press, 2015. 274, 278

[Rin13] Felipe Rincón, *Computing tropical linear spaces*, J. Symbolic Comput. **51** (2013), 86–98, DOI 10.1016/j.jsc.2012.03.008. MR3005783 ↑331

[RS02] Qazi I. Rahman and Gerhard Schmeisser, *Analytic theory of polynomials*, London Mathematical Society Monographs. New Series, vol. 26, The Clarendon Press, Oxford University Press, Oxford, 2002. MR1954841 ↑251

[RSS03] Günter Rote, Francisco Santos, and Ileana Streinu, *Expansive motions and the polytope of pointed pseudo-triangulations*, Discrete and computational geometry, Algorithms Combin., vol. 25, Springer, Berlin, 2003, pp. 699–736, DOI 10.1007/978-3-642-55566-4_33. MR2038499 ↑185

[RST99] Neil Robertson, P. D. Seymour, and Robin Thomas, *Permanents, Pfaffian orientations, and even directed circuits*, Ann. of Math. (2) **150** (1999), no. 3, 929–975, DOI 10.2307/121059. MR1740989 ↑95

[San12] Francisco Santos, *A counterexample to the Hirsch conjecture*, Ann. of Math. (2) **176** (2012), no. 1, 383–412, DOI 10.4007/annals.2012.176.1.7. MR2925387 ↑251

[SB17] Randall T. Schuh and Andrew Van Zandt Brower, *Biological systematics: Principles and applications*, Cornell University Press, 2017. 319

[Sch86] Alexander Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, A Wiley-Interscience Publication, John Wiley & Sons, Ltd., Chichester, 1986. MR874114 ↑39, 56, 217, 218, 333, 334

[Sch03] Alexander Schrijver, *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, Paths, flows, matchings; Chapters 1–38, Algorithms and Combinatorics, vol. 24, Springer-Verlag, Berlin, 2003. MR1956924 ↑64, 94, 178

[Sei18] Raimund Seidel, *Convex hull computations*, Handbook of Discrete and Computational Geometry (Csaba D. Tóth, Jacob E. Goodmann, and Joseph O'Rourke, eds.), 3rd ed., CRC Press, Boca Raton, FL, 2018. 212

[SGHL07] Helmut Salzmann, Theo Grundhöfer, Hermann Hähl, and Rainer Löwen, *The classical fields*, Structural features of the real and rational numbers, Encyclopedia of Mathematics and its Applications, vol. 112, Cambridge University Press, Cambridge, 2007, DOI 10.1017/CBO9780511721502. MR2357231 ↑55, 56

[Shi15] Yoshinori Shiozawa, *International trade theory and exotic algebras*, Evolut. Inst. Econ. Rev. **12** (2015), no. 1, 177–212. 125, 274, 277

[Sim78] Imre Simon, *Limited subsets of a free monoid*, 19th Annual Symposium on Foundations of Computer Science (Ann Arbor, Mich., 1978), IEEE, Long Beach, Calif., 1978, pp. 143–150. MR539835 ↑94

[Sim88] Imre Simon, *Recognizable sets with multiplicities in the tropical semiring*, Mathematical foundations of computer science, 1988 (Carlsbad, 1988), Lecture Notes in Comput. Sci., vol. 324, Springer, Berlin, 1988, pp. 107–120, DOI 10.1007/BFb0017135. MR1023416 ↑24

[Sma00] Steve Smale, *Mathematical problems for the next century*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 271–294. MR1754783 ↑247, 344

[Spe08] David E. Speyer, *Tropical linear spaces*, SIAM J. Discrete Math. **22** (2008), no. 4, 1527–1558, DOI 10.1137/080716219. MR2448909 ↑328, 331

[Spe09] David E. Speyer, *A matroid invariant via the K-theory of the Grassmannian*, Adv. Math. **221** (2009), no. 3, 882–913, DOI 10.1016/j.aim.2009.01.010. MR2511042 ↑329

[SS04] David Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Adv. Geom. **4** (2004), no. 3, 389–411, DOI 10.1515/advg.2004.023. MR2071813 ↑xiv, 56, 251, 329, 330

[SSB09] Sergeĭ Sergeev, Hans Schneider, and Peter Butkovič, *On visualization scaling, subeigenvectors and Kleene stars in max algebra*, Linear Algebra Appl. **431** (2009), no. 12, 2395–2406, DOI 10.1016/j.laa.2009.03.040. MR2563030 ↑94

[ST13] Bernd Sturmfels and Ngoc Mai Tran, *Combinatorial types of tropical eigenvectors*, Bull. Lond. Math. Soc. **45** (2013), no. 1, 27–36, DOI 10.1112/blms/bds058. MR3033951 ↑190

[Sta86] Richard P. Stanley, *Two poset polytopes*, Discrete Comput. Geom. **1** (1986), no. 1, 9–23, DOI 10.1007/BF02187680. MR824105 ↑89, 90

[Sta12] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR2868112 ↑368

[Ste10] Ernst Steinitz, *Algebraische Theorie der Körper* (German), Neu herausgegeben, mit Erläuterungen und einem Anhang: Abriß der Galoisschen Theorie versehenen von R. Baer und H. Hasse, Berlin, W. de Gruyter & Co, 1930, J. Reine Angew. Math. **137** (1910), 167–309, DOI 10.1515/crll.1910.137.167. MR1580791 ↑30

[Str35] Stefan Straszewicz, *Über exponierte Punkte abgeschlossener Punktmengen*, Fundam. Math. **24** (1935), 139–143. 156

[Str69] Volker Strassen, *Gaussian elimination is not optimal*, Numer. Math. **13** (1969), 354–356, DOI 10.1007/BF02165411. MR248973 ↑95

[Stu94] Bernd Sturmfels, *On the Newton polytope of the resultant*, J. Algebraic Combin. **3** (1994), no. 2, 207–236, DOI 10.1023/A:1022497624378. MR1268576 ↑123

[Stu96] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996, DOI 10.1090/ulect/008. MR1363949 ↑xiv, 339

[Stu02] Bernd Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Mathematics, vol. 97, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2002, DOI 10.1090/cbms/097. MR1925796 ↑xiv, 56, 104

[SW21] David Speyer and Lauren K. Williams, *The positive Dressian equals the positive tropical Grassmannian*, Trans. Amer. Math. Soc. Ser. B **8** (2021), 330–353, DOI 10.1090/btran/67. MR4241765 ↑331

[SY04] Bernd Sturmfels and Josephine Yu, *Classification of six-point metrics*, Electron. J. Combin. **11** (2004), no. 1, Research Paper 44, 16. MR2097310 ↑332

[Tao08] Terence Tao, *Structure and randomness*, Pages from year one of a mathematical blog, American Mathematical Society, Providence, RI, 2008, DOI 10.1090/mbk/059. MR2459552 ↑251

[Tar51] Alfred Tarski, *A decision method for elementary algebra and geometry*, 2nd ed., University of California Press, Berkeley and Los Angeles, Calif., 1951. MR0044472 ↑57

[Tho06] Rekha R. Thomas, *Lectures in geometric combinatorics*, Student Mathematical Library, vol. 33, American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2006. IAS/Park City Mathematical Subseries, DOI 10.1090/stml/033. MR2237292 ↑333

[Tit74] Jacques Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin-New York, 1974. MR0470099 ↑323

[Tra17] Ngoc Mai Tran, *Enumerating polytropes*, J. Combin. Theory Ser. A **151** (2017), 1–22, DOI 10.1016/j.jcta.2017.03.011. MR3663485 ↑190

[TWY20] Xiaoxian Tang, Houjie Wang, and Ruriko Yoshida, *Tropical support vector machine and its applications to phylogenomics*, 2020, Preprint `arXiv:2003.00677`. 332

[TY19] Ngoc Mai Tran and Josephine Yu, *Product-mix auctions and tropical geometry*, Math. Oper. Res. **44** (2019), no. 4, 1396–1411, DOI 10.1287/moor.2018.0975. MR4032448 ↑xv, 122, 125

[TZ98] Henryk Trappmann and Günter M. Ziegler, *Shellability of complexes of trees*, J. Combin. Theory Ser. A **82** (1998), no. 2, 168–178, DOI 10.1006/jcta.1997.2844. MR1620865 ↑331

[vdDS98] Lou van den Dries and Patrick Speissegger, *The real field with convergent generalized power series*, Trans. Amer. Math. Soc. **350** (1998), no. 11, 4377–4421, DOI 10.1090/S0002-9947-98-02105-9. MR1458313 ↑57 [vdW93] Bartel L. van der Waerden, *Algebra II. Unter Benutzung von Vorlesungen von E. Artin und E. Noether. Mit einem Geleitwort von Jürgen Neukirch* (German), 6th ed., Springer, Berlin, 1993. 56, 368

[Vir01] Oleg Viro, *Dequantization of real algebraic geometry on logarithmic paper*, European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 135–146. MR1905317 ↑57

[Vor67] Nikolaĭ Nikolaevič Vorob′ev, *Extremal algebra of positive matrices* (Russian, with English and Russian summaries), Elektron. Informationsverarb. Kybernet. **3** (1967), 39–71. MR216854 ↑94, 156, 190

[vzGG03] Joachim von zur Gathen and Jürgen Gerhard, *Modern computer algebra*, 2nd ed., Cambridge University Press, Cambridge, 2003. MR2001757 ↑45, 57

[Wai19] Martin J. Wainwright, *High-dimensional statistics*, A non-asymptotic viewpoint, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 48, Cambridge University Press, Cambridge, 2019, DOI 10.1017/9781108627771. MR3967104 ↑322

[War62] Stephen Warshall, *A theorem on boolean matrices*, J. Assoc. Comput. Mach. **9** (1962), 11–12, DOI 10.1145/321105.321107. MR149688 ↑70, 94

[Whi86] Neil White (ed.), *Theory of matroids*, Encyclopedia of Mathematics and its Applications, vol. 26, Cambridge University Press, Cambridge, 1986, DOI 10.1017/CBO9780511629563. MR849389 ↑330

[Whi87] Neil White (ed.), *Combinatorial geometries*, Encyclopedia of Mathematics and its Applications, vol. 29, Cambridge University Press, Cambridge, 1987, DOI 10.1017/CBO9781107325715. MR921064 ↑330

[Whi92] Neil White (ed.), *Matroid applications*, Encyclopedia of Mathematics and its Applications, vol. 40, Cambridge University Press, Cambridge, 1992, DOI 10.1017/CBO9780511662041. MR1165537 ↑330

[Wig07] Avi Wigderson, *P, NP and mathematics—a computational complexity perspective*, International Congress of Mathematicians. Vol. I, Eur. Math. Soc., Zürich, 2007, pp. 665–712, DOI 10.4171/022-1/25. MR2334207 ↑344, 345

[Wil14] Ryan Williams, *Faster all-pairs shortest paths via circuit complexity*, STOC'14—Proceedings of the 2014 ACM Symposium on Theory of Computing, ACM, New York, 2014, pp. 664–673. MR3238994 ↑95

[Woe] Gerhard J. Woeginger, *The* P*-versus-*NP *page*, `https://www.win.tue.nl/~gwoegi/P-versus-NP.htm`, retrieved 02 Feb 2021. 344

[YY07] Josephine Yu and Debbie S. Yuster, *Representing tropical linear spaces by circuits*, Formal Power Series and Algebraic Combinatorics, 2007, `http://igm.univ-mlv.fr/~fpsac/FPSAC07/SITE07/PDF-Proceedings/Talks/78.pdf`. 330

[YZ14] H. Peyton Young and Shmuel Zamir (eds.), *Handbook of game theory. Vol. 4*, Handbooks in Economics, Elsevier/North-Holland, Amsterdam, 2015. MR3752664 ↑280

[YZZ19] Ruriko Yoshida, Leon Zhang, and Xu Zhang, *Tropical principal component analysis and its application to phylogenetics*, Bull. Math. Biol. **81** (2019), no. 2, 568–597, DOI 10.1007/s11538-018-0493-4. MR3902912 ↑322, 323

[Zha21] Leon Zhang, *Computing Min-Convex Hulls in the Affine Building of* SL_d, Discrete Comput. Geom. **65** (2021), no. 4, 1314–1336, DOI 10.1007/s00454-020-00223-x. MR4249906 ↑332

[Zie95] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995, DOI 10.1007/978-1-4613-8431-1. MR1311028 ↑xiii, 223, 238, 333, 334, 336

[Zim77] Karel Zimmermann, *A general separation theorem in extremal algebras* (English, with Czech summary), Ekonom.-Mat. Obzor **13** (1977), no. 2, 179–201. MR453607 ↑94, 156, 212

[Ziv18] Rade T. Zivaljevič, *Topological methods in discrete geometry*, Handbook of Discrete and Computational Geometry (Csaba D. Tóth, Jacob E. Goodmann, and Joseph O'Rourke, eds.), 3rd ed., CRC Press, Boca Raton, FL, 2018. 189

[ZNL18] Liwen Zhang, Gregory Naitzat, and Lek-Heng Lim, *Tropical geometry of deep neural networks*, Proceedings of the 35th International Conference on Machine Learning (Jennifer Dy and Andreas Krause, eds.), Proceedings of Machine Learning Research, vol. 80, PMLR, 10–15 Jul 2018, pp. 5824–5832. 332

[Zon06] Chuanming Zong, *The cube: a window to convex and discrete geometry*, Cambridge Tracts in Mathematics, vol. 168, Cambridge University Press, Cambridge, 2006, DOI 10.1017/CBO9780511543173. MR2221660 ↑125

[ZP96] Uri Zwick and Mike Paterson, *The complexity of mean payoff games on graphs*, Theoret. Comput. Sci. **158** (1996), no. 1-2, 343–359, DOI 10.1016/0304-3975(95)00188-3. MR1388974 ↑279, 370

Index

Selected Published Titles in This Series

219 **Michael Joswig,** Essentials of Tropical Combinatorics, 2021
217 **Marius Crainic, Rui Loja Fernandes, and Ioan Mărcuţ,** Lectures on Poisson Geometry, 2021
215 **Tai-Ping Liu,** Shock Waves, 2021
214 **Ioannis Karatzas and Constantinos Kardaras,** Portfolio Theory and Arbitrage, 2021
213 **Hung Vinh Tran,** Hamilton–Jacobi Equations, 2021
211 **Mateusz Michałek and Bernd Sturmfels,** Invitation to Nonlinear Algebra, 2021
210 **Bruce E. Sagan,** Combinatorics: The Art of Counting, 2020
209 **Jessica S. Purcell,** Hyperbolic Knot Theory, 2020
208 **Vicente Muñoz, Ángel González-Prieto, and Juan Ángel Rojo,** Geometry and Topology of Manifolds, 2020
207 **Dmitry N. Kozlov,** Organized Collapse: An Introduction to Discrete Morse Theory, 2020
206 **Ben Andrews, Bennett Chow, Christine Guenther, and Mat Langford,** Extrinsic Geometric Flows, 2020
205 **Mikhail Shubin,** Invitation to Partial Differential Equations, 2020
204 **Sarah J. Witherspoon,** Hochschild Cohomology for Algebras, 2019
203 **Dimitris Koukoulopoulos,** The Distribution of Prime Numbers, 2019
202 **Michael E. Taylor,** Introduction to Complex Analysis, 2019
201 **Dan A. Lee,** Geometric Relativity, 2019
200 **Semyon Dyatlov and Maciej Zworski,** Mathematical Theory of Scattering Resonances, 2019
199 **Weinan E, Tiejun Li, and Eric Vanden-Eijnden,** Applied Stochastic Analysis, 2019
198 **Robert L. Benedetto,** Dynamics in One Non-Archimedean Variable, 2019
197 **Walter Craig,** A Course on Partial Differential Equations, 2018
196 **Martin Stynes and David Stynes,** Convection-Diffusion Problems, 2018
195 **Matthias Beck and Raman Sanyal,** Combinatorial Reciprocity Theorems, 2018
194 **Seth Sullivant,** Algebraic Statistics, 2018
193 **Martin Lorenz,** A Tour of Representation Theory, 2018
192 **Tai-Peng Tsai,** Lectures on Navier-Stokes Equations, 2018
191 **Theo Bühler and Dietmar A. Salamon,** Functional Analysis, 2018
190 **Xiang-dong Hou,** Lectures on Finite Fields, 2018
189 **I. Martin Isaacs,** Characters of Solvable Groups, 2018
188 **Steven Dale Cutkosky,** Introduction to Algebraic Geometry, 2018
187 **John Douglas Moore,** Introduction to Global Analysis, 2017
186 **Bjorn Poonen,** Rational Points on Varieties, 2017
185 **Douglas J. LaFountain and William W. Menasco,** Braid Foliations in Low-Dimensional Topology, 2017
184 **Harm Derksen and Jerzy Weyman,** An Introduction to Quiver Representations, 2017
183 **Timothy J. Ford,** Separable Algebras, 2017
182 **Guido Schneider and Hannes Uecker,** Nonlinear PDEs, 2017
181 **Giovanni Leoni,** A First Course in Sobolev Spaces, Second Edition, 2017
180 **Joseph J. Rotman,** Advanced Modern Algebra: Third Edition, Part 2, 2017
179 **Henri Cohen and Fredrik Strömberg,** Modular Forms, 2017
178 **Jeanne N. Clelland,** From Frenet to Cartan: The Method of Moving Frames, 2017
177 **Jacques Sauloy,** Differential Galois Theory through Riemann-Hilbert Correspondence, 2016

For a complete list of titles in this series, visit the
AMS Bookstore at **www.ams.org/bookstore/gsmseries/**.

The goal of this book is to explain, at the graduate student level, connections between tropical geometry and optimization. Building bridges between these two subject areas is fruitful in two ways. Through tropical geometry optimization algorithms become applicable to questions in algebraic geometry. Conversely, looking at topics in optimization through the tropical geometry lens adds an additional layer of structure. The author covers contemporary research topics that are relevant for applications such as phylogenetics, neural networks, combinatorial auctions, game theory, and computational complexity. This self-contained book grew out of several courses given at Technische Universität Berlin and elsewhere, and the main prerequisite for the reader is a basic knowledge in polytope theory. It contains a good number of exercises, many examples, beautiful figures, as well as explicit tools for computations using `polymake`.

ISBN 978-1-4704-6653-4

9 781470 466534

GSM/219

For additional information and updates on this book, visit

www.ams.org/bookpages/gsm-219

AMS
www.ams.org