

GRADUATE STUDIES  
IN MATHEMATICS **220**

# **Ultrafilters Throughout Mathematics**

**Isaac Goldbring**



# Ultrafilters Throughout Mathematics



GRADUATE STUDIES **220**  
IN MATHEMATICS

# Ultrafilters Throughout Mathematics

Isaac Goldbring



AMERICAN  
MATHEMATICAL  
SOCIETY

Providence, Rhode Island

## EDITORIAL COMMITTEE

Marco Gualtieri  
Bjorn Poonen  
Gigliola Staffilani (Chair)  
Jeff A. Viaclovsky  
Rachel Ward

2020 *Mathematics Subject Classification*. Primary 03C20, 54D80, 03H05, 03E55, 03C50.

During the writing of this book, the author was partially supported by NSF grants DMS-1349399 (CAREER) and DMS-2054477.

---

For additional information and updates on this book, visit  
[www.ams.org/bookpages/gsm-220](http://www.ams.org/bookpages/gsm-220)

---

### Library of Congress Cataloging-in-Publication Data

Names: Goldbring, Isaac, author.

Title: Ultrafilters throughout mathematics / Isaac Goldbring.

Description: Providence, Rhode Island : American Mathematical Society, [2022] | Series: Graduate studies in mathematics, 1065-7339 ; 220 | Includes bibliographical references and index.

Identifiers: LCCN 2021055552 | ISBN 9781470469009 (hardback) | ISBN 9781470469610 (paperback) | ISBN 9781470469603 (ebook)

Subjects: LCSH: Ultrafilters (Mathematics) | AMS: Mathematical logic and foundations – Model theory – Ultraproducts and related constructions. | General topology – Fairly general properties of topological spaces – Special constructions of topological spaces (spaces of ultrafilters, etc.). | Mathematical logic and foundations – Nonstandard models – Nonstandard models in mathematics. | Mathematical logic and foundations – Set theory – Large cardinals. | Mathematical logic and foundations – Model theory – Models with special properties (saturated, rigid, etc.).

Classification: LCC QA248 .G574 2022 | DDC 511.3–dc23/eng/20220112

LC record available at <https://lcn.loc.gov/2021055552>

---

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy select pages for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for permission to reuse portions of AMS publication content are handled by the Copyright Clearance Center. For more information, please visit [www.ams.org/publications/pubpermissions](http://www.ams.org/publications/pubpermissions).

Send requests for translation rights and licensed reprints to [reprint-permission@ams.org](mailto:reprint-permission@ams.org).

© 2022 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights  
except those granted to the United States Government.

Printed in the United States of America.

⊗ The paper used in this book is acid-free and falls within the guidelines  
established to ensure permanence and durability.

Visit the AMS home page at <https://www.ams.org/>

10 9 8 7 6 5 4 3 2 1      27 26 25 24 23 22

To Karina, Kaylee, and Daniella



---

# Contents

Preface	xiii
<b>Part 1. Ultrafilters and their applications</b>	
Chapter 1. Ultrafilter basics	3
§1.1. Basic definitions	3
§1.2. The ultrafilter quantifier	6
§1.3. The category of ultrafilters	7
§1.4. The number of ultrafilters	11
§1.5. The ultrafilter number $\mathfrak{u}$	13
§1.6. The Rudin-Keisler order	14
§1.7. Notes and references	17
Chapter 2. Arrow's theorem on fair voting	19
§2.1. Statement of the theorem	19
§2.2. The connection with ultrafilters	21
§2.3. Block voting	23
§2.4. Finishing the proof	24
§2.5. Notes and references	26
Chapter 3. Ultrafilters in topology	27
§3.1. Ultralimits	27
§3.2. The Stone-Čech compactification: the discrete case	31
§3.3. $z$ -ultrafilters and the Stone-Čech compactifications in general	33
§3.4. The Stone representation theorem	38



---

§3.5. Notes and References	44
Chapter 4. Ramsey theory and combinatorial number theory	45
§4.1. Ramsey's theorem	45
§4.2. Idempotent ultrafilters and Hindman's theorem	46
§4.3. Banach density, means, and measures	50
§4.4. Furstenberg's correspondence principle	53
§4.5. Jin's sunset theorem	55
§4.6. Notes and references	57
Chapter 5. Foundational concerns	59
§5.1. The ultrafilter theorem and the axiom of choice: Part I	59
§5.2. Can there exist a "definable" ultrafilter on $\mathbb{N}$ ?	62
§5.3. The ultrafilter game	66
§5.4. Selective ultrafilters and P-points	70
§5.5. Notes and references	77
<b>Part 2. Classical ultraproducts</b>	
Chapter 6. Classical ultraproducts	81
§6.1. Motivating the definition of ultraproducts	82
§6.2. Ultraproducts of sets	83
§6.3. Ultraproducts of structures	85
§6.4. Łoś's theorem	86
§6.5. The ultrafilter theorem and the axiom of choice: Part II	89
§6.6. Countably incomplete ultrafilters	92
§6.7. Revisiting the Rudin-Keisler order	94
§6.8. Cardinalities of ultraproducts	96
§6.9. Iterated ultrapowers	98
§6.10. A category-theoretic perspective on ultraproducts	101
§6.11. The Feferman-Vaught theorem	104
§6.12. Notes and references	108
Chapter 7. Applications to geometry, commutative algebra, and number theory	109
§7.1. Ax's theorem on polynomial functions	109
§7.2. Bounds in the theory of polynomial rings	111
§7.3. The Ax-Kochen theorem and Artin's conjecture	116

---

§7.4. Notes and references	122
Chapter 8. Ultraproducts and saturation	123
§8.1. Saturation	123
§8.2. First saturation properties of ultraproducts	127
§8.3. Regular ultrafilters	128
§8.4. Good ultrafilters: Part 1	134
§8.5. Good ultrafilters: Part 2	141
§8.6. Keisler's order	145
§8.7. Notes and references	155
Chapter 9. Nonstandard analysis	157
§9.1. Naïve axioms for nonstandard analysis	157
§9.2. Nonstandard numbers big and small	159
§9.3. Some nonstandard calculus	161
§9.4. Ultrapowers as a model of nonstandard analysis	163
§9.5. Complete extensions and limit ultrapowers	164
§9.6. Many-sorted structures and internal sets	168
§9.7. Nonstandard generators of ultrafilters	173
§9.8. Hausdorff ultrafilters	177
§9.9. Notes and references	178
Chapter 10. Limit groups	181
§10.1. Introducing the class of limit groups	181
§10.2. First examples and properties of limit groups	183
§10.3. Connection with fully residual freeness	186
§10.4. Explaining the terminology: the space of marked groups	189
§10.5. Notes and references	191
 <b>Part 3. Metric ultraproducts and their applications</b>	
Chapter 11. Metric ultraproducts	195
§11.1. Definition of the metric ultraproduct	195
§11.2. Metric ultraproducts and nonstandard hulls of metric spaces	198
§11.3. Completeness properties of the metric ultraproduct	199
§11.4. Continuous logic	201
§11.5. Reduced products of metric structures	208
§11.6. Notes and references	209

---

Chapter 12. Asymptotic cones and Gromov's theorem	211
§12.1. Some group-theoretic preliminaries	212
§12.2. Growth rates of groups	213
§12.3. Gromov's theorem on polynomial growth	216
§12.4. Definition of asymptotic cones	221
§12.5. General properties of asymptotic cones	223
§12.6. Growth functions and properness of the asymptotic cones	226
§12.7. Properness of asymptotic cones revisited	229
§12.8. Nonhomeomorphic asymptotic cones	231
§12.9. Notes and references	232
Chapter 13. Sofic groups	233
§13.1. Ultraproducts of bi-invariant metric groups	233
§13.2. Definition of sofic groups	235
§13.3. Examples of sofic groups	240
§13.4. An application of sofic groups	245
§13.5. Notes and references	248
Chapter 14. Functional analysis	249
§14.1. Banach space ultraproducts	249
§14.2. Applications to local geometry of Banach spaces	254
§14.3. Commutative $C^*$ -algebras and ultracoproducts of compact spaces	259
§14.4. The tracial ultraproduct construction	264
§14.5. The Connes embedding problem	273
§14.6. Notes and references	277
<b>Part 4. Advanced topics</b>	
Chapter 15. Does an ultrapower depend on the ultrafilter?	281
§15.1. Statement of results	281
§15.2. The case when $\mathcal{M}$ is unstable	284
§15.3. The case when $\mathcal{M}$ is stable	292
§15.4. Notes and references	295
Chapter 16. The Keisler-Shelah theorem	297
§16.1. The Keisler-Shelah theorem	297
§16.2. Application: Elementary classes	305

---

§16.3. Application: Robinson's joint consistency theorem	306
§16.4. Application: Elementary equivalence of matrix rings	307
§16.5. Notes and references	307
Chapter 17. Large cardinals	309
§17.1. Worldly cardinals	309
§17.2. Inaccessible cardinals	311
§17.3. Measurable cardinals	314
§17.4. Strongly and weakly compact cardinals	320
§17.5. Ramsey cardinals	325
§17.6. Measurable cardinals as critical points of elementary embeddings	327
§17.7. An application of large cardinals	333
§17.8. Notes and references	338
<b>Part 5. Appendices</b>	
Appendix A. Logic	341
§A.1. Languages and structures	341
§A.2. Syntax and semantics	342
§A.3. Embeddings	344
§A.4. References	346
Appendix B. Set theory	347
§B.1. The axioms of ZFC	347
§B.2. Ordinals	350
§B.3. Cardinals	351
§B.4. $V$ and $L$	353
§B.5. Relative consistency statements	353
§B.6. Relativization and absoluteness	355
§B.7. References	356
Appendix C. Category theory	357
§C.1. Categories	357
§C.2. Functors, natural transformations, and equivalences of categories	359
§C.3. Limits	360
§C.4. References	362

Appendix D. Hints and solutions to selected exercises	363
Bibliography	385
Index	395

---

# Preface

## What is this book about?

This book is about **ultrafilters**. So what is an ultrafilter? Given a set  $X$ , an ultrafilter on  $X$  is simply a “sensible” division of all of the subsets of  $X$  into two categories: small and large. For this division to be sensible, one should impose some axioms:

- $X$  should be a large subset of  $X$ , while  $\emptyset$  should be a small subset of  $X$ .
- If  $Y$  is a large subset of  $X$  and  $Y \subseteq Z \subseteq X$ , then  $Z$  should also be large; that is, a set containing a large set should also be large.
- If  $Y$  and  $Z$  are two large subsets of  $X$ , then so is  $Y \cap Z$ .

The last axiom is perhaps not entirely intuitive, but becomes more intuitive when stated in terms of small sets: the union of two small sets is once again small. The axioms also imply that a set is large precisely when its complement is small.

Why write a book about such a seemingly simple notion? It turns out that this notion is very useful for describing limits of various objects. For example, much to the chagrin of many calculus students, one knows that there are many sequences  $(a_n)_{n \in \mathbb{N}}$  from  $[0, 1]$  that have no limit. However, limit in the usual sense is very restrictive in that it requires  $a_n$  to be close to the limit for a large number of  $n$ , where large here means for all but finitely many  $n$ . Note that this restrictive notion of largeness does not lead to an ultrafilter on  $\mathbb{N}$  as there are certainly sets that are infinite and which have infinite complement. However, if one works with a notion of largeness as given by an ultrafilter, then all of a sudden *every sequence in  $[0, 1]$  has a*

*limit!* This fact can be used as a powerful tool in analytic and topological endeavors.

The notion of ultrafilter also allows one to consider limits of families of structures like groups, rings, graphs, or Banach spaces. The limiting structures alluded to here are called **ultraproducts** and will become a central part of this book. These limiting objects can be very useful in solving problems, for often various desirable properties are approximately true in the individual structures of the family, while in the limit they become exactly true.

### **Who should read this book?**

The short answer is: everyone! More precisely, the thesis of this book is that, while ultrafilters and ultraproducts are often relegated to graduate-level courses in logic, we believe that this practice is entirely misguided. Indeed, the notion of ultrafilter and ultraproduct are entirely accessible to an advanced undergraduate or beginning graduate student in mathematics (the target audience of this book). Moreover, as we will see throughout the course of this book, ultrafilters and ultraproducts have had numerous applications to nearly every area of mathematics. Thus, no matter what area of mathematics the reader is interested in, it is quite likely that ultrafilters and ultraproducts have made an impact in that area. An attempt has been made to present as diverse a sample of such applications as possible.

That being said, this book is being written by a logician, and ultrafilters present numerous fascinating foundational concerns, many of which are discussed in this book. If the reader is purely interested in mathematical applications, they may safely skip the portions of this book discussing these metamathematical issues.

### **What is in this book?**

Let us briefly summarize the contents of this book. Part 1 is entirely devoted to ultrafilters. Chapter 1 introduces the basic facts about ultrafilters, including what it means for them to be isomorphic and how many of them there are. Chapter 2 provides one with a first application of ultrafilters, namely to a proof of Arrow's theorem on fair voting. This application is nice in the sense that it requires little to no mathematical background and yet exemplifies a perfect use of ultrafilters. Chapter 3 introduces the use of ultrafilters in topology, including the aforementioned facts about generalized limits. This chapter also shows how ultrafilters can be used to describe the important Stone-Ćech compactification construction. Chapter 4 is a brief introduction to how ultrafilters can be used in certain parts of combinatorics; a much more detailed investigation of that line of research can be found in

the book [42], written by the author with Mauro Di Nasso and Martino Lupini. Chapter 5, the last chapter in Part 1 of the book, discusses many of the interesting foundational issues presented by the existence of ultrafilters.

Part 2 of the book is concerned with the classical ultraproduct construction. As alluded to above, this construction allows one to take the limit of families of objects such as groups, rings, graphs, etc., . . . The lengthy Chapter 6 introduces this construction and proves the Fundamental Theorem of Ultraproducts (otherwise known as Łoś's theorem), which states that the truth of a first-order sentence in an ultraproduct is determined by whether or not the sentence is true in a large (as measured by the ultrafilter) number of the individual structures. This chapter includes many other important facts about ultraproducts, including cardinalities of ultraproducts and a discussion of what happens when one tries to iterate the ultraproduct construction.

Chapter 7 gives one a first look at how ultraproducts can be used "in practice." The applications in this chapter are all algebraic in nature, and include Ax's theorem on polynomial functions and the Ax-Kochen theorem relating the rings  $\mathbb{Z}_p$  of  $p$ -adic integers with the power series rings  $\mathbb{F}_p[[T]]$ . One important feature of ultraproducts is that they are often very "rich" in the precise sense of being saturated. Chapter 8 gives a detailed discussion of exactly how saturated ultraproducts can be. Chapter 9 gives a brief introduction to nonstandard analysis. While nonstandard analysis is a subject of its own, it is often presented using ultraproducts and we discuss this approach here. This chapter is far from a complete story on nonstandard analysis and we refer the interested reader to [42] for a more thorough discussion. Chapter 10 discusses the class of subgroups of nonstandard (in the sense of Chapter 9) free groups; the finitely generated such subgroups are called limit groups and have become a widely studied class of groups in geometric group theory.

The ultraproduct construction referred to above is suitable for discrete spaces such as those arising in algebra and combinatorics, but is not very useful for structures appearing in analysis. Part 3 of the book is concerned with a modification of the ultraproduct construction for structures based on metric spaces. Chapter 11 introduces this metric ultraproduct and discusses some of its basic properties. That chapter also includes a discussion of a relatively new logic, aptly called continuous logic, which is the logic naturally connected to this metric ultraproduct construction.

The remainder of Part 3 details several applications of the metric ultraproduct construction. Chapter 12 describes a fascinating theorem of Gromov from geometric group theory, where the key ingredient to the proof is a particular metric ultraproduct called an asymptotic cone. Chapter 13



discusses the class of sofic groups, which can be defined in terms of metric ultraproducts of symmetric groups. Chapter 14, the final chapter of Part 3, discusses some applications of metric ultraproducts to functional analysis. One might argue that functional analysis is an area of mathematics where ultraproducts have played an increasingly more important role. Unfortunately, the mathematical background needed by the reader is much larger in this area of mathematics and thus this section cannot quite do justice to the importance of ultraproducts in functional analysis.

Part 4, the last part of this book, is devoted to three advanced topics. Chapter 15 discusses a question that often arises to many people seeing ultraproducts for the first time: does the ultraproduct depend on the ultrafilter being used? The answer to this question is surprisingly subtle and a more or less complete answer to a specific case of this question is discussed. Chapter 16 discusses the fantastic Keisler-Shelah theorem, which shows how elementary equivalence, a notion from logic, can be reformulated in terms of isomorphic ultrapowers, a purely algebraic notion. This chapter also includes a few applications of the Keisler-Shelah theorem. Chapter 17, the final chapter of the book, shows how the study of large cardinals in set theory can be recast in terms of ultrafilters satisfying certain extra properties. This part of the book might require a bit more maturity and/or background from the reader.

### **What are the prerequisites for reading this book?**

We have no illusions that any one student has all of the prerequisites necessary to read the entire book. However, this fact is by design! As discussed above, we are trying to convey to the reader that ultrafilters and ultraproducts are applicable in most areas of mathematics and thus we have tried to describe a wide variety of applications.

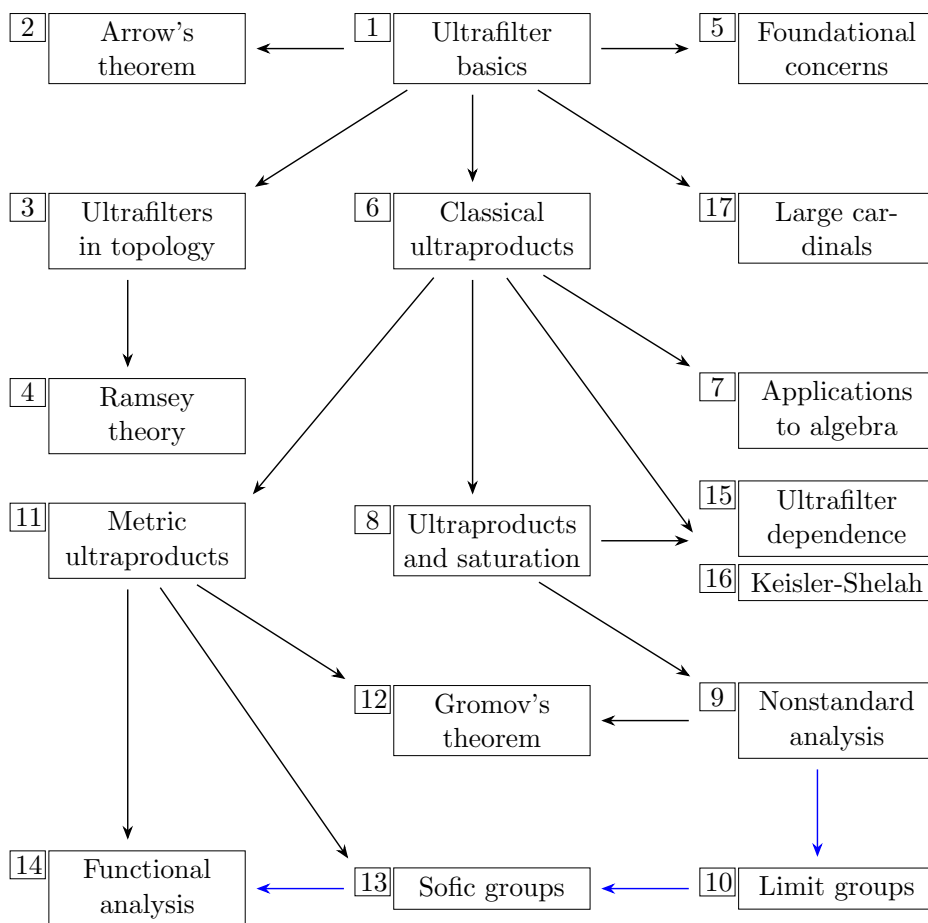
That being said, we have assumed that the reader is familiar with some basic facts from real analysis, topology, and algebra. Any facts that we believe are not part of the usual curricula from those disciplines are often described in full detail here. Sometimes certain topics are outside of the scope of this book and we provide references to the reader for places in the literature where they can learn more. It is also our hope that a reader interested in, for example, algebra sees the chapter on, say, functional analysis, and finds the general idea interesting enough that they decide to learn more about this area. In today's mathematical world, breadth is everything and an aspiring mathematician should keep their eyes open to all areas of mathematics.

In discussing ultrafilters, one cannot hide the fact that logic and set theory play an important role. Moreover, there is a high probability that

the average reader might not have the requisite knowledge in these areas to follow the main parts of this book. For the reader's convenience, appendices on these subjects are included in this book. Also, occasionally in the text, very basic parts of category theory are needed and the necessary facts from category theory are collected in the final appendix.

### How to read this book

Some later chapters rely somewhat heavily on earlier chapters. The following flowchart lists some of these dependencies. The blue arrows indicate dependencies that are not strictly necessary but possibly helpful.



### Exercises

Rather than ending each section or chapter with a list of exercises, we have instead sprinkled them throughout the text itself. Some of the exercises are simply checks for understanding, but others are more involved. Often,

the exercises themselves will be used in the proofs of later results. We recommend that the reader stop reading when they encounter an exercise and attempt a solution at that moment. Solutions to a handful of exercises appear in Appendix D but we urge the reader not to consult these solutions unless the situation becomes dire!

### **Acknowledgements**

We thank Andreas Blass, Bradd Hart, Alessandro Sisto, Eleftherios Tachtsis, and Martin Zeman for helpful discussions during the writing of this book.

We also extend our deepest gratitude to Ryan Burkhart, Alec Fox, Michael Hehmann, Jennifer Pi, Brian Ransom, and Jessica Schirle for a very thorough reading of an earlier draft of this book and for offering many, many corrections and suggestions for improvement.

Isaac Goldbring  
Irvine, CA

---

# Bibliography

- [1] C. Anantharaman and S. Popa, *An introduction to  $II_1$  factors*, book in preparation.
- [2] K. J. Arrow, *A difficulty in the concept of social welfare*, Journal of political economy, **58** (1950), pp. 328–346.
- [3] James Ax, *The elementary theory of finite fields*, Ann. of Math. (2) **88** (1968), 239–271, DOI 10.2307/1970573. MR229613
- [4] James E. Baumgartner, *A short proof of Hindman's theorem*, J. Combinatorial Theory Ser. A **17** (1974), 384–386, DOI 10.1016/0097-3165(74)90103-4. MR354394
- [5] James E. Baumgartner and Richard Laver, *Iterated perfect-set forcing*, Ann. Math. Logic **17** (1979), no. 3, 271–288, DOI 10.1016/0003-4843(79)90010-X. MR556894
- [6] Benjamin Baumslag, *Residually free groups*, Proc. London Math. Soc. (3) **17** (1967), 402–418, DOI 10.1112/plms/s3-17.3.402. MR215903
- [7] Gilbert Baumslag, Alexei Myasnikov, and Vladimir Remeslennikov, *Discriminating completions of hyperbolic groups*, Geom. Dedicata **92** (2002), 115–143, DOI 10.1023/A:1019687202544. Dedicated to John Stallings on the occasion of his 65th birthday. MR1934015
- [8] Mathias Beiglböck, *An ultrafilter approach to Jin's theorem*, Israel J. Math. **185** (2011), 369–374, DOI 10.1007/s11856-011-0114-5. MR2837141
- [9] Itai Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 350, Cambridge Univ. Press, Cambridge, 2008, pp. 315–427, DOI 10.1017/CBO9780511735219.011. MR2436146
- [10] Itai Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, Trans. Amer. Math. Soc. **362** (2010), no. 10, 5213–5259, DOI 10.1090/S0002-9947-10-04837-3. MR2657678
- [11] Vitaly Bergelson, *Combinatorial and Diophantine applications of ergodic theory*, Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, pp. 745–869, DOI 10.1016/S1874-575X(06)80037-8. Appendix A by A. Leibman and Appendix B by Anthony Quas and Máté Wierdl. MR2186252
- [12] Vitaly Bergelson, *Ultrafilters, IP sets, dynamics, and combinatorial number theory*, Ultrafilters across mathematics, Contemp. Math., vol. 530, Amer. Math. Soc., Providence, RI, 2010, pp. 23–47, DOI 10.1090/conm/530/10439. MR2757532

- [13] Andreas Blass, *A model without ultrafilters* (English, with Russian summary), Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **25** (1977), no. 4, 329–331. MR476510
- [14] Andreas Blass, *Combinatorial cardinal characteristics of the continuum*, in Handbook of Set Theory, Springer, 2010, pp. 395–489.
- [15] Andreas Blass and Saharon Shelah, *Ultrafilters with small generating sets*, Israel J. Math. **65** (1989), no. 3, 259–271, DOI 10.1007/BF02764864. MR1005010
- [16] Andreas Raphael Blass, *Orderings of ultrafilters*, ProQuest LLC, Ann Arbor, MI, 1970. Thesis (Ph.D.)—Harvard University. MR2939947
- [17] William Boos, *Infinitary compactness without strong inaccessibility*, J. Symbolic Logic **41** (1976), no. 1, 33–38, DOI 10.2307/2272942. MR409184
- [18] David Booth, *Ultrafilters on a countable set*, Ann. Math. Logic **2** (1970/71), no. 1, 1–24, DOI 10.1016/0003-4843(70)90005-7. MR277371
- [19] N. Bourbaki, *Éléments de mathématique. Topologie algébrique. Chapitres 1 à 4* (French), Springer, Heidelberg, 2016. MR3617167
- [20] Emmanuel Breuillard, Ben Green, and Terence Tao, *The structure of approximate groups*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 115–221, DOI 10.1007/s10240-012-0043-9. MR3090256
- [21] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Introductory notes on Richard Thompson’s groups*, Enseign. Math. (2) **42** (1996), no. 3–4, 215–256. MR1426438
- [22] Valerio Capraro and Martino Lupini, *Introduction to sofic and hyperlinear groups and Connes’ embedding conjecture*, Lecture Notes in Mathematics, vol. 2136, Springer, Cham, 2015. With an appendix by Vladimir Pestov, DOI 10.1007/978-3-319-19333-5. MR3408561
- [23] H. Cartan, *Filtres et ultrafiltres*, Comptes Rendus de l’Académie des Sciences, **205** (1937), pp. 777–779.
- [24] H. Cartan, *Théorie des filtres*, Comptes Rendus de l’Académie des Sciences, **205** (1937), pp. 595–598.
- [25] Eduard Čech, *On bicomact spaces*, Ann. of Math. (2) **38** (1937), no. 4, 823–844, DOI 10.2307/1968839. MR1503374
- [26] Christophe Champetier and Vincent Guirardel, *Limit groups as limits of free groups*, Israel J. Math. **146** (2005), 1–75, DOI 10.1007/BF02773526. MR2151593
- [27] Chen-chung Chang and H. Jerome Keisler, *Continuous model theory*, Annals of Mathematics Studies, No. 58, Princeton Univ. Press, Princeton, N.J., 1966. MR0231708
- [28] Chen-chung Chang and H. Jerome Keisler, *Model theory*, Dover Publications, 1990.
- [29] Charles Chidume, *Geometric properties of Banach spaces and nonlinear iterations*, Lecture Notes in Mathematics, vol. 1965, Springer-Verlag London, Ltd., London, 2009. MR2504478
- [30] Ian Chiswell, *Introduction to  $\Lambda$ -trees*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001, DOI 10.1142/4495. MR1851337
- [31] Matt Clay and Dan Margalit, *Groups*, Office hours with a geometric group theorist, Princeton Univ. Press, Princeton, NJ, 2017, pp. 3–20. MR3587214
- [32] E. Coleman, *Banach space ultraproducts*, Irish Math. Soc. Bull. **18** (1987), 30–39. MR885794
- [33] W. W. Comfort, *Ultrafilters: some old and some new results*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 417–455, DOI 10.1090/S0002-9904-1977-14316-4. MR454893
- [34] A. Connes, *Classification of injective factors. Cases  $II_1$ ,  $II_\infty$ ,  $III_\lambda$ ,  $\lambda \neq 1$* , Ann. of Math. (2) **104** (1976), no. 1, 73–115, DOI 10.2307/1971057. MR454659
- [35] John B. Conway, *A course in functional analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR1070713
- [36] Abraham Robinson, *A result on consistency and its application to the theory of definition*, Nederl. Akad. Wetensch. Proc. Ser. A. **59** = Indag. Math. **18** (1956), 47–58. MR0078307

- [37] V. B. Dem'yanov, *On cubic forms in discretely normed fields* (Russian), Doklady Akad. Nauk SSSR (N.S.) **74** (1950), 889–891. MR0037836
- [38] Keith J. Devlin, *Fundamentals of contemporary set theory*, Universitext, Springer-Verlag, New York-Heidelberg, 1979. MR541746
- [39] Mauro Di Nasso, *Hypernatural numbers as ultrafilters*, Nonstandard analysis for the working mathematician, Springer, Dordrecht, 2015, pp. 443–474. MR3409522
- [40] Mauro Di Nasso, *Iterated hyper-extensions and an idempotent ultrafilter proof of Rado's theorem*, Proc. Amer. Math. Soc. **143** (2015), no. 4, 1749–1761, DOI 10.1090/S0002-9939-2014-12342-2. MR3314087
- [41] Mauro Di Nasso and Marco Forti, *Hausdorff ultrafilters*, Proc. Amer. Math. Soc. **134** (2006), no. 6, 1809–1818, DOI 10.1090/S0002-9939-06-08433-4. MR2207497
- [42] Mauro Di Nasso, Isaac Goldbring, and Martino Lupini, *Nonstandard methods in Ramsey theory and combinatorial number theory*, Lecture Notes in Mathematics, vol. 2239, Springer, Cham, 2019, DOI 10.1007/978-3-030-17956-4. MR3931702
- [43] Mauro Di Nasso and Eleftherios Tachtsis, *Idempotent ultrafilters without Zorn's lemma*, Proc. Amer. Math. Soc. **146** (2018), no. 1, 397–411, DOI 10.1090/proc/13719. MR3723149
- [44] Seán Dineen, *The second dual of a  $JB^*$  triple system*, Complex analysis, functional analysis and approximation theory (Campinas, 1984), North-Holland Math. Stud., vol. 125, North-Holland, Amsterdam, 1986, pp. 67–69. MR893410
- [45] Hans-Dieter Donder, *Regularity of ultrafilters and the core model*, Israel J. Math. **63** (1988), no. 3, 289–322, DOI 10.1007/BF02778036. MR969944
- [46] Alan Dow, *Saturated Boolean algebras and their Stone spaces*, Topology Appl. **21** (1985), no. 2, 193–207, DOI 10.1016/0166-8641(85)90104-X. MR813288
- [47] Cornelia Druțu and Mark Sapir, *Tree-graded spaces and asymptotic cones of groups*, Topology **44** (2005), no. 5, 959–1058, DOI 10.1016/j.top.2005.03.003. With an appendix by Denis Osin and Mark Sapir. MR2153979
- [48] Gábor Elek and Endre Szabó, *On sofic groups*, J. Group Theory **9** (2006), no. 2, 161–171, DOI 10.1515/JGT.2006.011. MR2220572
- [49] Herbert B. Enderton, *Elements of set theory*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1977. MR0439636
- [50] Herbert B. Enderton, *A mathematical introduction to logic*, Elsevier, 2001.
- [51] Herbert B. Enderton, *Computability theory: An introduction to recursion theory*, Academic Press, 2010.
- [52] P. Enflo, *Uniform structures and square roots in topological groups. I, II*, Israel J. Math. **8** (1970), 230–252; *ibid.* **8** (1970), 253–272, DOI 10.1007/bf02771561. MR263969
- [53] Ilijas Farah, Bradd Hart, and David Sherman, *Model theory of operator algebras I: stability*, Bull. Lond. Math. Soc. **45** (2013), no. 4, 825–838, DOI 10.1112/blms/bdt014. MR3081550
- [54] Ilijas Farah, Bradd Hart, and David Sherman, *Model theory of operator algebras II: model theory*, Israel J. Math. **201** (2014), no. 1, 477–505, DOI 10.1007/s11856-014-1046-7. MR3265292
- [55] Ilijas Farah and Saharon Shelah, *A dichotomy for the number of ultrapowers*, J. Math. Log. **10** (2010), no. 1-2, 45–81, DOI 10.1142/S0219061310000936. MR2802082
- [56] S. Feferman and R. L. Vaught, *The first order properties of products of algebraic systems*, Fund. Math. **47** (1959), 57–103, DOI 10.4064/fm-47-1-57-103. MR108455
- [57] Ben Fine, Anthony M. Gaglione, Gerhard Rosenberger, and Dennis Spellman, *On CT and CSA groups and related ideas*, J. Group Theory **19** (2016), no. 5, 923–940, DOI 10.1515/jgth-2016-0005. MR3545911
- [58] Edward R. Fisher, *Abelian structures. I*, Abelian group theory (Proc. Second New Mexico State Univ. Conf., Las Cruces, N.M., 1976), Springer, Berlin, 1977, pp. 270–322. Lecture Notes in Math., Vol. 616. MR0540014

- [59] T. Frayne, A. C. Morel, and D. S. Scott, *Reduced direct products*, Fund. Math. **51** (1962/63), 195–228, DOI 10.4064/fm-51-3-195-228. MR142459
- [60] Tobias Fritz, *Tsirelson's problem and Kirchberg's conjecture*, Rev. Math. Phys. **24** (2012), no. 5, 1250012, 67, DOI 10.1142/S0129055X12500122. MR2928100
- [61] Zdeněk Frolík, *Non-homogeneity of  $\beta P - P$* , Comment. Math. Univ. Carolinae **8** (1967), 705–709. MR266160
- [62] Harry Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. Analyse Math. **31** (1977), 204–256, DOI 10.1007/BF02813304. MR498471
- [63] Anthony M. Gaglione and Dennis Spellman, *Even more model theory of free groups*, Infinite groups and group rings (Tuscaloosa, AL, 1992), Ser. Algebra, vol. 1, World Sci. Publ., River Edge, NJ, 1993, pp. 37–40. MR1377955
- [64] D. Galvin, *Ultrafilters, with applications to analysis, social choice and combinatorics*, unpublished notes, 2009.
- [65] Murray Gerstenhaber and Oscar S. Rothaus, *The solution of sets of equations in groups*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 1531–1533, DOI 10.1073/pnas.48.9.1531. MR166296
- [66] Saeed Ghasemi, *Reduced products of metric structures: a metric Feferman-Vaught theorem*, J. Symb. Log. **81** (2016), no. 3, 856–875, DOI 10.1017/jsl.2016.20. MR3569108
- [67] Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, Graduate Texts in Mathematics, No. 43, Springer-Verlag, New York-Heidelberg, 1976. Reprint of the 1960 edition. MR0407579
- [68] Robert Goldblatt, *Lectures on the hyperreals: An introduction to nonstandard analysis*, Graduate Texts in Mathematics, vol. 188, Springer-Verlag, New York, 1998, DOI 10.1007/978-1-4612-0615-6. MR1643950
- [69] I. Goldbring and H. J. Keisler, *Continuous sentences preserved under reduced products*, to appear in Journal of Symbolic Logic.
- [70] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985. MR764305
- [71] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73. MR623534
- [72] M. Gromov, *Endomorphisms of symbolic algebraic varieties*, J. Eur. Math. Soc. (JEMS) **1** (1999), no. 2, 109–197, DOI 10.1007/PL00011162. MR1694588
- [73] A. Grothendieck and J. A. Dieudonné, *Éléments de géométrie algébrique. I* (French), Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 166, Springer-Verlag, Berlin, 1971. MR3075000
- [74] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes* (French), Inst. Hautes Études Sci. Publ. Math. **8** (1961), 222. MR217084
- [75] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. **11** (1961), 167. MR217085
- [76] R. Gurevič, *On ultraproducts of compact Hausdorff spaces*, J. Symbolic Logic **53** (1988), no. 1, 294–300, DOI 10.2307/2274446. MR929393
- [77] Eric J. Hall, Kyriakos Keremedis, and Eleftherios Tachtsis, *The existence of free ultrafilters on  $\omega$  does not imply the extension of filters on  $\omega$  to ultrafilters*, MLQ Math. Log. Q. **59** (2013), no. 4-5, 258–267, DOI 10.1002/malq.201100092. MR3100753
- [78] C. Ward Henson, *Foundations of nonstandard analysis: a gentle introduction to nonstandard extensions*, Nonstandard analysis (Edinburgh, 1996), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., vol. 493, Kluwer Acad. Publ., Dordrecht, 1997, pp. 1–49. MR1603228

- [79] Horst Herrlich, Kyriakos Keremedis, and Eleftherios Tachtsis, *Remarks on the Stone spaces of the integers and the reals without AC*, Bull. Pol. Acad. Sci. Math. **59** (2011), no. 2, 101–114, DOI 10.4064/ba59-2-1. MR2852866
- [80] Edwin Hewitt, *Rings of real-valued continuous functions. I*, Trans. Amer. Math. Soc. **64** (1948), 45–99, DOI 10.2307/1990558. MR26239
- [81] Neil Hindman, *The existence of certain ultra-filters on  $N$  and a conjecture of Graham and Rothschild*, Proc. Amer. Math. Soc. **36** (1972), 341–346, DOI 10.2307/2039156. MR307926
- [82] Neil Hindman, *Finite sums from sequences within cells of a partition of  $N$* , J. Combinatorial Theory Ser. A **17** (1974), 1–11, DOI 10.1016/0097-3165(74)90023-5. MR349574
- [83] Neil Hindman and Dona Strauss, *Algebra in the Stone-Čech compactification: Theory and applications*, De Gruyter Textbook, Walter de Gruyter & Co., Berlin, 2012. Second revised and extended edition [of MR1642231]. MR2893605
- [84] Paul Howard and Jean E. Rubin, *Consequences of the axiom of choice*, Mathematical Surveys and Monographs, vol. 59, American Mathematical Society, Providence, RI, 1998. With 1 IBM-PC floppy disk (3.5 inch; WD), DOI 10.1090/surv/059. MR1637107
- [85] Paul E. Howard, *Los' theorem and the Boolean prime ideal theorem imply the axiom of choice*, Proc. Amer. Math. Soc. **49** (1975), 426–428, DOI 10.2307/2040659. MR384548
- [86] Ehud Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), no. 1, 189–243, DOI 10.1090/S0894-0347-2011-00708-X. MR2833482
- [87] José Iovino, *Applications of model theory to functional analysis*, Dover Publications, Inc., Mineola, NY, 2014. Revised reprint of the 2002 original; With a new preface, notes, and an updated bibliography. MR3362124
- [88] C. Ward Henson, José Iovino, Alexander S. Kechris, and Edward Odell, *Analysis and logic*, London Mathematical Society Lecture Note Series, vol. 262, Cambridge University Press, Cambridge, 2002. Lectures from the mini-courses offered at the International Conference held at the University of Mons-Hainaut, Mons, August 25–29, 1997; Edited by Catherine Finet and Christian Michaux. MR1967837
- [89] T. Jech, *Set theory*, Springer Science & Business Media, 2002.
- [90] Thomas J. Jech, *The axiom of choice*, Studies in Logic and the Foundations of Mathematics, Vol. 75, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. MR0396271
- [91] Z. Ji, A. Natarajan, T. Vidick, J. Wright, and H. Yuen, *MIP\*=RE*, arXiv preprint arXiv:2001.04383, 2020.
- [92] Renling Jin, *The sumset phenomenon*, Proc. Amer. Math. Soc. **130** (2002), no. 3, 855–861, DOI 10.1090/S0002-9939-01-06088-9. MR1866042
- [93] M. Junge, M. Navasques, C. Palazuelos, D. Perez-Garcia, V. B. Scholz, and R. F. Werner, *Connes embedding problem and Tsirelson's problem*, J. Math. Phys. **52** (2011), no. 1, 012102, 12, DOI 10.1063/1.3514538. MR2790067
- [94] Akihiro Kanamori, *The higher infinite: Large cardinals in set theory from their beginnings*, 2nd ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2009. Paperback reprint of the 2003 edition. MR2731169
- [95] A. Karrass and D. Solitar, *On a theorem of Cohen and Lyndon about free bases for normal subgroups*, Canadian J. Math. **24** (1972), 1086–1091, DOI 10.4153/CJM-1972-112-0. MR320150
- [96] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995, DOI 10.1007/978-1-4612-4190-4. MR1321597
- [97] H. Jerome Keisler, *Ultraproducts and elementary classes*, Nederl. Akad. Wetensch. Proc. Ser. A **64** = Indag. Math. **23** (1961), 477–495. MR0140396
- [98] H. Jerome Keisler, *Limit ultrapowers*, Trans. Amer. Math. Soc. **107** (1963), 382–408, DOI 10.2307/1993808. MR148547



- [99] H. Jerome Keisler, *Good ideals in fields of sets*, Ann. of Math. (2) **79** (1964), 338–359, DOI 10.2307/1970549. MR166105
- [100] H. Jerome Keisler, *Ultraproducts and saturated models*, Nederl. Akad. Wetensch. Proc. Ser. A 67 = Indag. Math. **26** (1964), 178–186. MR0168483
- [101] H. Jerome Keisler, *Ultraproducts which are not saturated*, J. Symbolic Logic **32** (1967), 23–46, DOI 10.2307/2271240. MR218224
- [102] H. Jerome Keisler, *The ultraproduct construction*, Ultrafilters across mathematics, Contemp. Math., vol. 530, Amer. Math. Soc., Providence, RI, 2010, pp. 163–179, DOI 10.1090/conm/530/10444. MR2757537
- [103] Juliette Kennedy, Saharon Shelah, and Jouko Väänänen, *Regular ultrapowers at regular cardinals*, Notre Dame J. Form. Log. **56** (2015), no. 3, 417–428, DOI 10.1215/00294527-3132788. MR3373611
- [104] Kyriakos Keremedis, *Tychonoff products of two-element sets and some weakenings of the Boolean prime ideal theorem*, Bull. Pol. Acad. Sci. Math. **53** (2005), no. 4, 349–359, DOI 10.4064/ba53-4-1. MR2214925
- [105] Jonathan Kirby, *An invitation to model theory*, Cambridge University Press, Cambridge, 2019, DOI 10.1017/9781316683002. MR3967730
- [106] Eberhard Kirchberg, *On nonsemitrivial extensions, tensor products and exactness of group  $C^*$ -algebras*, Invent. Math. **112** (1993), no. 3, 449–489, DOI 10.1007/BF01232444. MR1218321
- [107] V. L. Klee Jr., *Invariant metrics in groups (solution of a problem of Banach)*, Proc. Amer. Math. Soc. **3** (1952), 484–487, DOI 10.2307/2031907. MR47250
- [108] Péter Komjáth and Vilmos Totik, *Ultrafilters*, Amer. Math. Monthly **115** (2008), no. 1, 33–44, DOI 10.1080/00029890.2008.11920493. MR2375774
- [109] Linus Kramer, Saharon Shelah, Katrin Tent, and Simon Thomas, *Asymptotic cones of finitely presented groups*, Adv. Math. **193** (2005), no. 1, 142–173, DOI 10.1016/j.aim.2004.04.012. MR2132762
- [110] J. L. Krivine, *Sous-espaces et cônes convexes dans les espaces  $L_p$* , PhD thesis, Centre national de la recherche scientifique, 1967.
- [111] Kenneth Kunen, *Ultrafilters and independent sets*, Trans. Amer. Math. Soc. **172** (1972), 299–306, DOI 10.2307/1996350. MR314619
- [112] Kenneth Kunen, *Weak  $P$ -points in  $\mathbb{N}^*$* , Colloquia Mathematica Societatis Janos Bolyai, **23** (1978), pp. 741–749.
- [113] Jerzy Łoś, *Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres* (French), Mathematical interpretation of formal systems, North-Holland Publishing Co., Amsterdam, 1955, pp. 98–113. MR0075156
- [114] D. J. Lewis, *Cubic homogeneous polynomials over  $p$ -adic number fields*, Ann. of Math. (2) **56** (1952), 473–478, DOI 10.2307/1969655. MR49947
- [115] W. A. J. Luxemburg, *A general theory of monads*, Applications of Model Theory to Algebra, Analysis, and Probability (Internat. Sympos., Pasadena, Calif., 1967), Holt, Rinehart and Winston, New York, 1969, pp. 18–86. MR0244931
- [116] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR1712872
- [117] Dugald Macpherson, *Model theory of finite and pseudofinite groups*, Arch. Math. Logic **57** (2018), no. 1-2, 159–184, DOI 10.1007/s00153-017-0584-1. MR3749405
- [118] Anatolii Ivanovič Mal'cev, *The metamathematics of algebraic systems. Collected papers: 1936–1967*, Studies in Logic and the Foundations of Mathematics, Vol. 66, North-Holland Publishing Co., Amsterdam-London, 1971. Translated, edited, and provided with supplementary notes by Benjamin Franklin Wells, III. MR0349383

- [119] Maryanthe Elizabeth Malliaris, *Persistence and regularity in unstable model theory*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)—University of California, Berkeley. MR2713926
- [120] M. E. Malliaris, *Hypergraph sequences as a tool for saturation of ultrapowers*, J. Symbolic Logic **77** (2012), no. 1, 195–223, DOI 10.2178/jsl/1327068699. MR2951637
- [121] M. Malliaris and S. Shelah, *Cofinality spectrum theorems in model theory, set theory, and general topology*, J. Amer. Math. Soc. **29** (2016), no. 1, 237–297, DOI 10.1090/jams830. MR3402699
- [122] M. Malliaris and S. Shelah, *An example of a new simple theory*, arXiv preprint arXiv:1804.03254, 2018.
- [123] Maryanthe Malliaris and Saharon Shelah, *Keisler’s order has infinitely many classes*, Israel J. Math. **224** (2018), no. 1, 189–230, DOI 10.1007/s11856-018-1647-7. MR3799754
- [124] M. Malliaris and S. Shelah, *Keisler’s order is not simple (and simple theories may not be either)*, arXiv preprint arXiv:1906.10241, 2019.
- [125] M. Malliaris and S. Shelah, *A new look at interpretability and saturation*, Ann. Pure Appl. Logic **170** (2019), no. 5, 642–671, DOI 10.1016/j.apal.2019.01.001. MR3926500
- [126] David Marker, *Model theory: An introduction*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002. MR1924282
- [127] Donald A. Martin, *The axiom of determinateness and reduction principles in the analytical hierarchy*, Bull. Amer. Math. Soc. **74** (1968), 687–689, DOI 10.1090/S0002-9904-1968-11995-0. MR227022
- [128] Donald A. Martin, *Measurable cardinals and analytic games*, in Mathematical Logic in the 20th Century, World Scientific, 2003, pp. 264–268.
- [129] D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic **2** (1970), no. 2, 143–178, DOI 10.1016/0003-4843(70)90009-4. MR270904
- [130] Donald A. Martin and John R. Steel, *Projective determinacy*, Proc. Nat. Acad. Sci. U.S.A. **85** (1988), no. 18, 6582–6586, DOI 10.1073/pnas.85.18.6582. MR959109
- [131] Antonio Martínez-Abejón, *An elementary proof of the principle of local reflexivity*, Proc. Amer. Math. Soc. **127** (1999), no. 5, 1397–1398, DOI 10.1090/S0002-9939-99-04687-0. MR1476378
- [132] A. R. D. Mathias, *Solution of problems of Choquet and Puritz*, Conference in Mathematical Logic—London ’70 (Bedford Coll., London, 1970), Springer, Berlin, 1972, pp. 204–210. Lecture Notes in Math., Vol. 255. MR0363911
- [133] Hideyuki Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid. MR1011461
- [134] Dusa McDuff, *Central sequences and the hyperfinite factor*, Proc. London Math. Soc. (3) **21** (1970), 443–461, DOI 10.1112/plms/s3-21.3.443. MR281018
- [135] Arnold W. Miller, *There are no  $Q$ -points in Laver’s model for the Borel conjecture*, Proc. Amer. Math. Soc. **78** (1980), no. 1, 103–106, DOI 10.2307/2043048. MR548093
- [136] John Milnor, *Growth of finitely generated solvable groups*, J. Differential Geometry **2** (1968), 447–449. MR244899
- [137] Deane Montgomery and Leo Zippin, *Topological transformation groups*, Interscience Publishers, New York-London, 1955. MR0073104
- [138] Itay Neeman, *Ultrafilters and large cardinals*, Ultrafilters across mathematics, Contemp. Math., vol. 530, Amer. Math. Soc., Providence, RI, 2010, pp. 181–200, DOI 10.1090/conm/530/10445. MR2757538
- [139] Alexander Yu. Ol’shanskii and Mark V. Sapir, *A finitely presented group with two non-homeomorphic asymptotic cones*, Internat. J. Algebra Comput. **17** (2007), no. 2, 421–426, DOI 10.1142/S021819670700369X. MR2310154

- [140] D. Osin and A. O. Houchine, *Finitely presented groups with infinitely many non-homeomorphic asymptotic cones*, to appear in Algebraic and Geometric Topology.
- [141] Narutaka Ozawa, *Tsirelson's problem and asymptotically commuting unitary matrices*, J. Math. Phys. **54** (2013), no. 3, 032202, 8, DOI 10.1063/1.4795391. MR3059438
- [142] Pierre Pansu, *Croissance des boules et des géodésiques fermées dans les nilvariétés* (French, with English summary), Ergodic Theory Dynam. Systems **3** (1983), no. 3, 415–445, DOI 10.1017/S0143385700002054. MR741395
- [143] Vladimir G. Pestov, *Hyperlinear and sofic groups: a brief guide*, Bull. Symbolic Logic **14** (2008), no. 4, 449–480, DOI 10.2178/bsl/1231081461. MR2460675
- [144] Bedřich Pospíšil, *Remark on bicomact spaces*, Ann. of Math. (2) **38** (1937), no. 4, 845–846, DOI 10.2307/1968840. MR1503375
- [145] Florin Rădulescu, *The von Neumann algebra of the non-residually finite Baumslag group  $\langle a, b | ab^3a^{-1} = b^2 \rangle$  embeds into  $R^\omega$* , Hot topics in operator theory, Theta Ser. Adv. Math., vol. 9, Theta, Bucharest, 2008, pp. 173–185. MR2436761
- [146] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. (2) **30** (1929), no. 4, 264–286, DOI 10.1112/plms/s2-30.1.264. MR1576401
- [147] V. N. Remeslennikov,  *$\exists$ -free groups* (Russian), Sibirsk. Mat. Zh. **30** (1989), no. 6, 193–197, DOI 10.1007/BF00970922; English transl., Siberian Math. J. **30** (1989), no. 6, 998–1001 (1990). MR1043446
- [148] Abraham Robinson, *Non-standard analysis*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1996. Reprint of the second (1974) edition; With a foreword by Wilhelmus A. J. Luxemburg, DOI 10.1515/9781400884223. MR1373196
- [149] Walter Roelcke and Susanne Dierolf, *Uniform structures on topological groups and their quotients*, Advanced Book Program, McGraw-Hill International Book Co., New York, 1981. MR644485
- [150] Mary Ellen Rudin, *Partial orders on the types in  $\beta N$* , Trans. Amer. Math. Soc. **155** (1971), 353–362, DOI 10.2307/1995690. MR273581
- [151] Volker Runde, *Lectures on amenability*, Lecture Notes in Mathematics, vol. 1774, Springer-Verlag, Berlin, 2002, DOI 10.1007/b82937. MR1874893
- [152] Mark Sapir, *On groups with locally compact asymptotic cones*, Internat. J. Algebra Comput. **25** (2015), no. 1-2, 37–40, DOI 10.1142/S0218196715400020. MR3325875
- [153] Hans Schoutens, *The use of ultraproducts in commutative algebra*, Lecture Notes in Mathematics, vol. 1999, Springer-Verlag, Berlin, 2010, DOI 10.1007/978-3-642-13368-8. MR2676525
- [154] D. Scott, *Measurable cardinals and constructible sets*, in Mathematical Logic in the 20th Century, World Scientific, 2003, pp. 407–410.
- [155] Zlil Sela, *Diophantine geometry over groups. I. Makanin-Razborov diagrams*, Publ. Math. Inst. Hautes Études Sci. **93** (2001), 31–105, DOI 10.1007/s10240-001-8188-y. MR1863735
- [156] Zlil Sela, *Diophantine geometry over groups VIII: Stability*, Annals of Mathematics, **177** (2013), pp. 787–868.
- [157] Saharon Shelah, *Every two elementarily equivalent models have isomorphic ultrapowers*, Israel J. Math. **10** (1971), 224–233, DOI 10.1007/BF02771574. MR297554
- [158] Saharon Shelah, *Saturation of ultrapowers and Keisler's order*, Ann. Math. Logic **4** (1972), 75–114, DOI 10.1016/0003-4843(72)90012-5. MR294113
- [159] Saharon Shelah, *Proper forcing*, in Proper Forcing, Springer, 1982, pp. 73–113.
- [160] Saharon Shelah, *Vive la différence I: Nonisomorphism of ultrapowers of countable models*, in Set Theory of the Continuum, Springer, 1992, pp. 357–405.
- [161] Saharon Shelah, *Proper and improper forcing*, vol. 5 of Perspectives in Logic, Cambridge University Press, 2017.

- [162] S. Shelah and M. E. Rudin, *Unordered types of ultrafilters*, *Topology Proc.* **3** (1978), no. 1, 199–204 (1979). MR540490
- [163] W. Sierpiński, *Fonctions additives non complètement additives et fonctions non mesurables*, *Fundamenta Mathematicae*, **1** (1938), pp. 96–99.
- [164] T. Skolem, *Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen*, *Fundamenta mathematicae*, **23** (1934), pp. 150–161.
- [165] R. Solovay, *Measurable cardinals and the axiom of determinateness*, *Lecture Notes of UCLA Summer Institute on Axiomatic Set Theory*, Los Angeles, (1967).
- [166] R. M. Solovay,  $2^{\aleph_0}$  can be anything it ought to be, in *The Theory of Models*, *Proceedings of the 1963 International Symposium at Berkeley*, *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1965, p. 435.
- [167] R. M. Solovay, *On the cardinality of  $\sigma_2^1$  sets of reals*, in *Foundations of Mathematics*, Springer, 1969, pp. 58–73.
- [168] Robert M. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, *Ann. of Math. (2)* **92** (1970), 1–56, DOI 10.2307/1970696. MR265151
- [169] M. H. Stone, *The theory of representations for Boolean algebras*, *Trans. Amer. Math. Soc.* **40** (1936), no. 1, 37–111, DOI 10.2307/1989664. MR1501865
- [170] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, *Trans. Amer. Math. Soc.* **41** (1937), no. 3, 375–481, DOI 10.2307/1989788. MR1501905
- [171] E. Szemerédi, *On sets of integers containing no  $k$  elements in arithmetic progression*, *Acta Arith.* **27** (1975), 199–245, DOI 10.4064/aa-27-1-199-245. MR369312
- [172] Terence Tao, *Hilbert’s fifth problem and related topics*, *Graduate Studies in Mathematics*, vol. 153, American Mathematical Society, Providence, RI, 2014, DOI 10.1088/0253-6102/41/3/335. MR3237440
- [173] A. Tarski, *Une contribution à la théorie de la mesure*, *Fundamenta Mathematicae*, **15** (1930), pp. 42–50.
- [174] Katrin Tent and Martin Ziegler, *A course in model theory*, *Lecture Notes in Logic*, vol. 40, Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012, DOI 10.1017/CBO9781139015417. MR2908005
- [175] Guy Terjanian, *Un contre-exemple à une conjecture d’Artin* (French), *C. R. Acad. Sci. Paris Sér. A-B* **262** (1966), A612. MR197450
- [176] Simon Thomas and Boban Velickovic, *Asymptotic cones of finitely generated groups*, *Bull. London Math. Soc.* **32** (2000), no. 2, 203–208, DOI 10.1112/S0024609399006621. MR1734187
- [177] Stevo Todorćević, *Introduction to Ramsey spaces*, *Annals of Mathematics Studies*, vol. 174, Princeton University Press, Princeton, NJ, 2010, DOI 10.1515/9781400835409. MR2603812
- [178] Douglas Ulrich, *Keisler’s order is not linear, assuming a supercompact*, *J. Symb. Log.* **83** (2018), no. 2, 634–641, DOI 10.1017/jsl.2018.1. MR3835081
- [179] Lou van den Dries, *Lectures on the model theory of valued fields*, *Model theory in algebra, analysis and arithmetic*, *Lecture Notes in Math.*, vol. 2111, Springer, Heidelberg, 2014, pp. 55–157, DOI 10.1007/978-3-642-54936-6\_4. MR3330198
- [180] L. van den Dries and K. Schmidt, *Bounds in the theory of polynomial rings over fields. A nonstandard approach*, *Invent. Math.* **76** (1984), no. 1, 77–91, DOI 10.1007/BF01388493. MR739626
- [181] L. van den Dries and A. J. Wilkie, *Gromov’s theorem on groups of polynomial growth and elementary logic*, *J. Algebra* **89** (1984), no. 2, 349–374, DOI 10.1016/0021-8693(84)90223-0. MR751150
- [182] Nik Weaver, *Forcing for mathematicians*, *World Scientific Publishing Co. Pte. Ltd.*, Hackensack, NJ, 2014, DOI 10.1142/8962. MR3184751

- [183] Benjamin Weiss, *Sofic groups and dynamical systems*, Sankhyā Ser. A **62** (2000), no. 3, 350–359. Ergodic theory and harmonic analysis (Mumbai, 1999). MR1803462
- [184] Fred B. Wright, *A reduction for algebras of finite type*, Ann. of Math. (2) **60** (1954), 560—570, DOI 10.2307/1969851. MR65037

---

# Index

- $\Delta_r$ -set, 55
- $\Delta_r^*$ -set, 55
- $\alpha$ -inaccessible cardinal, 313
- $\epsilon$ -isomorphism, 254
- $\kappa$ -regularizing set, 129
- $\kappa$ -saturated structure, 124
- $\kappa$ -universal structure, 126
- $C^*$ -algebra, 104, 260
  - faithful tracial state on  $a$ , 267
  - projection in  $a$ , 262
  - real rank 0, 263
  - tracial state on  $a$ , 267
  - unitary in  $a$ , 275
- $p$ -adic integers, 119
- $z$ -filter, 36
- $z$ -ultrafilter, 33, 36
- Łoś's theorem
  - for  $\mathcal{L}_\kappa$ , 321
  - for continuous logic, 206
  - for first-order logic, 87
  
- abelianization of a group, 212
- amenable group, 240
- approximate subgroup, 229
  - property, 229
- Arrow's theorem, 19, 20
- Artin's conjecture, 121
- asymptotic cone, 221
- Ax's theorem, 110
- Ax-Kochen-Ershov theorem, 121
  
- axiom of determinacy, 68
  - for a pointclass, 68
  
- Baire category theorem, 64
- Baire measurable set, 64
- Baire space, 63
- Banach  $*$ -algebra, 259
- Banach algebra, 253
- Banach density, 50, 57
- Banach space, 250
  - complemented subspace of  $a$ , 257
  - dual space of  $a$ , 255
  - reflexive, 256
  - super-reflexive, 256
  - uniformly convex, 250
  - weak\*-topology on  $a$ , 258
- block decisive set of voters, 23
- Boolean algebra, 39
  - concrete, 39
- Borel determinacy, 69
- Borel sets, 63
- bounded linear transformation, 252
  - adjoint of  $a$ , 260
  
- Cantor space, 63
- category, 357
- club filter, 313
- club subset of a cardinal, 313
- cofinality, 352
- comeager, 64
- commutative transitive group, 184

- compactification, 32
- compactness theorem, 89
- complete embedding, 164
- complete expansion, 164
- complete extension, 164
- complete structure, 164
- completely regular space, 34
- completely separated, 37
- computability theory, 318
- Connes embedding problem, 274
- consistency strength, 310
- consistent pair, 142, 298
- consistent triple, 300
- constructible set, 111, 116
- continuous language, 203
- continuous logic, 201
- continuum hypothesis, 13, 72
- critical point, 329
- CSA group, 185
  
- decisive set of voters, 22
- definable
  - function, 111
  - set, 111
- density character of a metric space, 200
- derived subgroup, 212
- descriptive set theory, 62
- determined game, 67
- diagonal embedding, 84
  - for metric ultrapowers, 197
- diagram, 345
  - elementary, 345
- dictator, 20
- distribution, 136
  - accurate, 137
- election, 19
  - state of  $a$ , 20
- election procedure, 20
  - fair, 20
- elementary class, 305
- elementary equivalence, 344
- embedding, 345
  - elementary, 345
- equivalent categories, 360
- Erdős cardinal, 334
- existentially closed structure, 128
- expansion, 342
- external set, 170
  
- Følner sequence, 240
- Følner set, 240
- faithfully flat ring extension, 115
- Feferman-Vaught theorem, 104
- filter, 3
  - base for  $a$ , 4
  - Fréchet, 4
  - generated by a base, 4
  - on a Boolean algebra, 39
- finite cover property (fcp), 147
- finite intersection property (FIP), 4
- finitely representable, 254
- flat ring extension, 112
- formulae, 343
  - atomic, 343
- Frayne's theorem, 131
- FS-set, 47
- fully residually free group, 186
- functor, 359
- Furstenberg's correspondence
  - principle, 53
- Furstenberg's recurrence theorem, 54
  
- galaxy, 161
- Gale-Stewart theorem, 334
- gap, 284
- Gelfand duality, 104, 261
- geodesic metric space, 224
- GNS construction, 271
- Greenleaf-Ax-Kochen theorem, 120
- Gromov's theorem on polynomial
  - growth, 216
- growth function of a group, 213
  - exponential growth, 215
  - near polynomial growth, 226
  - polynomial growth, 214
  
- Hausdorff dimension, 227
- Heisenberg group, 212
- Hilbert space, 252
- Hilbert-Schmidt metric, 235
- Hindman's theorem, 47
  - strong version, 49
- homogeneous metric space, 223
- hyperfinite  $\text{II}_1$  factor, 274

- hyperfinite set, 173
  - internal cardinality of  $\mathfrak{a}$ , 173
- hyperlinear group, 247
- hyperreal numbers, 158
  - finite elements, 159
  - infinite elements, 159
  - infinitesimal elements, 158, 159
- ideal on a set, 64
  - $\sigma$ -, 64
- idempotent hyperinteger, 175
- independence property, 153
- independence relation, 293
- independent sequence, 293
- indicable group, 212
  - kernel of, 212
- indiscernible sequence, 293
- induced embedding between
  - ultrapowers, 165
- inner product space, 252
- internal definition principle, 171
- internal set, 170, 198
- iterated ultrapower, 98, 99, 166
- Jin's sunset theorem, 55, 56
- Keisler's order, 146, 288
- Keisler-Shelah theorem, 100, 141, 297
- Kervaire-Lajudembach conjecture, 246
- KL-group, 246
  - strong, 246
- Kleene-Brouwer ordering, 335
- language, 341
- large oscillation, 142
- Lebesgue measurable set, 63
- Lie group, 218
- limit group, 182
- limit ultrapower, 166
- Lindenbaum algebra, 82
- local ring, 117
  - henselian, 119
- lower cofinality, 285
- Mahlo cardinal, 314
- malnormal subgroup, 185
- marked group(s), 189
  - isomorphic, 189
  - space of, 189
  - ultraproducts of, 190
- meager set, 64
- mean, 51
  - invariant, 52
- measurable cardinal, 69, 93, 314
- measure-preserving transformation, 53
- metric group, 233
  - bi-invariant, 234
  - uniform, 233
- metric structure, 201, 203
- Milnor-Wolf theorem, 216
- modulus of uniform continuity, 202
- Morley sequence, 293
- natural transformation, 360
- nfcf, 147, 283
- nilpotent group, 212
- nonstandard hull, 198
- normalized Hamming metric, 235
- normed algebra, 253
  - unital, 253
- normed space(s), 250
  - ultraproduct of, 250
- nowhere dense set, 64
- null set, 63
- operator norm, 252
- ordinal, 350
- orthogonal projection, 262
- overflow principle, 171
- P-point, 73
- partition regular property, 49, 56
- perfectly normal space, 35
- piecewise syndetic set, 55, 57
- point-homogeneous space, 75
- pointed metric space, 196
- Polish space, 62, 63
- polycyclic group, 216
- powerset algebra, 39
- principle of local reflexivity, 257
- projective set, 65
- proper metric space, 198
- pseudo-intersection, 72



- Ramsey cardinal, 325
- Ramsey's theorem
  - finite version, 89
  - infinite version, 46
- reduced power
  - of sets, 84
  - of structures, 85
- reduced product
  - of metric spaces, 208
  - of metric structures, 209
  - of sets, 84
  - of structures, 85
- reduct, 342
- regular cardinal, 352
- regular equation, 246
- residually finite group, 243
- residually free group, 186
- residue field, 117
- Robinson's joint consistency
  - theorem, 306
- Rudin-Keisler order, 14, 94, 98
  
- S-subsemigroup, 175
- S-topology, 174
- saturated structure, 124
- Scott's theorem, 330
- sentence, 344
- singular cardinal, 352
- sofic group, 235
- solvable group, 212
- stable
  - formula, 282
  - structure, 283
  - theory, 149, 283
- standard part, 160
- stationary subset of a cardinal, 313
- Stone duality theorem, 43
- Stone representation theorem, 38, 44
- Stone space, 38
  - of a Boolean algebra, 40
- Stone-Čech compactification, 31, 32
- strict order property, 153
- strong limit cardinal, 311
- strong operator topology, 265
- strongly compact cardinal, 321
- strongly inaccessible cardinal, 311
- structure, 341
- substructure, 345
  - elementary, 345
- supercompact cardinal, 323
- supercompact elementary
  - embedding, 331
- syndetic set, 54
- Szemerédi's theorem, 54
  
- tail set, 64
- terms, 342
- theory, 344
  - complete, 344
  - of a class of structures, 305
  - satisfiable, 344
- thick set, 50
- tracial ultraproduct, 270
- tree, 335
  - ill founded, 335
  - infinite branch in  $a$ , 335
  - root of  $a$ , 335
  - well founded, 335
- Turing degree(s), 318
  - cone of, 318
- Turing reducible, 318
- Tychonoff space, 34
- Tychonoff's theorem, 30, 61
- type, 292
  
- ultrafilter theorem, 6
  - for Boolean algebras, 40
  - intermediate version, 60
  - weak version, 60
- ultrafilter(s), 4
  - $\kappa$ -regular, 129
  - $\kappa^+$ -good, 136
  - countably complete, 92
  - countably incomplete, 92
  - fine, 322
  - game, 66
  - good, 136
  - Hausdorff, 177
  - idempotent, 48, 61, 75
  - induced, 5, 11
  - isomorphic, 9
  - minimal, 70, 95
  - morphism between, 8
  - nonprincipal, 5
  - normal, 316, 322

- number, 13, 15
- on a Boolean algebra, 39
- principal, 5
- product of, 16, 98
- pushforward, 8, 41
- quantifier, 6
- quasi-normal, 71
- Ramsey, 70
- regular, 101, 129
- saturating a theory, 146
- selective, 70
- strongly summable, 75
- sum of, 47
- uniform, 6
- union, 76
- weakly selective, 73
- weakly summable, 75
- ultralimit, 27
- ultrapower
  - chain, 100
  - extension, 100
  - of sets, 84
  - of structures, 85
  - of the set-theoretic universe, 327
  - system, 166
- ultraproduct
  - embedding, 88
  - of metric structures, 204
  - of sets, 84
  - of structures, 85
- uncountably categorical theory, 146
- underflow principle, 171
- universal structure, 126
- universal theory, 132
- universally free group, 182
  
- virtual property of a group, 212
- von Neumann algebra, 265
  - normal tracial state on a, 271
  - of a group, 266
  - trace on a, 271
  - tracial, 271
  
- weakly compact cardinal, 324
- weakly inaccessible cardinal, 311
- Woodin cardinals, 337
- word-length function, 213
- worldly cardinal, 310
  
- Zariski closed set, 111
- zero-dimensional, 31
- zeroset, 35



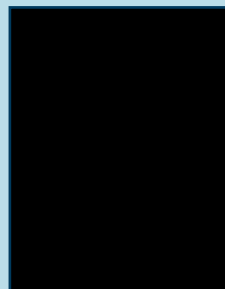
## Selected Published Titles in This Series

- 220 **Isaac Goldbring**, *Ultrafilters Throughout Mathematics*, 2022
- 219 **Michael Joswig**, *Essentials of Tropical Combinatorics*, 2021
- 218 **Riccardo Benedetti**, *Lectures on Differential Topology*, 2021
- 217 **Marius Crainic, Rui Loja Fernandes, and Ioan Mărcuț**, *Lectures on Poisson Geometry*, 2021
- 216 **Brian Osserman**, *A Concise Introduction to Algebraic Varieties*, 2021
- 215 **Tai-Ping Liu**, *Shock Waves*, 2021
- 214 **Ioannis Karatzas and Constantin Kardaras**, *Portfolio Theory and Arbitrage*, 2021
- 213 **Hung Vinh Tran**, *Hamilton–Jacobi Equations*, 2021
- 212 **Marcelo Viana and José M. Espinar**, *Differential Equations*, 2021
- 211 **Mateusz Michałek and Bernd Sturmfels**, *Invitation to Nonlinear Algebra*, 2021
- 210 **Bruce E. Sagan**, *Combinatorics: The Art of Counting*, 2020
- 209 **Jessica S. Purcell**, *Hyperbolic Knot Theory*, 2020
- 208 **Vicente Muñoz, Ángel González-Prieto, and Juan Ángel Rojo**, *Geometry and Topology of Manifolds*, 2020
- 207 **Dmitry N. Kozlov**, *Organized Collapse: An Introduction to Discrete Morse Theory*, 2020
- 206 **Ben Andrews, Bennett Chow, Christine Guenther, and Mat Langford**, *Extrinsic Geometric Flows*, 2020
- 205 **Mikhail Shubin**, *Invitation to Partial Differential Equations*, 2020
- 204 **Sarah J. Witherspoon**, *Hochschild Cohomology for Algebras*, 2019
- 203 **Dimitris Koukoulopoulos**, *The Distribution of Prime Numbers*, 2019
- 202 **Michael E. Taylor**, *Introduction to Complex Analysis*, 2019
- 201 **Dan A. Lee**, *Geometric Relativity*, 2019
- 200 **Semyon Dyatlov and Maciej Zworski**, *Mathematical Theory of Scattering Resonances*, 2019
- 199 **Weinan E, Tiejun Li, and Eric Vanden-Eijnden**, *Applied Stochastic Analysis*, 2019
- 198 **Robert L. Benedetto**, *Dynamics in One Non-Archimedean Variable*, 2019
- 197 **Walter Craig**, *A Course on Partial Differential Equations*, 2018
- 196 **Martin Stynes and David Stynes**, *Convection-Diffusion Problems*, 2018
- 195 **Matthias Beck and Raman Sanyal**, *Combinatorial Reciprocity Theorems*, 2018
- 194 **Seth Sullivant**, *Algebraic Statistics*, 2018
- 193 **Martin Lorenz**, *A Tour of Representation Theory*, 2018
- 192 **Tai-Peng Tsai**, *Lectures on Navier-Stokes Equations*, 2018
- 191 **Theo Bühler and Dietmar A. Salamon**, *Functional Analysis*, 2018
- 190 **Xiang-dong Hou**, *Lectures on Finite Fields*, 2018
- 189 **I. Martin Isaacs**, *Characters of Solvable Groups*, 2018
- 188 **Steven Dale Cutkosky**, *Introduction to Algebraic Geometry*, 2018
- 187 **John Douglas Moore**, *Introduction to Global Analysis*, 2017
- 186 **Bjorn Poonen**, *Rational Points on Varieties*, 2017
- 185 **Douglas J. LaFountain and William W. Menasco**, *Braid Foliations in Low-Dimensional Topology*, 2017
- 184 **Harm Derksen and Jerzy Weyman**, *An Introduction to Quiver Representations*, 2017
- 183 **Timothy J. Ford**, *Separable Algebras*, 2017
- 182 **Guido Schneider and Hannes Uecker**, *Nonlinear PDEs*, 2017
- 181 **Giovanni Leoni**, *A First Course in Sobolev Spaces, Second Edition*, 2017

For a complete list of titles in this series, visit the  
AMS Bookstore at [www.ams.org/bookstore/gsmseries/](http://www.ams.org/bookstore/gsmseries/).

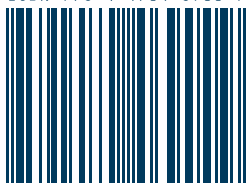
Ultrafilters and ultraproducts provide a useful generalization of the ordinary limit processes which have applications to many areas of mathematics. Typically, this topic is presented to students in specialized courses such as logic, functional analysis, or geometric group theory. In this book, the basic facts about ultrafilters and ultraproducts are presented to readers with no prior knowledge of the subject and then these techniques are applied to a wide variety of topics. The first part of the book deals solely with ultrafilters and presents applications to voting theory, combinatorics, and topology, while also dealing also with foundational issues. The second part presents the classical ultraproduct construction and provides applications to algebra, number theory, and nonstandard analysis. The third part discusses a metric generalization of the ultraproduct construction and gives example applications to geometric group theory and functional analysis. The final section returns to more advanced topics of a more foundational nature.

The book should be of interest to undergraduates, graduate students, and researchers from all areas of mathematics interested in learning how ultrafilters and ultraproducts can be applied to their specialty.



© 2022 Laurel Hungerford Photography

ISBN 978-1-4704-6900-9



9 781470 469009

GSM/220.H



For additional information  
and updates on this book, visit

[www.ams.org/bookpages/gsm-220](http://www.ams.org/bookpages/gsm-220)

