Non-Euclidean Geometry in the Theory of Automorphic Functions

Jacques Hadamard

Jeremy J. Gray
and
Abe Shenitzer
Editors

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Jeremy J. Gray and Abe Shenitzer, Editors

Translated by Abe Shenitzer

With Historical Introduction by Jeremy J. Gray

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by Jacques Hadamard


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Introduction by the Publishers of the Russian Translation

In the Editors’ Introduction to B.A. Fuks’ book *Non-euclidean geometry in the theory of conformal and pseudoconformal mappings*, volume V in this series, it was mentioned that the eminent French mathematician J. Hadamard wrote in the 1920s a monograph in connection with the preparation of an edition of the collected works of N.I. Lobachevski. The author’s manuscript was translated by A.V. Vasil’ev and edited by B.A. Fuks.

Hadamard’s monograph is a survey, and most of its propositions are stated without proof. This called for a number of notes. The notes were written by B.A. Fuks.

Fuks’ book can serve as an introduction to the Hadamard monograph.

The possibility of establishing a Lobachevskian metric in a simply connected region of the complex plane provided in the past the stimulus for the discovery of automorphic functions. The metric in question played an essential, and at times crucial, role in all stages of the construction of the grand edifice of these functions which have such important applications to many problems of mathematical analysis.

The basic aim of Hadamard’s small book is to demonstrate the fundamental importance of the Lobachevskian metric for the theory of automorphic functions.

While the author usually omits proofs, in most cases he provides their underlying ideas or outlines. When he does this, he tends to emphasize the significance of the relevant propositions or facts of Lobachevskian geometry for each argument.

By now it is clear that Hadamard’s book cannot serve as a textbook on the theory of automorphic functions and that it can only be recommended to readers who have a certain amount of knowledge of this theory. A good source for the required background knowledge is chapters II and III of the recently published second edition of V.V. Golubev’s *Lectures on the analytic theory of differential equations*, Gostekhizdat, M.-L., 1950. Chapters II and III of B.A. Fuks’ *Non-euclidean geometry in the theory of conformal and pseudoconformal mappings* (Gostekhizdat 1951), volume V in this series, are a good source of indispensable information about the Lobachevskian metric in a simply connected domain of the complex plane, about the group of motions generated by this metric, and about its properly discontinuous subgroups.

Chapter I of Hadamard’s book, “The group of motions of the Lobachevskian plane and its properly discontinuous subgroups”, is of an introductory nature. It describes the subsequently used realizations of the Lobachevskian plane, brings together the most important properties of the properly discontinuous subgroups of its group of motions, and partly explains the characteristic singular features of their fundamental domains.
Chapter II, “Properly discontinuous groups of 3 geometries. Fuchsian groups”, is devoted to the study of the properly discontinuous subgroups of the groups of motions of the geometries of Riemann, Euclid, and Lobachevski. (In particular, it deals with the conditions under which a polygon in the Lobachevskian plane can serve as a fundamental region of a properly discontinuous subgroup of its group of motions.)

Chapter III deals with Fuchsian functions and Chapter IV with Kleinian functions; in particular, these chapters include the theory of Poincaré’s theta series.

Chapter V is devoted to applications of the automorphic functions constructed in the earlier chapters to the problem of uniformization of algebraic curves and to the solution of ordinary linear differential equations with algebraic coefficients.

The last chapter, Chapter VI, is titled “Fuchsian groups and geodesic lines”. It is relatively short and is in the nature of an appendix. The book includes references to works that contain comprehensive accounts of touched-on issues. Additional references are found in the editor’s notes.

REMARK. In the English translation, the Russian footnotes appear as Notes at the end of each of the six chapters. (Eds.)
References


Additional References


Non-Euclidean Geometry in the Theory of Automorphic Functions
Jacques Hadamard

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This is the English translation of a volume originally published only in Russian and now out of print. The book was written by Jacques Hadamard on the work of Poincaré.

Poincaré’s creation of a theory of automorphic functions in the early 1880s was one of the most significant mathematical achievements of the nineteenth century. It directly inspired the uniformization theorem, led to a class of functions adequate to solve all linear ordinary differential equations, and focused attention on a large new class of discrete groups. It was the first significant application of non-Euclidean geometry. The implications of these discoveries continue to be important to this day in numerous different areas of mathematics.

Hadamard begins with hyperbolic geometry, which he compares with plane and spherical geometry. He discusses the corresponding isometry groups, introduces the idea of discrete subgroups, and shows that the corresponding quotient spaces are manifolds. In Chapter 2 he presents the appropriate automorphic functions, in particular, Fuchsian functions. He shows how to represent Fuchsian functions as quotients and how Fuchsian functions invariant under the same group are related, and indicates how these functions can be used to solve differential equations. Chapter 4 is devoted to the outlines of the more complicated Kleinian case. Chapter 5 discusses algebraic functions and linear algebraic differential equations, and the last chapter sketches the theory of Fuchsian groups and geodesics.

This unique exposition by Hadamard offers a fascinating and intuitive introduction to the subject of automorphic functions and illuminates its connection to differential equations, a connection not often found in other texts.

This book is the second in an informal sequence of works called “History of Mathematics, Sources”, to be included within the History of Mathematics series, co-published by the AMS and the London Mathematical Society. Volumes to be published within this subset are classical mathematical works that served as cornerstones for modern mathematical thought. (For another historical translation on this topic, see Sources of Hyperbolic Geometry, volume 10 in the History of Mathematics series.)