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Extension Theory
Die Ausdehnungslehre.

Vollständig und in strenger Form bearbeitet von

Hermann Grassmann, Professor am Gymnasium zu Stettin.

BERLIN, 1862.
VERLAG VON TH. CHR. FR. ENSLIN.
(ADOLPH ENSLIN.)
Extension Theory

Hermann Grassmann

Translated by Lloyd C. Kannenberg

American Mathematical Society
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by Hermann Grassmann

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Translator’s Note

GRASSMANN’s Ausdehnungslehre of 1862 is his most mature presentation of his system, and is unique in capturing the full sweep of his mathematical achievement. Stripped of the philosophical drapery of his earlier Lineale Ausdehnungslehre of 1844, the “Second Ausdehnungslehre” also contains an enormous amount of material not included (or only suggested) in the earlier book (which, as its title suggests, was deliberately restricted to the “lineale” aspects of the theory, by which he meant that part associated with linear constructions, or, in its two-dimensional realization, with constructions using the straightedge of elementary geometry). Among this new material is a detailed development of the inner product and its relation to the concept of angle, the “theory of functions” from the point of view of extension theory, and GRASSMANN’s fundamental contributions to the problem of PFaff. Curiously, one topic mentioned in the Foreword of the “First Ausdehnungslehre” that does not appear in the Second is the geometric exponential magnitude and the related geometric logarithm. In many ways, not least of which is its “rigorous Euclidean form”, the Second Ausdehnungslehre is the version of GRASSMANN’s system most accessible to contemporary readers.

This translation is based on the version of the Ausdehnungslehre ("A2") appearing in Hermann Grassmann’s gesammelte mathematische und physikalische Werke, published by B. G. Teubner under the general editorship of F. ENGEL (which volumes will be referred to here as the “Teubner Edition”). Since the editors of this volume (ENGEL and one of GRASSMANN’s sons, Hermann Junior) hoped that it would become a working reference for active mathematicians rather than remain primarily of historical interest, they allowed themselves liberties with the A2 that were not taken with the other publications in the collection (aside from the Lehrbuch der Mathematik, only excerpts of which were published in the Teubner Edition). Thus, in addition to the usual editorial work of correcting misprints, altering a word or construction here and there for clarity, and appending Editorial Comments as end notes, they rearranged some of GRASSMANN’s theorems to smooth the flow of the argument (e.g. No. 132 was moved and renumbered 116b), interpolated new theorems of their own to flesh out and complete the discussion (e.g. No. 230a), replaced some of GRASSMANN’s Proofs with versions of their own, where they regarded the originals as unsatisfactory (e.g. Part 1 of the Proof of No. 172), and in at least one instance deleted one of GRASSMANN’s Remarks as being unclear (the Remark following No. 213). In fact, at the time of publication (1896) of this Volume their hope was not unreasonable. As Michael CROWE has documented in his magisterial History of Vector Analysis (Notre Dame, 1967; Dover reprint, 1985), there was at that time a “struggle for existence” between several vectorial systems, and such an “improved” A2 might have been expected to promote the cause of the GRASSMANNian system. In the event, of course, the GRASSMANNian system
was not generally adopted, and thus today interest in the $A_2$ is indeed more as a historical document (albeit an extraordinarily interesting one) than as a tool for mathematical research.

These circumstances left me with decisions to make about how to handle such editorial "improvements". Since my purpose has been to make Grassmann available to an English-speaking audience, I had no qualms about letting stand corrections to misprints and minor stylistic improvements; similarly, the rearrangements and even interpolations remained in place, although they are identified as such, either in Footnotes or by being enclosed \{between braces\}. Similarly, the Editorial Notes, excepting one or two lengthy digressions, have been included almost intact (more on the Editorial Notes below). On the other hand, the replacements and deletions could not pass muster as they stood. Grassmann's originals are restored to their proper places in this translation, but they are marked \{with braces\}, and the materials that replaced them in the Teubner Edition now appear in the Editorial Notes. Naturally there was no question of excluding the helpful figures that the editors of the Teubner Edition added to the Work.

Grassmann's nomenclature presents problems of an altogether different order. While a translator ought not inflict his technical problems on the patient reader, it is necessary to say at least a word or two here about my treatment of Grassmann's novel technical vocabulary. There is, unfortunately, no consistency among the experts about the proper English equivalents for Grassmann's technical terms. In fact Grassmann himself wavered somewhat in his native German; thus in the Ausdehnungslehre of 1844 (the "A$_1$") he used "Abschattung" for what, in the $A_2$, he called "Zurückleitung". Here and in my translation I have used "shadow" for this concept, which seems to have been the term preferred by Whitehead in his Universal Algebra. Two other contentious terms are "Strecke" and "Verknüpfung", which I have translated as "displacement" and "conjunction", respectively. English-speaking Grassmann enthusiasts can and do argue about the best equivalents, and I am afraid my choices may not please them much; but if one were to wait until all such controversies were settled, at least another century would surely pass before an Englished Grassmann appeared. In general I have tried to be consistent with the choices used in my translation of the $A_1$. Finally, it is necessary to mention explicitly two expressions that might possibly lead to some confusion. First, the word "Hauptgebiet" is rendered throughout as "principal domain", and is so indicated in the Index of Technical Terms; it refers to the underlying vector space of the system, and must not be confused with the expression "principal ideal domain": Second, the word "Stufe", consistently translated as "order", is used in two distinct ways, the first as the "order of a (principal) domain", roughly the dimension of the underlying space, and second as the "order of an elementary magnitude", approximately its rank in tensor-like terms.

The Editorial Notes have their own interest, both as elaborations and elucidations of Grassmann's text, and as illustrating turn-of-the-century attitudes toward his work by competent and reasonably sympathetic mathematicians.

Following the Editorial Notes is a set of Supplementary Notes keyed to the Chapters in the two Parts of the book. These Notes are intended as a guide to help elucidate, in modern terms, what Grassmann actually did.

It is a pleasure to acknowledge the deep debt I owe to Prof. Alvin Swimmer and Prof. George B. Seligman, both of whom very kindly read through the entire manuscript and provided extensive lists of corrigenda together with invaluable
suggestions and comments. The addition of Supplementary Notes was inspired by Prof. Seligman. With their help this will be a much more useful work. Whatever defects remain are entirely my responsibility.

To conclude: This completes the second installment of my project of translating GRASSMANN’s principal works into English. It is pleasant to contemplate collecting the ten pounds (plus accrued interest?) that R. W. GENES offere( Nature 48, 517 (1893)) for the translation of GRASSMANN’s Ausdehnungslehre; but that is, I suppose, rather too much to hope for. On the other hand, if these translations produce even a few new GRASSMANN enthusiasts in the English-speaking community, the effort will have been worth the trouble. This translation I dedicate to the memory of my paternal grandfather, my first German teacher.

L. Kannenberg
Weston, 1999
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Foreword

The present work comprises the complete theory of extension, a mathematical science, the first volume of which I already published seventeen years ago under the title *Linear Extension Theory, a New Branch of Mathematics; Otto Wigand Co.*, Leipzig 1844. In the Foreword to that work I had also indicated the principal topics which, according to my plan, were to be included in the contents of the second volume. Now, instead of publishing the second volume as a continuation of the first, and thus concluding the work in conformity with that plan, I have preferred to take up those topics previously treated in this new work as well, thus to provide a [single] coherent whole.

The principal reason that moved me to this decision is the difficulty which, according to all the mathematicians whose opinions I had the opportunity to hear, the study of that work caused the reader on account of what they believed to be its more philosophical than mathematical form. And this difficulty must in fact have been considerable, since the geometric articles that I have written on the interpretation of that work (Crelles Journal 31, 111–137 (1846), 36, 177–184 (1848), 42, 187–212 (1851), 44, 1–25 (1852), 49, 1–65, 123–141 (1855), 52, 254–275 (1856); Geometrische Analyse, Leipzig 1847)¹ are often cited and applied by other mathematicians, but the domain treated in that work itself is nowhere touched upon or applied, if I except an interesting short article by Kysaeus (Bedeutung und Anwendung der Zahlen in Geometrie, Siegen 1850). As a consequence of this, no review of the book, or statement of its contents, has ever appeared, indeed not a single mention of it outside of the publisher’s list, except for one I wrote myself (Grunerts Archiv 6, 237–250 (1845)).²

To remove that difficulty thus became an essential task for me if I wanted the book to be read and understood by others as well as myself. This difficulty could not however be removed without substantially changing the plan of the whole. For it is implicit not in some arbitrarily chosen form but in the plan I had envisaged: To formulate the whole from the ground up, independent of other branches of mathematics. The direct implementation of this plan, although it must be the most expeditious for science as such, must, unless it has also become subjective, offer considerable difficulty in that type of presentation, particularly in a science such as extension theory, which extends and intellectualizes the sensual intuitions of geometry into general, logical concepts, and, with regard to abstract generality, is not simply one among the other branches of mathematics, such as algebra, combination theory, and function theory, but rather far surpasses them, in that all fundamental elements are unified under this branch, which thus as it were forms the keystone of the entire structure of mathematics.

Thus I had to abandon this plan, and in the present work I have presupposed the other branches of mathematics, at least in their elementary development. In
addition I have adopted exactly the opposite method in the form of presentation, as I have applied the most rigorous mathematical form we know, the Euclidean, to the present work, and have relegated to the Remarks everything that serves to illustrate or motivate the method chosen.

A necessary consequence of the plan as altered was that all the results of the first part, insofar as they did not include applications to physics, had to be taken up and derived anew according to the altered plan of the new edition (as is seen in Nos. 1–136, 216–329). Yet because of the differences in method of the two editions the same topic becomes so dissimilar between them that one scarcely finds any duplication, with the exception of the results obtained themselves, from which in the nature of the subject no deviation can appear. Thus the previous edition is by no means made superfluous by the new one. Indeed, even the new method is by no means to be preferred to the old; on the contrary, the methods of the first edition that derive from the original idea, and proceed thence completely independently, probe more deeply into the essence of the subject, and thus from a purely scientific viewpoint have a decided advantage relative to the latter. On the other hand this latter method is more acceptable to, and in any case more easily understood by mathematicians, who do not willingly see lying fallow the wealth of mathematical knowledge obtained otherwise by their studies. Thus the two presentations are mutually complementary and illustrative.

The presentation chosen here very closely follows arithmetic, in the sense that it assumes the numerical magnitude as a continuous quantity. Now just as arithmetic develops all other magnitudes from a single magnitude, arbitrarily chosen from the rest, which is defined as the unit and may be symbolized by $e$ (see my *Lehrbuch der Arithmetik; Ensilver, Berlin 1861*), so, within the framework given here, extension theory first defines several unit magnitudes, $e_1, e_2, \ldots$, none of which is derivable from the others (thus for example $e_2$ cannot be developed from $e_1$ by multiplication of $e_1$ with some numerical magnitude), and then considers the magnitudes, which I have called extensive magnitudes, resulting from these units by multiplication with numerical magnitudes and addition of these products. In this way there easily follow the laws of addition, subtraction, multiples, and fractions, discussed in Chapter 1.

It may seem surprising that such a simple idea, which basically consists in no more than regarding multiple sums of different magnitudes (which is how extensive magnitudes appear) as autonomous magnitudes, can in fact be developed into a new science, and indeed in this connection it has been objected that all of extension theory is merely an abbreviated method of notation, and that it is erroneous to regard as magnitudes expressions that are not magnitudes at all. But this objection is based on a complete misunderstanding of the essence of mathematics and of magnitudes. According to this argument all of arithmetic, indeed, one may say all of pure mathematics, is merely an abbreviated method of notation; for number is just an abbreviated expression for a sum of units, the product for a sum of equal numbers, the power for a product of them, and so on; however, without this abbreviated method of notation, or better expressed, without this {formal} unification, no progress toward a unification of concepts is imaginable. Without this unification, for example, it would not be possible to attain the concepts of reductive types of calculation (subtraction, division, roots, logarithms) and the new numerical forms obtained by means of them: negative, fractional, irrational, and imaginary. Thus it is important above all that one actually unify that which
in its essence forms a unity, and which therefore must also lead to new results to which one would not be led without that unification.

Now in fact extension theory leads to an inexhaustible wealth of such relations which, without the formation of that conceptual unity appearing in the extensive magnitude, would in no way be conceived or derived. Whether one regards the name “magnitude” as appropriate for this concept is in itself of very subordinate importance, since little depends on names here. The question is only whether this new concept unites with the general concept of magnitude in such a way that in their essence they amalgamate into a general concept, and that a dividing line drawn between the two domains would separate related matters arbitrarily, and divide the subject inconsistently. Thus in this latter case it would even be erroneous not to attribute the name “magnitude” to this new concept.

Now I believe that between that which I have called an extensive magnitude and the general numerical magnitude, and particularly the imaginary magnitude \((a + bi)\), there exists a relation so intimate that it would be absurd to treat one and not the other as a magnitude, since the imaginary magnitude is derivable from two units, 1 and \(i = \sqrt{-1}\), by real numerical coefficients, just as are extensive magnitudes from two or more units (cf. No. 413, Remark). Thus it seems to me perfectly justified if I designate the extensive magnitude as a magnitude. But I go even further, in that I designate them not just as magnitudes generally but as elementary magnitudes. Thus they contrast with other magnitudes that likewise bear as decided a character of unified magnitudes as such, as do those elementary ones, and which only enter upon addition of higher structures (cf. Nos. 77, 364, and 377).

I now proceed to clarify the course of development of the present work.

To addition, subtraction, multiples and fractions is adjoined (Chapter 2) the general concept of the multiplication of extensive magnitudes, which is based on the relation of multiplication to addition (that is, that one may multiply the summands in place of the sum). The multiplication of these magnitudes thereby reduces to that of their units, \((e_1, e_2, \ldots)\), and from consideration of the products of these units there then follow the different species of multiplication. Now it is possible to separate from these species two, to which all the others can be reduced.

The laws of one of these coincide precisely with ordinary multiplication in algebra, and therefore I will call it algebraic. However it is by far the more complicated with respect to the magnitudes generated and can only be brought to full clarity by consideration of functions, as a consequence of which I have relegated it to the second part of the work. The symbol for this algebraic multiplication must in the nature of the subject coincide with the usual symbol, since it would be absurd to symbolize differently conjunctions that are subject to the same laws in all relations.

The second of those multiplications, which is treated in Chapter Three, is shown to be characteristic of extension theory, and as essential in developing it, as it yields the different orders of elementary magnitudes that enter into extension theory. Thus it is characterized by the condition that two elementary factors of the product can only be interchanged if one simultaneously changes the algebraic sign \((+\text{ or } -)\) of the product. While the relation of this product to addition is indeed the same as with standard multiplication, its other laws deviate substantially from that of standard multiplication, whence it is necessary to distinguish it by its symbol. In this work I have therefore chosen as its symbol brackets enclosing the product, whence \([ab] = -[ba]\) if \(a\) and \(b\) are elementary factors of this product. This product
leads to an extraordinary manifold of apparent forms {Erscheinungsformen}, and permits an enormous abundance of relations to appear which throw a new and unsuspected light on all branches of mathematics, so that this product forms the true core of the new science. Once the concept of the supplementary magnitude is introduced, that product appears in a completely new aspect as the inner product (Chapter 4), so that the range of the subject presented in the first edition (cf. the Foreword to that work) emerges fully in this form. The first part of this volume concludes with applications to geometry (Chapter 5).

Now in the second part there appear combined magnitudes that we can characterize collectively as functions of elementary magnitudes. The first chapter of this part treats functions in general linked by multiplication and division, the second differential calculus, the third the theory of series, and finally the fourth integral calculus, all of these only to the extent that they relate to extensive magnitudes, to be sure. Even so I believe that the corresponding branches of ordinary mathematics (relating to numerical magnitudes), and in particular integral calculus, are not only essentially simplified by this presentation, but are also variously supplemented and developed further.

Because the material is considerably increased from the first edition I have found it necessary to drop all applications to physics; but I hope, if my time and strength permit, to be able to pursue a mathematical treatment of the most important branches of physics in an independent work, in which I will make use of the science reported upon here.¹

I have earnestly endeavored to avoid superfluous technical words and to limit myself to the smallest possible number of new technical expressions; but since one cannot speak without any words, and thus must either employ new word formations or word combinations for new concepts, or else bestow new meanings on old words, there unavoidably still remains a considerable collection of technical expressions. To aid in comprehensibility I have above all chosen the technical expressions so that, I hope, they immediately suggest the concepts they represent by their own forms; and at the end I have provided an alphabetical index, giving references to the places where they are defined.

It still remains for me to indicate related efforts of other mathematicians. Almost without exception these refer to those topics that I have designated as applications of extension theory to geometry (thus to §§ 24, 28–30, 37–40, 56, 74–79, 91, 92, 101, 102, 114–119, 144–148, 159–170 of the Ausdehnungslehre of 1844 and Nos. 216–347 of the present work). In the first edition (1844) the only investigation included there was known to me was the celebrated work Barycentrische Kalkül by the founder of geometric analysis, Möbius, in which he treats the addition of points. Investigations on the geometric addition of displacements (of given length and direction) and of the significance of imaginary magnitudes were unknown to me.

This latter was first presented fully in an article by Gauss (Göttingische gelehrte Anzeigen, 1831), to which Gauss called my attention in a letter² because of a point handled in the same way in the Foreword to the Ausdehnungslehre. The concept of the geometric addition of displacements in the plane was already implicit in this presentation of imaginaries. The first to have dealt with the geometric addition of displacements in complete generality appears to have been Bellavitis, who had
already formulated the relevant calculus in 1835 (Annali delle scienze del regno Lombardo-Veneto 5, 244–250 (1835)); see No. 227, Remark.

MÖBIUS independently developed the laws for the geometric addition of displacements and applied them to problems of celestial mechanics in his Mechanik des Himmels (1843). After the appearance of my Ausdehnungslehre (1844), investigations in the domain of geometric analysis proliferated. In particular, it was again MÖBIUS and BELLAVITIS who materially advanced the science, and contributed significantly to its understanding and the wider circulation of the geometric methods of calculation on which I reported. There then appeared my own work on this subject, partly developed in my paper “Geometrische Analyse, geknüpft an die von Leibnitz erfundene geometrische Charakteristik, gekrönte Preisschrift, Leipzig 1847”, which MÖBIUS sought to make more accessible to mathematicians by a lucid presentation annexed thereto, and partly in Crelle’s Journal (in the articles cited above). In addition, a year after the appearance of my Lineale Ausdehnungslehre, SAINT-VENANT published on the geometric addition of lines (Comptes rendus 21, 620–625 (1845)), which was identical to the outer multiplication of displacements presented by me in my book (§ 28–40). Obvious he was not familiar with this work, and I sent two copies of it to CAUCHY with the request that one be delivered to SAINT-VENANT, whose address was unknown to me.

Later on, in several articles printed in the Comptes rendus of 1858*, CAUCHY published a method for solving algebraic equations and related problems by means of certain symbolic magnitudes that he called clefs algébriques, a method that agreed precisely with that presented in my Ausdehnungslehre of 1844 (§ 45, 46 and 93). I was far from accusing the celebrated mathematician of plagiarism, but I did believe it was owed me and the subject that I raise a priority claim to the Paris Academy. But the Commission to which this claim was referred in April 1854 for examination and report (Comptes rendus 38, 743f) has never been heard from, and in addition CAUCHY never afterward published anything on the subject.

These articles by CAUCHY are the only ones that had any point of contact with my Ausdehnungslehre (of 1844) outside the domain of geometry. Since however these articles claim an independent origin, it appears as if the characteristic core of my book, excepting the geometric incidentals themselves, has stimulated no related efforts. And therefore with renewed courage I have undertaken this new book, which encompasses the old one and brings it to a conclusion.

For I have every confidence that the effort I have applied to the science reported upon here, which has occupied a considerable span of my lifetime and demanded the most intense exertions of my powers, is not to be lost. Indeed I well know that the form I have given the science is, and must be, imperfect. But I also know and must declare, even at the risk of sounding presumptuous, — I know, that even if this work as well should lie idle yet another seventeen years or more without influencing the living development of science, a time will come when it will be drawn forth from the dust of oblivion and the ideas laid down here will bear fruit. I know that if I do not succeed to a position, as yet yearned for in vain, where I can gather a circle of students about me upon whom I can impress these ideas and who I can stimulate to develop and enrich them further, yet some day these ideas, even if in an altered form, will reappear and with the passage of time will participate in a lively intellectual exchange. For truth is eternal, it is divine; and no phase in

the development of truth, however small the domain it embraces, can pass away without a trace. It remains even if the garments in which feeble men clothe it fall into dust.

Stettin, 29 August 1861
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Editorial Notes

Numbers enclosed [in brackets] identify the text page where the Editorial Note is cited.

The Ausdehnungslehre of 1862, which, following Grassmann’s precedent, we denote by $A_2$, bears the date 29 August 1861 under the Foreword, and was also published in 1861. Grassmann had had it printed at his own expense in a run of 300 copies, giving the commission for the printing to Enslin in Berlin; in any case the year 1862 appearing on the title page was chosen for the sake of the book dealers.

[xiii] 1. The articles in Crelles Journal relate to the generation of algebraic curves and surfaces, the so-called higher projectivity in the plane and in space, and to the various types of multiplication.

[xiii] 2. This article was later reprinted as Appendix III to the Second Edition of the $A_1$.

[xiv] 3. The Lehrbuch der Arithmetik was originally published in 1860 by R. Grassmann in Stettin; it was then reprinted on commission to Enslin, and provided a new title with the date 1861.

[xvi] 4. The planned work on the important branches of physics was never published. Because the Ausdehnungslehre at first received no attention from professional mathematicians, for a number of years Grassmann turned completely to the philological studies he had begun earlier. Later works related to physical problems are two articles on mechanics and one each on electrodynamics and acoustics.

[xvi] 5. Gauss’s letter is dated 14 December 1844. An extract follows:

“... leafing through your book amongst a host of other heterogeneous works, I think I perceive that its tendency has in part been encountered on that path which I myself traveled almost half a century ago, of which admittedly only a small part is mentioned in the 1831 Comment. der Göttingischen Societat and again in the Göttingischen Gelehrten Anzeigen (1831, p. 64), that is the concentrated metaphysics of complex magnitudes; while the infinite fruitfulness of this principle for the investigation of spatial relations was abundantly dealt with in my lectures, proofs of it were communicated, with other motivations, only here and there, and discernible as such only to the observant eye. In them there is only a partial and distant similarity with your work; and indeed I perceive that, in order to discern the essential core of your work it is first necessary to become familiar with your special terminology. Since however that will require of me a time free of other duties, I must no longer delay expressing my sincere thanks to you for so kindly sending me your work ...”
The point in the *Göttingischen Anzeigen* to which GAUSS refers appears in his *Works* in Vol. II, p. 175ff.

[svii] 6. The full title of this journal is *Annali delle scienze del Regno Lombardo-Veneto*. Opera periodica di alcuni collaboratori. Padova (from 1831). In it appear numerous articles by BELLAVITIS, of which the following are relevant:

“Sulla Geometria derivata”, 2, 250 (1831). This includes the presentation of the sum of two straight lines by the diagonal and its known significance for the imaginary magnitudes; this means one is led from that theorem on points in a *straight line* to such a theorem on points in a *plane*, whence the *derivata*.

“Teoria delle figure inverse e loco uso nella Geometria elementare”, 6, 126 (1836). The theory of reciprocal radius-vectors and of equipollences are interpreted as special cases of a general principle of transference (Geometria derivata).


[svii] 7. The title of this article is “Mémoire sur les sommes et les différences géométriques, et sur leur usage pour simplifier la Mécanique”.


[svii] 9. The session of the Academy referred to was on 17 April 1854. One reads the following in the *Comptes Rendus* reference cited:

> *Mathematical analysis. Extract of a memoir of M. GRASSMANN.*

“I request that the Academy of Science take notice of the priority claim I find it necessary to make on the occasion of two articles: “Sur les clefs algébriques”, by M. CAUCHY, and “De l’interprétation {géométrique} des clefs algébriques et des déterminants”, by M. de SAINT-VENANT,”* inserted in the *Comptes Rendus* 36, pages 70, 129, 582. I have, in the year 1844, published the principles established in these articles, and the results deduced from them by the two geometers I have just named. I have the honor to present to the Academy the work in which these ideas are contained (1) and certain articles published subsequently on the same subject (2), and I would be pleased if the Academy of Sciences would accept these works. To support my claim I take the liberty of communicating to you an extract of my researches reporting on that subject and contained in the works cited, noting where in these works each question is to be found.

“All these researches are based on quantities that I call *extensive magnitudes*, which are, in essence, no different from the *facteurs symboliques* and *clefs algébriques*

*In this article SAINT-VENANT does in fact mention GRASSMANN’s name in a footnote on p. 584, but only with respect to the concept of the inner product, which GRASSMANN had developed in the *Geometrische Analyse*. 


of M. CAUCHY. But as the point of view of these quantities that I envisage is completely different from that of M. CAUCHY it is necessary to go into certain details. That is the object of the Note that I have the honor to submit today to the judgment of the Academy.'

"That note, in its nature scarcely susceptible to analysis, cannot be reproduced in extenso on account of its length. It was referred to the examination of a Commission composed of MM. CAUCHY{1}, LAMÉ, and BINET."

MÖBIUS brought the articles by CAUCHY and SAINT-VENANT to GRASSMANN's attention in a letter of 2 September 1853, and urged him to pursue his priority claim.


[27] 11. By the "principle" that GRASSMANN posits here he excludes from the outset any consideration of numerical relations between the unit products and the original units, and limits himself to numerical relations between unit products of equally many factors. In this way he is barred from proceeding to the system of higher complex numbers, whose concept HAMILTON had already presented in full generality in his Lectures on Quaternions of 1853, and whose theory has been further developed more recently by various mathematicians.

In addition it is remarkable that in this general investigation of the various types of product structure GRASSMANN considered only products of two factors, and not also those of three factors. Not once does he mention that the product structure in which every set of three units satisfies the equation

\[(e_ie_k)e_j = e_i(e_ke_j),\]

whence the so-called associative law holds, likewise belong to the linear product structures, and only later, in No. 78, does he introduce the associative law, but without mentioning its significance. This is the more surprising, since in the A1 he develops the associative law, or, as he says, the law of the combinability of the terms of a conjunction, right at the beginning (in §3) with the greatest acuteness and clarity.

Had GRASSMANN sought to define all possible species of linear product structures for products of three factors as well, he would perhaps have arrived at a different conception of the importance of his "principle", for he would then have hit upon completely new product structures, which he could not have provided for in his system. Naturally we ought not to engage upon treating the problem suggested here, so we will mention only two types of linear product structure that enter with products of three factors.

The defining equations of the first type have the form

\[(e_ie_k)e_j = ae_i(e_ke_j) \quad (i, k, j = 1, \ldots, n),\]

and those of the second read as follows:

\[
\begin{cases}
(e_ie_k)e_j + (e_ke_j)e_i + (e_je_i)e_k = 0, \\
(e_ie_k)e_j = be_j(e_ie_k),
\end{cases}
\]

where a and b denote numbers. In the first case one obtains the associative product structure for a = 1. In the second case, if one sets b = -1 and in addition adopts the defining equation

\[e_ie_k + e_ke_i = 0 \quad (i, k = 1, \ldots, n),\]
one obtains a linear product structure of a completely different type, which plays an important rôle in Lie's theory of transformation groups. In this product structure the units \( e_1, \ldots, e_n \) are independent infinitesimal transformations of an arbitrary space and the product \( e_ie_k \) is the POISSON bracket formed from two infinitesimal transformations; the first of equations (2) is then nothing but the famous JACOBI identity. In this connection, see Sophus Lie, *Theorie der Transformationsgruppen*, Vol. III, p. 747ff.

[33] 12. The sign convention chosen by GRASSMANN corresponds to the sign rule of CRAMER, cf. his *Introduction à l'analyse des lignes courbes*, Genf 1750, Appendix, p. 657f. CAUCHY's sign convention is found e.g. in BALTZER's *Determinanten*.

[33] 13. The multiplication theorem for determinants follows at once from this theorem. Thus if one sets

\[ a_k = \beta_1^{(k)}b_1 + \cdots + \beta_n^{(k)}b_n, \]

then according to No. 63 one can express both sides of the final equation of this number by the combinatorial product \([b_1 \cdots b_n]\), and if this does not vanish, one obtains, with regard to No. 32, an equation that is nothing but the multiplication theorem for determinants. In his *Theorie der complexen Zahlensysteme*, Leipzig 1867, p. 122f, HANKEL published this proof of the multiplication theorem, probably as a consequence of a letter from GRASSMANN.

[38] 14. The concept of the elementary linear evolution is in the following always applied to the case that the linearly evolved series of magnitudes is formed from \( n \) extensive magnitudes that are derivable from the \( n \) original units and stand in no numerical relation to one another. Under this assumption the elementary linear evolution is equivalent to a linear homogeneous transformation of a special form.

In fact the aggregate of all magnitudes derivable from \( a_1, \ldots, a_n \) form a domain of \( n \)th order, which (according to 21 and 24) coincides with the domain defined by \( e_1, \ldots, e_n \); the general form of a magnitude of this domain is \( x_1a_1 + \cdots + x_na_n \). Now if one subjects this series of magnitudes \( a_1, \ldots, a_n \) to an elementary linear evolution, perhaps by replacing \( a_m \) with \( a_m + \alpha a_{m+1} \) and leaving the remaining \( a_k \) unchanged, then each magnitude \( \sum x_ia_i \) of our domain of \( n \)th order is changed into a magnitude \( \sum x_ia_i + \alpha x_m a_{m+1} \), which likewise belongs to this domain. If we write these new magnitudes in the form \( \sum x'_ia_i \), we obtain between the \( x_i \) and the \( x'_i \) the relations

\[ x'_1 = x_1, \ldots, x'_m = x_m, x'_{m+1} = x_{m+1} + \alpha x_m, x'_{m+2} = x_{m+2}, \ldots, x'_n = x_n. \]

This however is evidently a linear homogeneous transformation of unit determinant, which indicates how the magnitudes of our domain of \( n \)th order interchange among themselves under that elementary linear evolution. If one applies several elementary linear evolutions, one after the other, this is equivalent to the execution of several of that type of linear homogeneous transformations, one after the other, and the linear evolution by which the first series is transformed into the last thus likewise amounts to a linear homogeneous transformation of unit determinant.

It should be explicitly observed that the \( \infty \) linear homogeneous transformations of the special form \((*)\) taken together form a *one-parameter group* in the sense of Lie.
15. That linear evolutions in geometry can be carried out by means of straightedges is demonstrated in No. 254. On circular evolution cf. Nos. 154 and 331.

16. If \([a_1 \cdots a_m] = [b_1 \cdots b_m] \neq 0\), then according to 70 \(b_1, \ldots, b_m\) can be derived from \(a_1, \ldots, a_m\), whence

\[
b_k = \alpha_{k1}a_1 + \cdots + \alpha_{km}a_m \quad (k = 1, \ldots, m)
\]

where the determinant of the \(\alpha_{kv}\) is certainly not zero and in addition by virtue of the equation \([a_1 \cdots a_m] = [b_1 \cdots b_m]\) has the value one according to Nos. 63 and 32. Simultaneously the domain of \(m\)th order defined by \(a_1, \ldots, a_m\) is identical to the domain defined by \(b_1, \ldots, b_m\). Thus if in an arbitrary magnitude \(\sum x_v a_v\) of this domain one replaces the \(a_v\) by the \(b_v\), then the magnitudes of the domain are interchanged among themselves and in fact, as one easily perceives, by a linear homogeneous substitution of unit determinant; for we can set \(\sum x_v b_v = \sum x'_v a_v\) and obtain thereby between the \(x\) and \(x'\) the relation

\[
x'_v = \alpha_{1v}x_1 + \cdots + \alpha_{mv}x_m \quad (v = 1, \ldots, m),
\]

that is a genuine linear homogeneous transformation of this type.

If one recalls Editorial Note 14, then one recognizes that Theorem 76 is equivalent to the following: Every linear homogeneous substitution of unit determinant can be obtained by successively applying a number of substitutions of a special form, that is substitutions of the form (*), which according to Editorial Note 14 express an elementary linear evolution.

17. This corollary is necessary since the numbers in Nos. 84 and 95 are treated as magnitudes of zeroth order without comment, and especially since No. 95 refers to No. 77. But we observe that the product of an elementary magnitude and a number is again an elementary magnitude. For according to 46 one can move the number to any given factor of first order of the product, but this factor remains a factor of first order.

18. Here it is assumed that \([pq)(pq)] = [pqpq]\), although as yet the multiplication of two magnitudes of higher order is not sufficiently defined that one knows whether the parentheses can be dropped or not; only in No. 87 will a stipulation on this point be found. GRASSMANN probably noticed this oversight later and consequently repeated the whole treatment in the Remark following No. 88.

19. From this theorem one easily obtains the following:

THEOREM. In a principal domain of \(n\)th order every magnitude of \((n - 1)\)th order is elementary.

Thus according to No. 88 it is clear, first of all, that every sum of arbitrarily many elementary magnitudes of \((n - 1)\)th order in a principal domain of \(n\)th order is again an elementary magnitude. Now since every magnitude of \((n - 1)\)th order can be represented as a sum of elementary magnitudes of \((n - 1)\)th order, that is as a sum of products of each unit of \((n - 1)\)th order and a numerical magnitude, and since (according to 77) these units and (according to the Remark following No. 77) these products are also elementary magnitudes, in general every magnitude of \((n - 1)\)th order in a principal domain of \(n\)th order is elementary.

In the Remark following No. 88 GRASSMANN mentions that a magnitude of \(m\)th order in a principal domain of \(n\)th order is not in general elementary if \(1 < m < n - 1\), but he demonstrates this only for the simplest case: \(n = 4, m = 2, and\)
passes over the case \( n > 4 \) in silence. Only at a \textit{single} later point in this work does GRASSMANN touch on the question, namely in No. 286, where he shows how one can determine whether a given sum of line elements in space is again a line element or not, or in other words, whether a given magnitude of second order in a domain of fourth order is elementary or not.

Moreover, precisely the same lacuna is already found in the \( \Lambda_1 \) (cf. there §51); there also only the case \( n = 4, m = 2 \) is considered (§124).

Later, in his last mathematical article,\(^*\) which was inspired by the work of REYE, GRASSMANN returned once more to the question and settled it by specifying the conditions which a magnitude of \( q \)th order, expressed in terms of the units of \( q \)th order, in a principal domain of \((q + s)\)th order, must satisfy if it is to be elementary. The conditions at which he arrived are certain equations of second, third, \ldots, \( q \)th degree between the coefficients of the magnitude of \( q \)th order under consideration.

[50] 20. The supplement symbol \(|\) was read as “in” by GRASSMANN’s sons, a reading that probably originated with GRASSMANN himself.

[50] 21. The transition from the magnitude of a principal domain to its supplement is an operation which from the standpoint of projective geometry is considered as belonging to the reciprocities (dualistic transformations), and indeed to the special reciprocities that one calls polar systems.

If \( a = \sum x_k e_k \) is an arbitrary magnitude of first order, then its supplement \( |a = \sum \{x_k e_k\} \) is a magnitude of \((n - 1)\)th order, which according to No. 88 and Editorial Note 19 is certainly elementary. The domain of these magnitudes consists of all magnitudes of first order \( b = \sum y_k e_k \) subordinate to them, and which therefore satisfy the equation \([ba] = 0\). But now according to No. 143,

\[
\sum y_k e_k \sum x_j e_j = \sum x_k y_k,
\]

whence the domain of \( | \sum x_j e_j \) consists of all magnitudes of first order \( \sum y_j e_j \) for which \( \sum x_k y_k = 0 \). Thus if we interpret the units \( e_1, \ldots, e_n \) as \( n \) points of an \((n - 1)\)-fold extended plane that do not lie in an \((n - 2)\)-fold extended plane, and thereby the \( x_1, \ldots, x_n \) as homogeneous coordinates of the points of this space (cf. Nos. 232 and 238), the domain of the supplement of \( \sum x_k e_k \) is nothing but the \((n - 2)\)-fold extended polar plane of the points \( x_1, \ldots, x_n \) with respect to the manifold of second degree \( \sum x_k^2 = 0 \).

We obtain another interpretation of the supplement if we take \( e_1, \ldots, e_n \) to be such displacements (indefinitely distant points) of an \( n \)-fold extended space as are not parallel to an \((n - 1)\)-fold extended plane of this space (cf. No. 229), and if at the same time we interpret the numbers \( x_1, \ldots, x_n \) as right- or oblique-angle parallel coordinates. Then the supplement of \( \sum x_k e_k \) is the product of \( n - 1 \) displacements that are parallel to the \((n - 1)\)-fold extended polar plane of the infinitely distant points of the displacement \( x_k e_k \) with respect to an arbitrary member of the \( \infty^1 \) manifolds of second degree

\[
\sum x_k^2 = \text{const}.
\]

Of these two, to be sure not essentially different, interpretations, GRASSMANN only applied the second in the $A_2$, and even this only for right-angle parallel coordinates. Cf. the Remark following No. 93 and Nos. 330, 331, 335. The first interpretation is found in GRASSMANN’s article in Crelles Journal 84, 273–283 (1878) cited in Editorial Note 19. Cf. also R. STURM’s remarks in GRASSMANN’s obituary notice, Math. Ann. 14, 16 (1878).

[53] 22. By this definition all outer products of two factors, the sum of whose orders exceeds that of the principal domain, are excluded from consideration. This seems appropriate, since all such products would have the value zero.

[53] 23. This may be claiming too much; cf. Editorial Note 11.

[54] 24. In the appeal to No. 79 there is a fundamental idea that deserves to be expressed as a special theorem between Nos. 94 and 95, perhaps thus:

94A. If the orders of the two elementary magnitudes $A$ and $B$ together are not greater than the order $n$ of the principal domain, then one obtains the progressive product of $A$ and $B$ by combinatorially multiplying the series of the elementary factors of $A$ with the elementary factors of $B$, thus for $q + r \leq n$:

$$[(a_1a_2 \cdots a_q)(b_1b_2 \cdots b_r)] = [a_1a_2 \cdots a_qb_1b_2 \cdots b_r].$$

**PROOF.** According to No. 94 the progressive product of $A$ and $B$ is an outer product of these magnitudes; the rest follows from No. 79.

[56] 25. In the original edition of the $A_2$ GRASSMANN did not explicitly consider the case of even $n$; the material enclosed by braces was inserted by the editors of the Teubner Edition to fill the gap.

[57] 26. The converse in braces is necessary because later it is used several times, for example in Nos. 119, 119b, 120, and 124.


[59] 28. The product $[QR]$ is certainly progressive. Thus if one denotes the orders of $E$, $F$, $G$, $Q$, $R$ by the corresponding lower case letters, then

$$q = n - (e + f), \quad r = n - (e + g),$$

whence

$$q + r = 2n - 2e - f - g,$$

or, since $e + f + g = n$, $q + r = n - e$, that is

$$q + r < n,$$

from which it follows that the product $[QR]$ is in fact progressive.

[61] 29. The beginning of the proof of No. 103 does not exactly harmonize with the rest, since later it is obviously assumed that each of the $n$ factors of first order $a_1$, $a_2$, … is only included in one of the three magnitudes $A$, $B$, $C$, which cannot easily be inferred from the introductory words. This difficulty is eliminated if one gives the beginning of the proof the following wording: “Suppose $a_1$, … , $a_n$ is any given series of $n$ magnitudes of first order, and assume that formula 103 always holds if the $n$ elementary factors included in $A$, $B$, $C$, taken in any given order, correspond to the series of magnitudes $a_1$, … , $a_n$; then I will show” etc. One must then observe that the $a_1$, … , $a_n$ are completely arbitrary, and thus may stand in
a numerical relation to one another, indeed they need not even be different from one another.

The proof of No. 103 is therefore the more remarkable, as it permits the invariant character of the equation

\[ [AB \cdot AC] = [ABC]A \]

to appear most clearly. It is based on the fact that this equation continues to hold if one replaces the elementary factors of \( A, B, C \) with arbitrary magnitudes numerically derivable from them, and since the equation is proved in No. 102 for products of units, it is thereby proved in general. If we recall Editorial Note 16, we can also obviously say: The above equation remains invariant if one subjects the magnitudes \( \sum x_k e_k \) of the principal domain \( e_1, \ldots, e_n \) to an arbitrary linear homogeneous transformation \( x'_i = \sum a_{ik} x_k \).

We must leave it to the reader to make clear this invariant character of the equations of extension theory in each individual case. We will only indicate an especially striking example.

It should not go unremarked that the equation of No. 103 is introduced in the \( A_1 \) in a completely different way than it is in the \( A_2 \). In the \( A_1 \) this equation serves to define the regressive product (\( A_1, \S 132 \) and 133). In the \( A_2 \) on the other hand the regressive product is defined in the simplest conceivable way on the basis of a new concept, the concept of the supplement, in No. 94, and then the equation of No. 103 is proved, whereby the concept of the regressive product is again made independent of that of the supplement.

One can use the formula of No. 103,

\[ [AB \cdot AC] = [ABC]A, \]

where the sum of the orders of \( A, B, C \) equals the order of the principal domain, for the derivation of a series of theorems about which indeed Grassmann said nothing explicitly, but which properly belong in his system, and of which he has implicitly made use, in part. We will formulate and prove these theorems here.

**Theorem 1.** A product of arbitrarily many elementary magnitudes is always again an elementary magnitude.

We obviously need only prove this theorem for a product of two factors. Thus suppose \( L \) and \( M \) are two nonvanishing elementary magnitudes with orders \( \lambda \) and \( \mu \). If \( n \) is the order of the principal domain and \( \lambda + \mu \leq n \), then the product \([LM]\) is progressive and is therefore certainly an elementary magnitude.* On the other hand, if \( \lambda + \mu > n \), then according to No. 87 \( L \) and \( M \) can be represented in the form

\[ L = [AL_1], \quad M = [AM_1], \]

where \( A, L_1, M_1 \) are elementary magnitudes and in particular \( A \) is of \((\lambda + \mu - n)\)th order. Moreover,

\[ [LM] = [AL_1 \cdot AM_1] = [AL_1 M_1]A, \]

since the assumptions of No. 103 are satisfied, and here \([AL_1 M_1]\) is a progressive product of \( n \)th order, and therefore a number. Consequently the product \([LM]\) is also an elementary magnitude in the case \( \lambda + \mu > n \). Our theorem is therefore proved.

---

*For \( \lambda + \mu = n \) the product is a number; but the numbers must be reckoned among the elementary magnitudes.
THEOREM 2. The supplement of an elementary magnitude is always again an elementary magnitude.

PROOF. That the supplement of a magnitude of first order is again an elementary magnitude is obvious; for this supplement is of \((n - 1)\)th order, and according to No. 88 and Editorial Note 19 every magnitude of \((n - 1)\)th order is elementary.

On the other hand, if \(A\) is an elementary magnitude of \(m\)th order, then one can set \(A = [a_1 \cdots a_m]\), where \(a_1, \ldots, a_m\) are magnitudes of first order. Then according to No. 98

\[|A| = |a_1|a_2 \cdots |a_m|.\]

Here, as shown above, \(|a_1|, |a_2|, \ldots\) are each elementary magnitudes of \((n - 1)\)th order, whence so also is their product according to Theorem 1 above, that is the supplement of \(A\) is an elementary magnitude.

The third theorem we will prove here is the converse of No. 79B, and can be expressed in the following way:*

THEOREM 3. If \(A\) and \(B\) are two nonzero elementary magnitudes and if \(A\) is subordinate to \(B\), then \(A\) can be represented in the form

\[A = [BC],\]

where \(C\) is an elementary magnitude and the product \([BC]\) is regressive.

PROOF. Suppose \(\alpha\) and \(\beta\) are the orders of \(A\) and \(B\), and according to 79B let \(B\) be represented as a progressive product in the form \(B = [AL]\), where \(L\) is an elementary magnitude of \((\beta - \alpha)\)th order. Further, suppose \(D\) is an elementary magnitude of \((n - \beta)\)th order that has no domain of first order in common with \(B\), nor therefore with \(A\), so that \([BD]\) is a number different from zero and \([AD]\) is a progressive product different from zero. In particular we imagine \(D\) so chosen that \([BD]\) = 1. The sum of the orders of \(A, L, D\) is then equal to \(n\) and therefore according to No. 103

\[[B \cdot AD] = [AL \cdot AD] = [ALD]A = [BD]A = A,\]

or, setting the progressive product \([AD]\) = \(C\),

\[A = [BC].\]

The representation claimed for \(A\) is thus constructed, for the product \([BC]\) is manifestly regressive, since the order of \(C\) is \(\alpha + n - \beta\).

From the last equation it follows according to 97 that

\[|A| = [|B|C],\]

where \(|A|, |B|, |C|\) are again elementary magnitudes according to Theorem 2, and where (according to 115) the product \(||B|C|\) is now regressive. From this there follows

THEOREM 4. If \(A\) and \(B\) are elementary and \(A\) is subordinate to \(B\), then the supplement \(|A|\) is superordinate to the supplement \(|B|\).

Finally, from Theorems 2 and 4 there follows

THEOREM 5. The supplements of two incident elementary magnitudes are again incident elementary magnitudes.

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*In the A1 this is found completely incidentally as a footnote to §153.
30. It is assumed here that
\[ [ACD] = [A \cdot CD] \quad \text{and} \quad [A \cdot BC] = [ABC], \]
which is not permitted without further details. One can remove this objection if,
as with the proof of formula 105, one distinguishes the two cases in which the sum
of the orders of \( A, B, C \) equals \( n \), and when it equals \( 2n \).

In the first case the proof follows exactly as given in the text; for if one sets
\[ [BC] = [CD], \]
then \( D \) is of the same order as \( B, \) whence
\[ [AC \cdot BC] = [AC \cdot CD] = [ACD]C. \]
Now here the products \([AC]\) and \([ACD]\) are progressive, since for both, by virtue of
the assumption, the sum of the orders of their factors is not greater than \( n \). Thus
(according to 80)
\[ [ACD] = [A \cdot CD], \]
or, since \([CD] = [BC],\)
\[ [ACD] = [A \cdot BC] = [ABC], \]
and thus as a matter of fact
\[ [AC \cdot BC] = [ABC]C. \]

In the second case the sum of the orders of \( A, B, C \) is equal to \( 2n \), whence one
proceeds exactly as in the proof of No. 105.

31. As before let \( \alpha, \beta, \gamma \) be the orders of the elementary factors \( A, B, C. \)
Since the two cases \( \alpha + \beta + \gamma = n \) and \( = 2n \) were already disposed of in Nos. 105–
107, we need only consider the cases in which \( \alpha + \beta + \gamma < n, n < \alpha + \beta + \gamma < 2n, \)
and \( \alpha + \beta + \gamma > 2n. \) However, by virtue of No. 101 the last immediately reduces to the
first if one takes the supplements on both sides of the equations to be proved. For
the supplements \([A, \) \([B, \) \([C \) are according to Theorem 2 of Editorial Note 29 again
elementary magnitudes, and of orders \( n - \alpha, n - \beta, n - \gamma, \) and from \( \alpha + \beta + \gamma > 2n \)
it follows that
\[ n - \alpha + n - \beta + n - \gamma < n. \]
Thus only the first two cases remain to be considered.

We begin with the proof of the first formula,
\[ [AB \cdot AC] = [A \cdot ABC]. \]

Case I. \( \alpha + \beta + \gamma < n. \) If \( 2\alpha + \beta + \gamma \leq n, \) then both products \([AB \cdot AC]\)
and \([A \cdot ABC]\) are progressive, and according to No. 109 both are zero, since the
domains of the two factors always have a domain of \( \alpha \)th order in common. If on the
other hand \( 2\alpha + \beta + \gamma > n, \) then both products are regressive and again according to
109 are zero, since the elementary magnitudes \( A, [AB], [AC], [ABC] \) can all be
contained in a domain of \( (\alpha + \beta + \gamma) \)th order at most.

Case II. \( n < \alpha + \beta + \gamma < 2n. \) If \([AB]\) vanishes, then both sides of the equation
to be proved are zero; on the other hand if \( \alpha + \beta = n, \) then \([AB]\) is a number that
can be placed anywhere in the product, and since
\[ [ABC] = [(AB)]C, \]
the equation to be proved is correct as well. We can therefore assume that \([AB] \neq 0 \)
and \( \alpha + \beta \neq n. \)

We must now consider separately four different subcases, when \( \alpha + \beta < n \) or
\( > n, \) and \( \alpha + \gamma \leq n \) or \( \geq n. \)
Subcase 1. $\alpha + \beta < n$, $\alpha + \gamma \leq n$, whence $2\alpha + \beta + \gamma < 2n$. If $A$, $B$, and $C$ are included in a domain of lower than $n$th order, then this is also true of the magnitudes $C$, $[AB]$, and $[AC]$; the regressive products $[AB \cdot AC]$ and $[(AB)C] = [ABC]$ are therefore according to 109 both zero. We therefore assume that $A$, $B$, $C$ can be contained in no domain of less than $n$th order.

Under this assumption $[AB]$ and $C$ have a domain of $(\alpha + \beta + \gamma - n)$th, but no higher, order in common, and therefore according to No. 87 can be represented in the form

$$[AB] = [DL], \quad C = [DC_1],$$

where the sum of the orders of $D$, $L$, and $C_1$ is equal to $n$. Thus according to No. 108

$$[ABC] = [DL \cdot DC_1] = [DLC_1]D,$$

where $[DLC_1] = [ABC_1]$ is a number.

Now it is a question of whether $A$ and $C$ have a domain in common or not. If $A$ and $C$ have a domain of first or higher order in common, then the same is true of $A$ and $D$, and thus according to No. 109 the progressive products $[AC]$ and $[AD]$ are both zero. Consequently

$$[A \cdot ABC] = [DLC_1][AD] = 0 = [AB \cdot AC],$$

that is, the equation to be proved is satisfied.

If however $A$ and $C$ have no domain of first or higher order in common, then the same is true of $A$ and $D$, so that $[AD]$ and $[AC]$ are both $\neq 0$, and in fact

$$[AC] = [A(DC_1)] = [(AD)C_1],$$

since the product $[A(DC_1)]$ is purely regressive (cf. Nos. 116 and 119). On the other hand, according to our assumption $[AB]$ and $[AC]$ are included in no domain of lower than $n$th order and therefore have a domain of $(2\alpha + \beta + \gamma - n)$th, but no higher, order in common; at the same time it is clear that $[AD]$ is a magnitude of $(2\alpha + \beta + \gamma - n)$th order in this domain. Consequently $[AB]$ can be represented in the form

$$[AB] = [(AD)B_1],$$

and it follows from 103 that

$$[AB \cdot AC] = [(AD)B_1 \cdot (AD)C_1] = [(AD)B_1C_1][AD] = [ABC_1][AD],$$

while the value previously found for $[ABC]$ yields

$$[A \cdot ABC] = [DLC_1][AD] = [ABC_1][AD].$$

Thus the two expressions are equal.

The first subcase is therefore completed.

Subcase 2. $\alpha + \beta > n$, $\alpha + \gamma \leq n$. Since $[AB] \neq 0$, the domains of $A$ and $B$ have a domain of $(\alpha + \beta - n)$th, but no higher, order in common. If $M$ is a magnitude in this domain, then we can set

$$A = [MA_1], \quad B = [MB_1],$$

and according to No. 103

$$[AB] = [MA_1 \cdot MB_1] = [MA_1B_1]M,$$

where $[MA_1B_1]$ is a number, whence

$$[AB \cdot AC] = [MA_1B_1][M \cdot AC].$$
On the other hand one has

\[ [A \cdot ABC] = [MA_1 B_1][A \cdot MC]. \]

But \([A \cdot MC] = [MA_1 \cdot MC]\), and here the sum of the orders of \(M, A_1,\) and \(C\) equals \(\alpha + \gamma\), and thus \(\leq n\), whence we come back either to Case I or to No. 103 and find that

\[ [A \cdot MC] = [MA_1 \cdot MC] = [M \cdot MA_1 C] = [M \cdot AC], \]

and thus again it follows that

\[ [AB \cdot AC] = [A \cdot ABC]. \]

Subcases 3 and 4. The two remaining subcases, \(\alpha + \beta > n,\) \(\alpha + \gamma \geq n\) and \(\alpha + \beta < n,\) \(\alpha + \gamma \geq n\), reduce to 1 and 2 if one takes the supplement on both sides of the equation to be proved.

The second formula is an immediate consequence of the first. Thus one has

\[ [ABC] = [(AB)C] \quad \text{and} \quad [AB] = \pm[BA], \]

whence from the first formula it follows that

\[ [BA \cdot AC] = [A \cdot (BA)C] = [A \cdot BAC], \]

which is precisely the second formula.

The third formula,

\[ [AC \cdot BC] = [C \cdot ABC], \]

can, for the cases \(\alpha + \beta + \gamma \leq n\) and \(\geq 2n\) be proved precisely as was the first. However, in the case \(n < \alpha + \beta + \gamma < 2n\) it is not always valid, as one can best see by considering an example.

Let \(n = 5\) and

\[ A = [e_1 e_2], \quad B = [e_1 e_3], \quad C = [e_4 e_5]; \]

then \([AB]\) and \([ABC]\) are both zero, but on the other hand

\[ [AC] = [e_1 e_2 e_4 e_5] = [e_1 e_4 e_5 e_2], \quad [BC] = [e_1 e_3 e_4 e_5] = [e_1 e_4 e_5 e_3], \]

and therefore

\[ [AC \cdot BC] = [e_1 e_4 e_5 e_2 \cdot e_1 e_4 e_5 e_3] \]
\[ = [e_1 e_4 e_5 e_2 e_3][e_1 e_4 e_5] \]
\[ = [e_1 e_4 e_5], \]

and is thus not zero, whence the third formula is proved incorrect for this case.

[63] 32. Both formulas still remain valid if \(B\) is superordinate to \(A,\) and thus generally if \(A\) and \(B\) are incident. Since the first of the two formulas in No. 129 applies in this more general case, we will at once produce the proof of the two formulas for the case not treated in the text.

Thus suppose \(A, B, C\) are elementary magnitudes with orders \(\alpha, \beta, \gamma,\) the sum of the orders of \(A\) and \(C,\) that is \(\alpha + \gamma,\) equals \(n,\) and \(A\) is subordinate to \(B.\)

Thus according to Editorial Note 29, \([A, |B, |C\) are elementary magnitudes and \(|B\) is subordinate to \(|A,\) and in addition the sum of the orders of \(|A\) and \(|C\) equals \(n - \alpha + n - \gamma = 2n - n = n.\) Consequently, according to No. 108,

\[ ||A \cdot (|B|C) = ||A|C||B, \]
\[ ||C|B \cdot |A| = ||C|A||B, \]

whence according to No. 101 the equations of No. 108 also remain valid in the case just considered.
33. The editors of the Teubner Edition replaced the material in braces with the following:

First, if $|AB| = 0$, then since $\alpha + \beta > n$, the domains of $A$ and $B$ must (according to 109) be included in a domain of lower than $n$th order. Therefore not all $n$ elementary factors $a_1, \ldots, a_n$ appear in $A$ and $B$, but rather only a certain number $\delta$ of them, where $\delta < n$, while the other $n - \delta$ are missing in both $A$ and $B$. Now since $IA$ includes only the factors not appearing in $A$, and correspondingly for $IB$, $IA$ and $IB$ have these $n - \delta$ factors missing from $A$ and $B$. Since in addition the sum of the orders of $IA$ and $IB$ has the value $n - \alpha + n - \beta$ and thus is less than $n$, it follows (according to 109) that

$$[IA \cdot IB] = 0.$$  

But since $|AB| = 0$ as well, and (according to b) the supplement of a number in the same sense is also equal to this number, so that that of zero is again zero, it follows that

$$I[AB] = [IA \cdot IB].$$

Second, if $|AB|$ differs from zero, then (according to 109) the covering domain of the domains of $A$ and $B$ has the order $n$, their common domain the order $\alpha + \beta - n$. Now since the magnitudes $A$ and $B$ are both products of factors from the series of $n$ magnitudes $a_1, \ldots, a_n$, it is clear that each of the $n$ factors $a_1, \ldots, a_n$ appears in at least one of the two magnitudes $A$ and $B$ and that in addition $A$ and $B$ have precisely $\alpha + \beta - n$ of these elementary factors in common. If $C$ is the product of these $\alpha + \beta - n$ factors, then $A$ and $B$ can be represented in the forms

$$A = [CA_1], \quad B = [CB_1],$$

where $[A_1CB_1]$ includes all elementary factors and is therefore a number. But (according to 105)

$$|AB| = |CA_1 \cdot CB_1| = |CA_1B_1|C,$$

and further (according to c)

$$I[AB] = |CA_1B_2|IC,$$

or, since (according to a)

$$IC = [CA_1B_1][A_1B_1],$$

one has

$$I[AB] = [CA_1B_1][CA_1B_1][A_1B_1].$$

On the other hand

$$IA = I[CA_1] = |CA_1B_1|B_1, \quad IB = I[CB_1] = |CB_1A_1|A_1,$$

whence

$$[IA \cdot IB] = [CA_1B_1][CB_1A_1][B_1A_1],$$

and since this expression (according to 58) does not change its value if one interchanges $A_1$ and $B_1$ twice, it follows again that

$$I[AB] = [IA \cdot IB].$$

34. By virtue of this theorem one can represent every elementary magnitude in a principal domain of $n$th order as a pure regressive product of elementary magnitudes of $(n - 1)$th order. Thus, given an elementary magnitude of $m$th order

$$C = [a_1, \ldots, a_n],$$

one need only define $n - m$ magnitudes of first order $a_{m+1}, \ldots, a_n$ such that they stand in no numerical relation to one another or to $a_1, \ldots, a_m$, and
that $|a_1a_2 \cdots a_m| = 1$. If one then defines the magnitudes $A_1, \ldots, A_n$ as in No. 112, one has $C = [A_nA_{n-1} \cdots A_{m+1}]$. At the same time the possibility is thereby given of representing the supplement of $C$ in a simple way as a combinatorial product of magnitudes of first order. Thus according to No. 98

$$|C| = ||A_n|A_{n-1} \cdots |A_{m+1}|,$$

where the $|A_k$ are magnitudes of first order that can be written down immediately, once one has ascertained in what way $A_1, \ldots, A_n$ are derived from the $n$ units of $(n - 1)$th order.

It is clear that one is now in a position to take either magnitudes of first order, or of $(n - 1)$th, as fundamental, at pleasure. In the language of modern projective geometry, it depends on whether one calculates with point or with plane coordinates.

Finally, two special cases of the general formula proved in theorem 112 should be mentioned. They are the two formulas

$$[A_nA_{n-1} \cdots A_2] = a_1$$

and

$$[A_nA_{n-1} \cdots A_1] = 1,$$

which will be much used later on (cf. Nos. 292, 299, and 300).

Theorem 112 was later employed by CLEBSCH in the Göttinger Abh. 17 (1872) and by F. MEYER in his book Apolarität und rationale Curven, Tubingen 1883, and in fact in the form of a theorem on matrices.

[68] 35. Since $A$ is of order $\alpha$, $n - \alpha$ is the order of $D_r$, and $m - (n-\alpha) = m + \alpha - n$ that of $C_r$; therefore $m + \alpha$ is necessarily $> n$ and consequently the product $[AB]$ is regressive. Now since the $[AD_r]$ are numbers and the $C_r$ belong to the domain of the elementary magnitude $B$, the whole theorem amounts to the following: If an arbitrary magnitude $A$ is multiplied by an elementary magnitude $B$ and the product $[AB]$ is regressive, then $[AB]$ is a magnitude belonging to the domain of $B$, for which an explicit representation is given in the theorem.

To grasp the theorem one notes that, in the case where the order of $C_r$ equals $m - 1$, each $D_r$ includes only one of the elementary factors of $B$, and that one can satisfy the equation $[C_rD_r] = B$ under these circumstances just by a change in the algebraic sign of the elementary factors in question.

There is a gap in the proof of No. 113. Thus if, following the prescription on p. 67, lines 7–6 from the bottom, one forms all combinations $A_1, A_2, \ldots$ on the condition that $A_r$ always consists of the magnitudes of $b_1, \ldots, b_n$ that are missing in $D_r$, then one by no means obtains all combinations of $b_1, \ldots, b_n$ of $\alpha$th class, although $A$ is in general only numerically derivable from all these combinations. This gap is however easily filled.

We can always write $A$ in the form

$$A = \sum \alpha_r A_r + \sum \alpha'_a A'_a,$$

where each $A_r$ includes precisely $m - (n-\alpha)$ and no more of the factors $b_1, \ldots, b_n$, while each $A'_a$ contains at least $m - (n-\alpha) + 1$ of these factors. Then each $A'_a$ has at least one factor in common with each $D_r$ and the progressive products $[A'_aD_r]$ are therefore all zero; furthermore, $B$ has in common with each $A'_a$ a domain of
higher than \((m + \alpha - n)\)th order, so that the regressive products \([A'_a B]\) are also all zero. Now if we form the product \([A B]\), then, as in the text,

\[
[A B] = \sum \alpha_r [A_r B] = \sum \alpha_r [A_r D_r] C_r = \sum \alpha_s [A_s D_r] C_r,
\]

which, since \([A'_a D] = 0\), one can also write as

\[
[A B] = \sum_r \left[ \left( \sum \alpha_s A_s + \sum \alpha'_s A'_s \right) D_r \right] C_r = \sum [AD_r] C_r.
\]

[73] 36. From this Number on, the concept of congruence is used with a somewhat broader meaning than was expressed in the definition of the congruence concept stated in No. 2. Thus while there only two nonzero magnitudes are called congruent if they stand in a numerical relation, from now on two vanishing magnitudes may be said to be congruent (see for example the second part of the proof of No. 121, and the Remark following No. 313).

[78] 37. If in the case \(e)\) \(q + r\) and \(q + s\) are both \(< n\), then

\[
[BAC] = (-1)^{qr}[ABC]
\]

and further

\[
[ACB] = (-1)^{(n-q-s)(n-r)}[B \cdot AC]
\]

and thus, since \(n - q - s = r - t\) and \(s - t + n - r = 2n - 2r - q\), one has in fact

\[
[BAC] = (-1)^{qt}[B \cdot AC].
\]

The case where \(q + r\) and \(q + s\) are both \(> n\) reduces to the one just settled if one takes the supplements on both sides.

If in the case \(f)\) \(q + r < n\) and \(q + s < n\), then \(q + s - n + r < n\), whence

\[
[BAC] = (-1)^{qr}[ABC] = (-1)^{qr}[ACB]
\]

and thus, since \(n - q - s = r - t\) and \(s - t + n - r = 2n - 2r - q\), one has in fact

\[
[BAC] = (-1)^{qt}[B \cdot AC].
\]

The case where \(q + r > n\) and \(q + s < n\) reduces to the one just settled if one takes the supplements on both sides.

If one of the two sums \(q + r\) or \(q + s\) equals \(n\), then \(B\) and \(C\) are also incident in the case \(e)\) and there is no longer a distinction between the cases \(e)\) and \(f)\). Thus for \(q + r = n\), where \(s = t\), the first formula of \(e)\) coincides with the second of \(f)\), and the second formula of \(e)\) with the first of \(f)\). If \(q + s = n\), the corresponding follows.

[79] 38. Refer to No. 123, Proof, Parts 2 and 3.

[81] 39. It is interesting to compare this expression for the shadow (Zurückleitungs) with that (Abschattung) given by GRASSMANN in the \(A_1\).

The magnitudes denoted by \(A, A', B, C\) in No. 129 are in the \(A_1\) denoted by \(A, A', G, L\) (\(A_1, \S 149\)), and upon using these characters the expression given in No. 129 for \(A'\) reads

\[
A' = \frac{[G \cdot AL]}{[GL]}.
\]
Now let \( p \) and \( m \) be the orders of \( A \) and \( G \) and thus \( n - m \) be the order of \( L \). Then (according to No. 128) for the case of the progressive shadow \( m \geq p \), whence \( p + (n - m) \leq n \), and thus (according to 58 and Part 1 of Proof, 120)

\[
[GL] = (-1)^{m(n-m)}[LG], \quad [AL] = (-1)^{p(n-m)}[LA],
\]

\[
[G \cdot LA] = (-1)^{(n-m)(m-p)}[LA \cdot G].
\]

On the other hand, for the case of the regressive shadow \( m \leq p \), whence \( p + (n-m) \leq n \), and thus

\[
[GL] = (-1)^{m(n-m)}[LG], \quad [AL] = (-1)^{(n-p)m}[LA],
\]

\[
[G \cdot LA] = (-1)^{m(p-m)}[LA \cdot G],
\]

since according to No. 95 \([LA]\) is a magnitude of \((p-m)\)th order and \(m+(p-m) \geq n\), so that the product \([G \cdot LA]\) is progressive. Thus in both cases

\[
A' = \frac{[LA \cdot G]}{[LG]},
\]

which is precisely the same expression as appears in the \( A_1 \).

[81] 40. “The products \([A_{u+1}C], \ldots, [A_uC]\) are however pure with respect to the factors \(a_1, \ldots, a_n\) (according to 114, 127, 128, and 119b).” Thus if one denotes the orders of \( A, B, C \) by \(\alpha, \beta, \gamma\) and first, if \(a_1, \ldots, a_n\) are magnitudes of first order, then (according to 127) the shadow is progressive and thus (according to 128)

\[
\alpha \leq \beta,
\]

or, since the order \(\gamma\) of the excluded domain \(C\) supplements the order of \(B\) up to that of the principal domain, that is \(\beta = n - \gamma\), it follows that

\[
\alpha \leq n - \gamma \quad \text{or} \quad \alpha + \gamma \leq n.
\]

The products \([A_{u+1}C], \ldots\) are therefore progressive with respect to the factors \(A_{u+1}\) and \(C, \ldots\), and thus (according to 119b) also remain purely progressive if one decomposes their factors into pure factors of first order.

Second, if the factors \(a, \ldots, a\) are magnitudes of \((n-1)\)th order, then (according to 127) the shadow is regressive and therefore (according to 128)

\[
\alpha \geq \beta \quad \text{or} \quad \alpha \geq n - \gamma, \quad \text{that is} \quad \alpha + \gamma \geq n.
\]

The products \([A_{u+1}C], \ldots\) are therefore regressive with respect to the factors \(A_{u+1}\) and \(C, \ldots\), and thus (according to 119b) also remain regressive if one decomposes their factors \(A_{u+1}\) and \(C\) into products of \((n-1)\)th order, that is, the products \([AC], \ldots\) are also purely regressive with respect to the factors \(a_1, \ldots, a_n\).

[82] 41. The formula utilized here is in No. 108 only proved for the case that \(B\) is subordinate to \(A\), but as is shown in Editorial Note 33 it is also valid if \(B\) is superordinate to \(A\).

[82] 42. {In this Note Grassmann’s son, Hermann Junior, illustrated the concept of the shadow in ordinary three-dimensional geometry with examples, since contemporary readers found the presentation in Numbers 127–129 too abstract. These are omitted here, as they are only of didactic interest. Tr.}

[84] 43. If \(m = n - 1\), each \(F_k\) includes only one of the units \(e_1, \ldots, e_n\) as a factor; one may therefore not always be able to satisfy the equations \([E_kF_k] = 1\) by
a suitable ordering of the factors of $F_k$; instead one must in certain circumstances take $F_k$ to be a negative unit. Cf. Editorial Note 35.

[87] 44. It still remains to be shown that the given values of $x_1, x_2, \ldots, x_r$ together with the completely arbitrary values of $x_{r+1}, \ldots, x_n$ actually satisfy equation (c).

It was already shown that, if equations (a) do not contain a contradiction, it must be possible to numerically derive the magnitude $b$ from the $a_1, \ldots, a_r$. Let

$$b = y_1a_1 + y_2a_2 + \cdots + y_ra_r;$$

and further, let

\[ x_{r+1}a_{r+1} + x_{r+2}a_{r+2} + \cdots + x_na_n = d. \tag{*} \]

Then, since by assumption the magnitudes $a_{r+1}, a_{r+2}, \ldots, a_n$ are numerically derivable from the $a_1, \ldots, a_r$, the magnitude $d$ is likewise derivable from these magnitudes, and thus can be represented in the form

$$d = z_1a_1 + z_2a_2 + \cdots + z_ra_r.$$ 

Now by virtue of (*), $c = b - d$, and thus

$$c = (y_1 - z_1)a_1 + (y_2 - z_2)a_2 + \cdots + (y_r - z_r)a_r.$$ 

But if one substitutes this value for $c$ into the expressions for $x_1, x_2, \ldots, x_r$ given in (f), one obtains

$$x_1 = y_1 - z_1, \quad x_2 = y_2 - z_2, \quad \ldots, \quad x_r = y_r - z_r;$$

and if one multiplies these equations respectively by $a_1, a_2, \ldots, a_r$ and adds them, there results

$$x_1a_1 + x_2a_2 + \cdots + x_ra_r = b - d,$$

that is, referring back to (*), the equation

\[ x_1a_1 + x_2a_2 + \cdots + x_ra_r + x_{r+1}a_{r+1} + x_{r+2}a_{r+2} + \cdots + x_na_n = b. \tag{c} \]

Thus equation (c) is actually satisfied by the values (f).

[89] 45. The two methods of solution of a system of linear equations admit a geometric interpretation in case $n < 4$, which will be developed here for the case $n = 3$.

In order to interpret the first solution geometrically, one regards the magnitudes $e^{(1)}, e^{(2)}, e^{(3)}$ as three noncoplanar displacements. The three magnitudes $a_1, a_2, a_3$, which according to equations (b) are numerically derived from the displacements $e^{(1)}, e^{(2)}, e^{(3)}$ by the coefficients standing in the columns of equations (a), are therefore likewise displacements, and, provided at first it is assumed that $[a_1a_2a_3] \neq 0$, they are noncoplanar. In addition, equation (c) implies that the displacement $b$ is numerically derivable from the three displacements $a_1, a_2, a_3$, and the derivation numbers $x_1, x_2, x_3$ are the unknowns sought. Thus if one decomposes the displacement $b$ into three summands $b_1, b_2, b_3$, which run parallel to the displacements $a_1, a_2, a_3$ respectively (see Fig. 22), which is only possible in one way so long as $[a_1a_2a_3] \neq 0$, then the three ratios of the three pairs of correlated displacements $b_i$ and $a_i$ are the unknowns $x_i$ sought, that is,

$$x_1 = \frac{b_1}{a_1}, \quad x_2 = \frac{b_2}{a_2}, \quad x_3 = \frac{b_3}{a_3}.$$
In case the combinatorial product \([a_1a_2a_3]\) = 0, but two of the magnitudes \(a_1, a_2, a_3\), say \(a_1\) and \(a_2\), still stand in no numerical relation to one another, one again constructs the four displacements \(a_1, a_2, a_3\) and \(b\), corresponding to equation (b). Thereby, if equations (a) do not contain a contradiction, one obtains four displacements in the same plane (see Fig. 23). One then assigns the magnitude \(x_3\) a value completely arbitrarily, reduces \(b\) by \(x_3a_3\) and sets \(b - x_3a_3 = c\). Finally, if one represents \(c\) as a multiple sum of \(a_1\) and \(a_2\), that is, as a sum of two displacements \(c_1\) and \(c_2\) that run parallel to \(a_1\) and \(a_2\) respectively, then \(x_1 = \frac{c_1}{a_1}\) and \(x_2 = \frac{c_2}{a_2}\) are the values of \(x_1\) and \(x_2\) corresponding to the arbitrary value of \(x_3\) chosen. If one chooses a common initial point for the displacements \(a_1, a_2, a_3, b,\) and \(c\) (as in Fig. 23), then one finds all values of \(x_1, x_2, x_3\) that satisfy the given equation (a) if one displaces the final point of the displacement \(c\) parallel to \(a_3\) and for each position of \(c\) decomposes the displacement \(b\) into the sum \(b = c + x_3a_3\) and decomposes the displacement \(c\) itself into the sum \(c = x_1a_1 + x_2a_2\). One proceeds similarly if each pair of the magnitudes \(a_1, a_2, a_3\) stand in a numerical relation.
In order to find a geometric interpretation for the second solution of the system of linear equations as well, one regards the four units $e_0, e_1, e_2, e_3$, whose combinatorial product was set to unity, as elementary points that form the vertices of a tetrahedron (see Fig. 24). Then the equations

\[
\begin{align*}
a^{(1)} &= \alpha_0^{(1)} e_0 + \alpha_1^{(1)} e_1 + \alpha_2^{(1)} e_2 + \alpha_3^{(1)} e_3, \\
a^{(2)} &= \alpha_0^{(2)} e_0 + \alpha_1^{(2)} e_1 + \alpha_2^{(2)} e_2 + \alpha_3^{(2)} e_3, \\
a^{(3)} &= \alpha_0^{(3)} e_0 + \alpha_1^{(3)} e_1 + \alpha_2^{(3)} e_2 + \alpha_3^{(3)} e_3,
\end{align*}
\]

which are derived from the units $e_0, e_1, e_2, e_3$ by the coefficients appearing in the rows of equations (α), are again points, which, provided the product $[a^{(1)} a^{(2)} a^{(3)}] \neq 0$, define a surface element $A = [a^{(1)} a^{(2)} a^{(3)}]$. If in addition one again sets

\[
X = x_0 |e_0| + x_1 |e_1| + x_2 |e_2| + x_3 |e_3|,
\]

whence $X$ represents a surface element, and where $x_0 = 1$, then the given equations (α) are identical to the equations

(δ) \[ [a^{(1)} X] = 0, \quad [a^{(2)} X] = 0, \quad [a^{(3)} X] = 0, \]

which express that the plane of the surface element $X$ goes through the three points $a^{(1)}, a^{(2)}, a^{(3)}$. This surface element can therefore be expressed in the form

(ε) \[ X = \lambda [a^{(1)} a^{(2)} a^{(3)}] = \lambda A, \]

where $\lambda$ denotes a number whose value can be ascertained with the help of the equation $|e_0 X| = x_0 = 1$. The expressions

(θ) \[ x_1 = \frac{e_1 A}{e_0 A}, \quad x_2 = \frac{e_2 A}{e_0 A}, \quad x_3 = \frac{e_3 A}{e_0 A} \]
for the unknowns $x_1, x_2, x_3$ resulting from equation (ε) then represent the unknowns as ratios of those four parallelepipeds defined by the surface element $A$ and each vertex of the basis tetrahedron. But two such parallelepipeds, for example $[e_1A]$ and $[e_0A]$, are to one another as are the two segments $e_1s_1$ and $e_0s_1$ into which the edge $e_1e_0$ of the tetrahedron is divided by its intersection $s_1$ with the plane $A$. The unknowns $x_1, x_2, x_3$ are therefore nothing but the ratios of the segments which the plane of the surface element $A$ produces on the edges of the tetrahedron $[e_1e_0], [e_2e_0], [e_3e_0]$, assuming that the segments are always reckoned from the vertices of the tetrahedron out to the point of the division.

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[92] 46. Cauchy encloses multiple products, such as would be denoted outer in Grassmann’s sense, between two vertical lines; he also uses brackets in a similar but not identical way.

[93] 47. With the help of the concept of the supplement Grassmann can now define the inner products of two magnitudes directly, whereas in the Geometrische Analyse he had always to provide outer products to which the inner products considered were proportional.

[96] 48. The second part of the proof of this theorem would be considerably shorter if one referred to Theorem 5 of Editorial Note 29, and made use of the Corollary to No. 90.

[97] 49. The editors of the Teubner Edition replaced the material in braces with the following:

149. If $E$, $F$, $G$ are units, and if $F$ is of higher order than $G$ and the product $[EF]$ is progressive and not zero, then

$$[EF]GE = [F]G.$$  

Further, if $F$ is of lower order than $G$ and the product $[GE]$ is progressive and not zero, then

$$[FE]GE = [F]G.$$  

Finally, if $F$ and $G$ are of the same order, and both products $[EF]$ and $[GE]$ are progressive and not zero, then both formulas apply.

Proof. 1. If $F$ and $G$ are not incident on one another, then $[EF]$ and $[EG]$ are also not incident on one another, and thus (according to 147) both sides of the equations to be proved are zero.

2. If on the other hand $F$ is incident on $G$, then the three cases distinguished in the theorem must be handled separately.

a. We assume, first, that the product $[EF]$ is progressive and not zero, and that $F$ is of higher order than $G$; then $G$, which is at the same time incident on $F$, is subordinate to $F$, and one can therefore represent $F$ in the form

$$F = [GH],$$

where the product $[GH]$ is progressive and $H$ is again a unit, or perhaps a negative unit. Consequently

$$[EF]GE = [E(GH)]GE,$$
or, since the product \( [E(GH)] \) is purely progressive, according to 119  
\[
[EF|EG] = [EGH|EG] \\
= H
\]
(according to 148), for \( [EGH] = [EF] \) is not zero by assumption, and \( [EG] \) is itself a unit (according to 77) as a nonvanishing progressive product of two units. But the right side of our equation is (again according to 148)  
\[
= [GH|G] = [F|G].
\]

b. Assuming, second, that the product \( [GE] \) is progressive and not zero, and that \( F \) is of lower order than \( G \), then \( F \), which is at the same time incident on \( G \), is subordinate to \( G \), and one can therefore represent \( G \) in the form  
\[
G = [HF],
\]
where the product \( [HF] \) is progressive and \( H \) is a unit, or perhaps a negative unit. Consequently  
\[
[FE|GE] = [FE]HFE,
\]
or, since the product \( [HFE] \) is purely progressive, (according to 119)  
\[
[FE|GE] = [FE]H(FE) \\
= |H
\]
(according to 148), for \( [H(FE)] = [HFE] = [GE] \) is not zero. But (again according to 148) the right side is again  
\[
= [F|HF] = [F|G].
\]

c. Finally, if we assume that the products \( [EF] \) and \( [GE] \) are both progressive and not zero, and that \( F \) and \( G \) are of the same order; then, since they are at the same time incident on one another, the domains of the two magnitudes must coincide; consequently \( G \) is subordinate to \( F \) and \( F \) to \( G \), and thus according to Parts 2a and b of the Proof both formulas apply.

**Remark.** A corresponding theorem also applies if \( [EF] \) and \( [EG] \) are regressive products, except that in this case interchange the expressions “higher” and “lower”. The proof is precisely as in No. 148.  

[97] 50. The formula also applies if \( q = r \); for then \( q(r - 1) \) is certainly even, and thus the formula of No. 150 takes the form \( [A|B] = [B|A] \), which corresponds with Nos. 141 and 144.  

[98] 51. That is, in the Proofs of Nos. 147 and 148.  

[98] 52. The expression “elements of a domain” has not previously been used in a precise sense; what is intended one can however infer from the names “line element” and “surface element” introduced later (cf. Nos. 249 and 257). Thus by an element of a domain of qth order GRASSMANN obviously means an elementary magnitude of qth order belonging to the domain; thus he ought properly to say:  

*A domain of mth and a domain of qth order in a principal domain of nth order are called normal to one another if an elementary magnitude of the first domain is normal to an elementary magnitude of qth order of the second.*

Thus one needs only a single “element” of the one domain to compare with a single “element” of the other, since according to No. 70 all elements of a domain are congruent in the sense of No. 2.
If one has two elementary magnitudes $A$ and $B$, of $m$th and $q$th order respectively, and if $m \leq q$, then the product $[A|B]$ is progressive;* if however $m \geq q$, the product $[B|A]$ is progressive. Thus according to No. 109 the two magnitudes $A$ and $B$ are normal to one another if and only if that one of them whose order is no greater than that of the other has a domain of first or higher order in common with the supplement of the other.

[98] 53. This definition is based on an assumption that ought properly be proved, namely that two completely normal domains or magnitudes are also normal in the sense previously defined. Since one can only justify this assumption with the help of a part of the development in Nos. 153 to 167, it would have been better to introduce the definition of completely normal domains and magnitudes only after No. 167, and then to formulate a theorem, somewhat as follows:

**Theorem 1.** Two completely normal domains or magnitudes of a principal domain of $n$th order are also normal to one another in the sense of No. 152.

This would have been the more appropriate as the concept “completely normal” does not appear at all in Nos. 153–167, and only plays a role in Nos. 171 and 172.

In order to prove Theorem 1 we will consider two completely normal domains of $m$th and $q$th orders respectively. Then according to No. 163 we can in each of the two domains find a normal system of the appropriate order and of numerical value one, and in fact let $u_1, \ldots, u_m$ and $v_1, \ldots, v_q$ be the two normal systems. According to the concept of complete normality every magnitude $\sum \alpha_k u_k$ is normal to every magnitude $\sum \beta_k v_k$, and thus in particular every $u_k$ to every $v_k$. It therefore follows that in addition $u_1, \ldots, u_m, v_1, \ldots, v_q$ all together form a normal system of numerical value one, and (according to 157) at the same time there is no numerical relation between $u_1, \ldots, u_m, v_1, \ldots, v_q$. Consequently, on the basis of No. 161, we can append to $u_1, \ldots, u_m, v_1, \ldots, v_q$ other $n - m - q$ magnitudes $w_1, \ldots, w_{n - m - q}$ in such a way that a complete normal system of numerical value one results.

Furthermore, according to No. 167

\[(*) \quad [v_1 \cdots v_q] = \pm [u_1 \cdots u_m \cdot w_1 \cdots w_{n - m - q}]\]

and thus

\[[(u_1 \cdots u_m)][v_1 \cdots v_q] = 0.\]

For, if $m \leq q$, the product $[(u_1 \cdots u_m)][v_1 \cdots v_q]$ is progressive (cf. the previous Editorial Note) and therefore vanishes on account of $(*)$, according to No. 60; but if $m \geq q$, the product is regressive and vanishes according to 109, since on account of $(*)$ the covering domain of its factors is of $(n - q)$th order, and thus is smaller than $n$. Consequently the two completely normal magnitudes $[u_1 \cdots u_m]$ and $[v_1 \cdots v_q]$ and likewise their domains are really normal to one another in the sense of No. 152.

From the above considerations it follows that one of two completely normal magnitudes is always subordinate to the supplement of the other. Conversely it is clear that every elementary magnitude that is subordinate to the supplement of a magnitude $[v_1 \cdots v_q]$, is completely normal to that magnitude. Consequently we can also say:

**Theorem 2.** Two magnitudes are completely normal to one another if and only if one is subordinate to the supplement of the other.

Finally we can express the following, no longer independent, theorem:

*Indeed, the supplement $|B$ is an elementary magnitude of $(n - q)$th order; cf. the Corollary to No. 90 and Editorial Note 29.*
THEOREM 3. If two domains are completely normal to one another, every magnitude of one of the domains is normal to every magnitude of the other.

Only with this theorem does the nomenclature “completely normal” appear in its proper light.

[98] 54. These words are somewhat astonishing, for the “must” makes no sense; one could equally well choose three arbitrary noncoplanar displacements as units.

[99] 55. The circular evolution is, just like the linear (cf. No. 71), equivalent to a linear homogeneous substitution of a particular form. Thus if one subjects the series of \( n \) magnitudes \( a_1, \ldots, a_n \), which stand in no numerical relation to one another, to a positive circular evolution, say by replacing \( a_1 \) and \( a_2 \) with \( \cos \alpha \cdot a_1 + \sin \alpha \cdot a_2 \) and \( \cos \alpha \cdot a_2 - \sin \alpha \cdot a_1 \), then the magnitudes \( \sum x_k a_k \) of the domain of \( n \)th order derivable from \( a_1, \ldots, a_n \) will be interchanged among themselves by the linear homogeneous transformation

\[
\begin{align*}
    x_1' &= x_1 \cos \alpha - x_2 \sin \alpha, \\
    x_2' &= x_1 \sin \alpha + x_2 \cos \alpha, \\
    x_3' &= x_3, \\
    \vdots \\
    x_n' &= x_n.
\end{align*}
\]

(1)

This transformation has determinant 1 and obviously leaves the quadratic form \( \sum x_k^2 \) invariant, that is, it is an orthogonal substitution. The totality of all \( \infty \) transformations of the form (1) constitute a one-parameter group in the sense of Lie.

Under a negative circular evolution the magnitudes \( a_1 \) and \( a_2 \) are replaced by the magnitudes \( \cos \alpha \cdot a_1 + \sin \alpha \cdot a_2 \) and \( -(\cos \alpha \cdot a_2 - \sin \alpha \cdot a_1) \). It corresponds to a linear homogeneous transformation of the form

\[
\begin{align*}
    x_1' &= x_1 \cos \alpha + x_2 \sin \alpha, \\
    x_2' &= x_1 \sin \alpha - x_2 \cos \alpha, \\
    x_3' &= x_3, \\
    \vdots \\
    x_n' &= x_n.
\end{align*}
\]

(2)

This is also orthogonal, but has determinant \(-1\). It therefore follows that the totality of all \( \infty \) transformations (2) do not form a group, but that the transformations (1) and (2) together form a non-continuous group.


[100] 57. If in a domain of \( m \)th order one has two normal systems of \( m \)th order, \( a_1, \ldots, a_m \) and \( b_1, \ldots, b_m \) of the same numerical value, then according to No. 157* neither the \( a_1, \ldots, a_m \) nor the \( b_1, \ldots, b_m \) stand in a numerical relation, and thus the \( b_1, \ldots, b_m \) can be numerically derived from the \( a_1, \ldots, a_m \):

\[
b_\mu = \alpha_\mu_1 a_1 + \cdots + \alpha_\mu_m a_m \quad (\mu = 1, \ldots, m),
\]

where according to No. 63 the determinant of the \( \alpha_\mu \) is \( \neq 0 \). Now if in an arbitrary magnitude \( \sum x_\mu a_\mu \) of the domain under consideration one replaces the magnitudes \( a_1, \ldots, a_m \) with the \( b_1, \ldots, b_m \), one obtains a new magnitude \( \sum x_\mu b_\mu = \sum x_\mu a_\mu \)

*Numbers 157–159 would have been better placed before No. 154.
of the domain. Consequently the transition from one normal system to the other is equivalent to the linear homogeneous transformation

$$x'_\mu = \alpha_{1\mu}x_1 + \cdots + \alpha_{m\mu}x_m \quad (\mu = 1, \ldots, m),$$

by virtue of which the magnitudes of first order of the domain under consideration will be interchanged among themselves. Since in addition

$$\left[ \sum x_\mu b_\mu \right]^2 = \left[ \sum x'_\mu a_\mu \right]^2$$

and under our assumptions the equations

$$b_1^2 = \cdots = b_m^2 = a_1^2 = \cdots = a_m^2,$$

$$[b_\mu | b_\nu] = 0, \quad [a_\mu | a_\nu] = 0 \quad (\mu \neq \nu)$$

are valid, it follows that \(\sum x'_\mu^2 = \sum x_\mu^2\). Thus the linear homogeneous transformation (1) leaves the quadratic form \(\sum x_\mu^2\) invariant. In other words, the substitution (1) is orthogonal.

Conversely, every real orthogonal substitution (1) corresponds to the transition from one normal system of \(m\)th order to another, numerically equal to it. In particular therefore, according to Editorial Note 55, a circular evolution must transform any given normal system into a numerically equal normal system; and this is demonstrated in 155.

[102] 58. If the two normal systems are of the same order, say the \(m\)th, then with this theorem the following is also proven:

Every real linear homogeneous substitution

$$x'_\mu = \alpha_{1\mu}x_1 + \cdots + \alpha_{m\mu}x_m \quad (\mu = 1, \ldots, m),$$

for which the quadratic form \(\sum x'_\mu^2\) remains invariant, can thus be obtained if one carries out, one after the other, a series of linear homogeneous substitutions of the special forms (1) and (2) displayed in Editorial Note 55. It therefore follows in addition that the determinant of the substitution is \(\pm 1\).

[102] 59. That is, replace a positive circular evolution with the corresponding negative one and conversely.

[103] 60. This only holds if the units are three equally long and mutually orthogonal displacements. Cf. Editorial Note 53.

[103] 61. One observes that the shadow is progressive or regressive according as the orders \(\alpha\) and \(\beta\) of \(A\) and \(B\) stand in the relation \(\alpha \leq \beta\) or \(\alpha \geq \beta\). If on both sides of the equation for \(A'\) one takes the supplement, then one recognizes that \(|A'|\) is the normal shadow of \(|A|\) on the domain of \(|B|\). If \(A'\) was a progressive shadow, the shadow \(|A'|\) would naturally be regressive and conversely.

[104] 62. It is indeed shown by this theorem that the generalization of the concept of the supplement introduced in the proof of No. 110 coincides with the supplement defined in Nos. 89 and 90, provided \(a_1, \ldots, a_n\) form a complete normal system of numerical value one, but a demonstration is still lacking that the normal system of numerical value one is the only one for which this happens (cf. p. 66, lines 3–4).
This demonstration is easily supplied. Thus if \( a_1, \ldots, a_n \) are \( n \) magnitudes of the principal domain \( e_1, \ldots, e_n \), whose combinatorial product has the value one, then according to the definition on p. 64,
\[
I a_k = [a_ka_1 \cdots a_{k-1}a_{k+1} \cdots a_n][a_1 \cdots a_{k-1}a_{k+1} \cdots a_n],
\]
where the first factor has one of the two values \( \pm 1 \); consequently
\[
[a_k I a_k] = 1, \quad [a_k I a_j] = 0 \quad (j \neq k).
\]
Now for any arbitrary magnitude \( A \) it is supposed that always \( IA = |A| \), and so it is certainly necessary that always \( I a_k = |a_k| \), and therefore that the equations
\[
[a_k |a_k] = 1, \quad [a_k |a_j] = 0 \quad (j \neq k)
\]
obtain. It is therefore necessary that \( a_1, \ldots, a_n \) form a normal system of numerical value one. That this is also sufficient is shown in No. 167.

[105] 63. Naturally the theorem also holds if \( B' \) is the regressive normal shadow of \( B \) on the domain of \( A \). For then (according to Editorial Note 61) \( |B' \) is the progressive normal shadow of \( |B \) on the domain of \( |A| \), and thus according to No. 169
\[
||A||B|| = ||A||B'||, \quad ||B||A|| = ||B'||||A||,
\]
and thus according to No. 101 the equations of No. 169 also apply in this case.

[106] 64. How this can be done is more precisely specified in the Proof of Theorem 1 in Editorial Note 52.

[107] 65. The editors of the Teubner Edition replaced the material in braces with the following:

One can now show that the product \( [A_1 B_1|A_1 C_r] \) satisfies all the conditions of the first formula of theorem 149. Let \( n \) be the order of the principal domain and \( A_1 \) the product of \( m \) magnitudes \( a_1, \ldots, a_m \) of the normal system. Then (according to 159) all the magnitudes of first order normal to \( a_1, \ldots, a_m \) belong to the domain of the \( n - m \) remaining magnitudes \( a_{m+1}, \ldots, a_n \) of this normal system. The sum of the orders of \( A_1 \) and \( B_1 \) can therefore be at most \( m + n - m \), that is \( = n \), whence the product \( [A_1 B_1] \) is necessarily progressive. But (according to 109) it is also not zero, since the domains of the magnitudes \( A_1 \) and \( B_1 \) have no magnitude of first order in common. Thus since the domains of \( A_1 \) and \( B_1 \) are perfectly normal to one another, it follows (according to 152) that every magnitude of first order of the domain of \( A_1 \) is normal to every magnitude of first order of the domain of \( B_1 \). Consequently, if the two domains had a nonzero magnitude in common, this magnitude would be normal to itself, whence its inner square would be zero, which (according to 146) is impossible for a nonzero magnitude. In addition, by assumption \( B \) is of order greater than or equal to that of \( C \), whence \( B_1 \) is also of order greater than or equal to that of \( C_r \). Finally (according to 168) theorem 149, proved initially for units of higher order, that is for combinatorial products of the original units, still holds if, in the place of products of units, there appear combinatorial products of the magnitudes of an elementary normal system, and thus in our case; that is
\[
[A_1 B_1|A_1 C_r] = [B_1|C_r],
\]
for \( r = 1, 2, \ldots \).

[108] 66. The editors of the Teubner Edition replaced the material in braces with the following:
First, let the elementary factors of \([AB]\) all be normal to one another and let the system of these factors \(a_1, \ldots, a_m\) be extended to a complete normal system by the addition of the magnitudes \(a_{m+1}, \ldots, a_n\); further, let \(A, A_1, \ldots, A_q, A_{q+1}, \ldots, A_t\) be those multiplicative combinations of \(a_1, a_2, \ldots, a_n\) whose orders are equal to that of \(A\), and in particular let \(A, A_1, \ldots, A_q\) be the combinations of the first \(m\) magnitudes \(a_1, \ldots, a_m\). Then \(A\), which is of the same order as \(A_i\), is numerically derivable from the multiplicative combinations \(A, A_1, \ldots, A_t\). Let

\[
A = \alpha A + \alpha_1 A_1 + \cdots + \alpha_q A_q + \alpha_{q+1} A_{q+1} + \cdots + \alpha_t A_t;
\]

then

\[
[AB|AB] = [AB](\alpha A + \alpha_1 A_1 + \cdots + \alpha_q A_q + \alpha_{q+1} A_{q+1} + \cdots + \alpha_t A_t)B = \alpha [AB|AB] + \alpha_1 [AB|A_1B] + \cdots + \alpha_q [AB|A_qB] + \alpha_{q+1} [AB|A_{q+1}B] + \cdots + \alpha_t [AB|A_tB].
\]

This development can however be simplified even further; for it can be shown that all the terms on the last line vanish.

Thus if one imagines the magnitudes \(a_1, \ldots, a_n\) as forming the magnitudes \(B_i\), whose order agrees with that of \(B\), and \(B\) as represented by a multiple sum of the \(B_i\), then from the terms of the last line of our development there result pure terms of the form

\[
\alpha_{q+k} B_i [AB|A_{q+k}B_i].
\]

But here all the \(A_{q+k}\) include as a factor one of the magnitudes \(a_{m+1}, \ldots, a_n\) that do not appear in the product \([AB]\), and these therefore also appear in the products \([A_{q+k}B_i]\), which by assumption are progressive. These products are therefore not subordinate to the product \([AB]\). But they are also not superordinate to them; for, since by assumption \(B\) is of order less than or equal to that of \(B\), every product \([A_{q+k}B_i]\) must lack at least one factor of \([AB]\). The products \([A_{q+k}B_i]\) are therefore not incident on \([AB]\), and thus (according to 147 and 168) generally

\[
[AB|A_{q+k}B_i] = 0.
\]

The above development therefore simplifies to the following:

\[
[AB|AB] = \alpha [AB|AB] + [AB|A_1B] + \cdots + \alpha_q [AB|A_qB].
\]

Now since (by assumption) \([AB] = [A_1B_1] = \cdots\), we obtain the expression

\[
= \alpha [AB|AB] + \alpha_1 [A_1B_1|A_1B] + \cdots + \alpha_q [A_qB_q|A_qB].
\]

Because the elementary factors of \([AB]\) are all normal to one another, and identical to those of \(\pm [A_1B_1], \ldots\) (by assumption), \(A\) is perfectly normal to \(B\), and likewise \(A_1\) to \(B_1\), \ldots. Consequently (according to 171) the previous expression

\[
= \alpha A^2[B|B] + \alpha_1 A_1^2[B_1|B] + \cdots + \alpha_q A_q^2[B_q|B].
\]

But now

\[
[A_r|A] = [A_r|\alpha A + \alpha_1 A_1 + \cdots + \alpha_t A_t)] = \alpha_r A_r^2,
\]

since (according to 147, 168) \(A_r\) inner multiplied with the magnitudes \(A, A_1, \ldots\) gives zero, except for \(A_r\). Thus in the equation previously established one can replace \(\alpha_r A_r^2\) by \([A_r|A]\), and that expression becomes

\[
= [A|A][B|B] + [A_1|A][B_1|B] + \cdots + [A_q|A][B_q|B],
\]

that is, formula \((a)\) holds under our assumption.
[116] 67. According to Nos. 148 and 150 (see also Editorial Note 51),
\[ |E|EG| = (-1)^{(q-1)}|E|G = (-1)^{(q-1)}|G = |E'|E'G|, \]
where \( p \) and \( q \) are understood to be the orders of \( E \) and \( EG \).

[117] 68. It is easy to show that the value of the expression for \( \cos \angle AB \) lies between the limits \(-1\) and \(+1\), and thus that \( \angle AB \) is real. Thus if \( A = \sum \alpha_k E_k \)
and \( B = \sum \beta_j E_j \), then according to Nos. 143, 151, and 146:
\[ [A|B] = \sum \alpha_k \beta_k, \quad \alpha^2 = \sum \alpha_k^2, \quad \beta^2 = \sum \beta_k^2 \]
and thence follows, as is known,
\[ [A|B]^2 \leq \alpha^2 \beta^2. \]

One misses here and in the following a definition of the angle between two magnitudes of differing order. Perhaps GRASSMANN, in the puzzling Remark following No. 213, had something of the sort in mind; but nothing certain on this point can be established.

If one wishes to define the angle between two elementary magnitudes \( A \) and \( B \) of orders \( \alpha \) and \( \beta \), two procedures offer themselves.* One can either define this angle as the angle between \( A \) and the normal shadow \( A' \) of \( A \) on the domain of \( B \), or as the angle between \( B \) and the normal shadow \( B' \) of \( B \) on the domain of \( A \). It can be shown that these two definitions express the same thing and at the same time include GRASSMANN’s definition of the angle between two magnitudes of the same order.

Thus according to 165,
\[ A' = \frac{B(A|B)}{B^2}, \quad B' = \frac{A(B|A)}{A^2}, \]
and thus according to Nos. 195 and 98,
\[ \cos \angle AA' = \frac{[A|A']}{\sqrt{A^2 A'^2}} = \frac{[A(|B||A|)]}{B^2 \sqrt{A^2 A'^2}}, \]
\[ \cos \angle B'B = \frac{[B|B']}{\sqrt{B^2 B'^2}} = \frac{[(A|B|)]|B|}{A^2 \sqrt{B^2 B'^2}}. \]

If we assume for simplicity that \( \alpha \leq \beta \), then the three magnitudes \( A, |B, |A||B \)
have the orders \( \alpha, n-\beta, \) and \( \beta-\alpha, \) respectively, whence their product is according to No. 116 purely progressive, and thus, according to Nos. 119 and 97
\[ [A(|B||A||B|)] = [(A|B)| (A|B)] = [A|B]^2. \]
Likewise it follows, considering No. 124, Case a, that
\[ [(A|B|)]|B| = (-1)^{(\beta-\alpha)(n-\beta)}[(A|B|)(B|A)], \]
and on the other hand, according to 150 and 92,
\[ | [A|B] = (-1)^{\alpha(\beta-1)}||[B|A] \]
\[ = (-1)^{\alpha(\beta-1)+(\beta-\alpha)(n-\beta+\alpha)}|B|A], \]

*In ordinary space one defines the angle between a straight line and a plane as the angle between the straight line and its orthogonal projection to the plane. The following treatment is just the natural generalization of this definition to a domain of nth order.
and consequently also
\[(A(B|A)|B) = [(A|B)|(A|B)] = |A|B|^2,\]
so that the numerators in the two expressions \(\cos \angle AA'\) and \(\cos \angle B'B\) correspond.

In order to prove the correspondence of the denominators as well, we must calculate \(A'^2\) and \(B'^2\).

One has
\[A'^2 = [A'|A'] = \frac{[A'|B|(A|B)]}{B^2}.
\]
Here the product of the three magnitudes \(A', |B|, \) and \(|A|B|\) is purely regressive, whence
\[A'^2 = \frac{[A'|B|(A|B)]}{B^2} = \frac{|A|B|^2}{B^2},\]
since according to No. 169 and Editorial Note 63, \([A'|B] = |A|B|\). Likewise
\[B^2 = \frac{|B|A|}{A^2} = \frac{|A|B|^2}{A^2}.
\]
On account of these formulas one obtains at once
\[
\cos \angle AA' = \cos \angle B'B = \sqrt{\frac{|A|B|^2}{A^2B^2}},
\]
where the positive value of the root is to be taken.

Thereby one may, if \(A\) and \(B\) are of arbitrary order, define the angle \(AB\) by the equation
\[\cos \angle AB = \sqrt{\frac{|A|B|^2}{A^2B^2}}.
\]
Here one can, so long as \(A\) and \(B\) are of different orders, leave the sign of the root unspecified; but if \(A\) and \(B\) are of the same order and thus \(|A|B|\) is a number, one must set the square root of the numerator = \(|A|B|\) and choose the positive square root in the denominator.

[117] 69. Here too it must be shown that \(\sin(abc\cdots)\) belongs to a real angle, thus that the numerical value of \([abc\cdots]\) is not greater than the product \(a\beta\gamma\cdots\) of the numerical values of the individual factors \(a, b, c, \ldots\). In GRASSMANN’s style the proof can be given as follows:

We imagine each of the magnitudes \(a, b, c, \ldots\) divided by its numerical value, so that a product \([a_1\cdots a_n] (m \leq n)\) results, whose factors of first order, \(a_1, \ldots, a_m\) are all numerically equal to one; then we need only show that the numerical value of \([a_1\cdots a_m]\) can be no greater than one. For this we may assume that the product \([a_1\cdots a_m]\) does not vanish, for if it were its numerical value would be zero.

If \(a_1, \ldots, a_m\) are normal to one another, then according to No. 175
\[
[(a_1\cdots a_m)|(a_1\cdots a_m)] = [a_1\cdots a_m]^2 = 1.
\]
If on the other hand they are not normal to one another, we may assume without loss of generality that \(a_1\), say, is not normal to all the \(m - 1\) magnitudes \(a_2, \ldots, a_m\), and in addition, according to Nos. 160 and 163, we can find a normal system of \(m\)th order \(a_1, u_2, \ldots, u_m\) of unit numerical value, whose domain coincides with the domain of the magnitudes \(a_1, \ldots, a_m\). Then
\[
a_k = \lambda_k a_1 + \sum_{v=2}^{m} \lambda_{k,v} u_v = \lambda_k a_1 + a'_k \quad (k = 2, \ldots, m),
\]
where the \( \lambda_2, \ldots, \lambda_n \) are certainly not all zero. From the equations
\[
a'^2_k = 1 = \lambda^2_k + \sum_v \lambda^2_{kv}, \quad a''^2_k = \sum_v \lambda^2_{kv}
\]
it therefore follows that the product \( \rho_2 \cdots \rho_n \) of the numerical values \( \rho_2, \ldots, \rho_n \) of \( a'^2_2, \ldots, a'^2_n \) is smaller than one. If we now set \( a'_k = \rho_k a''_k \), then \( a''_k \) is numerically equal to one, and according to No. 67 we get
\[
[a_1 \cdots a_m] = [a'_1 a'_2 \cdots a'_m] = \rho_2 \cdots \rho_m [a''_1 a''_2 \cdots a''_m].
\]
Since here the numerical value of the product \([a'_1 a'_2 \cdots a'_m]\) is obviously greater than that of \([a_1 \cdots a_m]\), we can say: “If one has a nonvanishing product of \( m \) magnitudes of first order \( (m \leq n) \), all of which have numerical value one, but which are not normal to one another, then in the domain of these magnitudes one can always find \( m \) magnitudes of first order of numerical value one, whose product has a greater numerical value than that of the given product.” One can now continue this increase in the numerical value until one is led to a product whose factors are normal to one another; since on the other hand a product of the nature considered, whose factors are normal to one another, possesses the numerical value one, the following theorem holds:

**Theorem.** If \( a_1, \ldots, a_m \) are nonzero magnitudes of first order in a principal domain of \( n \)th order \( (m \leq n) \), then the numerical value of the product \([a_1 \cdots a_m]\) is no greater than the product of the numerical values of \( a_1, \ldots, a_m \), and in fact it is equal to this product if the \( a_1, \ldots, a_m \) are normal to one another.

[118] 70. That here and in No. 199 \( a, b, c, d \) are supposed to be magnitudes of first order follows both from the notation as lower-case Latin characters and from the application of the theorem 127.

Otherwise it is not without interest that the theorems 175–182, 185–194, 196, 198–199, 201–205, 208–210, 213–215 remain applicable if one replaces the magnitudes of first order and their numerical values appearing therein with magnitudes of \((n-1)\)th order and their numerical values. Thus in the first place every magnitude of \((n-1)\)th order is the supplement of a perfectly unique magnitude of first order. Second, according to No. 98 the supplement of a product equals the product of the supplements of its factors, and thus in particular the supplement of an inner product is equal to the inner product of the supplements of its two factors, whence it follows at once that the supplement of a magnitude always has the same numerical value as the magnitude itself. Finally it is also obvious that \( \cos \angle ab = \cos \angle |a|b \), since \([a|b]\) is a number and thus \(|[a|b]| = |a|b|\). If one bears in mind that every equation in the Numbers referred to continue to hold if one takes the supplement on both sides, one perceives immediately that precisely the same equations also apply for magnitudes of \((n-1)\)th order.

[118] 71. Here the positive value of the square root is to be understood.

[121] 72. If in ordinary space one chooses for units three mutually orthogonal displacements of length one, then \( a, b, c \) are three arbitrary displacements in space which one may produce from a point \( O \), and \( \sin(abc) \) is precisely the expression that v. STAUDT has denoted the sine of the vertex \( a, b, c \) (Creille’s Journal 24, 255 (1842)).
[121] 73. The editors of the Teubner Edition dropped this Remark, since it is not clear what Grassmann really had in mind in this regard.

[121] 74. It seems worth while to say a few words in conclusion about the significance of the concepts introduced in Nos. 151–215.

If one thinks of the units $e_1, \ldots, e_n$ in an $n$-fold extended Euclidean space as $n$ mutually orthogonal displacements, all of whose lengths are equal to the unit of length, then every magnitude $\sum x_v e_v$ is likewise represented by a displacement whose length is equal to the numerical value of $\sum x_v e_v$. Mutually orthogonal magnitudes of first order are described by mutually orthogonal displacements, and every elementary normal system by $n$ mutually orthogonal displacements; the angle between two magnitudes of first order is equal to the angle between the corresponding displacements, and so forth. For the case $n = 3$ Grassmann himself has carried this out in detail.

Now, as Lie has emphasized,* all these considerations are grounds for saying that Grassmann worked on non-Euclidean geometry. Thus if Euclidean geometry applies in a space of $n$ dimensions, then among the manifold of all the displacements of this space, or, what is the same thing, in the infinitely distant $(n - 1)$-fold extended plane, the non-Euclidean geometry discovered by Riemann applies, and Grassmann has in fact developed this geometry here.

If we recall the connection between the normal system and the orthogonal substitutions (cf. Editorial Note 58) and consider that in an $n$-fold extended Euclidean space the orthogonal substitutions of unit determinant are nothing but the rotations about the origin of coordinates, then with study (Editorial Note 4 to Linear Extension Theory, translation of the A1) we can express the state of affairs thus: The investigations of Nos. 151–215 are contributions to the theory of the invariants of the group of all rotations about a point.

One is led to an interpretation somewhat more general but not essentially different from the above, if one interprets the coefficients $x_1, \ldots, x_n$ of the magnitude of first order $\sum x_v e_v$ as arbitrary homogeneous coordinates in an $(n - 1)$-fold extended plane space. The concept of numerical value no longer has a geometric significance, since its analytical expression is not homogeneous of zeroth order in the coordinates. On the other hand, for example, the angle between two magnitudes of first order $\sum x_v e_v$ and $\sum y_v e_v$ coincides with the distance between the two points $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ if one takes as fundamental for the measure of this distance Cayley’s definition of measure with respect to the fundamental manifold $\sum x_v^2 = 0$. To be sure, Cayley’s definition of measure is older than the A2, for it was originated in 1839.

[126] 75. This Number would be more easily understood if the special theorem denoted a Corollary had been placed at the head of the Number. Thus the thrust of the main theorem of No. 222 is only comprehensible from the Corollary; for it assumes one already knows that $A - R, B - R, \ldots$ are displacements, which only emerges from the Corollary. The transposition could be done without hesitation, since the proof of the Corollary is completely independent of the main theorem.


The demonstration “that there is no other addition of points and lines than that given here” cannot be regarded as adequate, for the entire subsequent proof is

* Theorie der Transformationsgruppen, Bd. III, P. 534f.
only justified if one, first, makes the various special assumptions that GRASSMANN introduces, which can have no basis in the nature of the subject; second, as is done in the text, inserts the word “simple” throughout; and finally, assumes Euclidean geometry from the outset. Other assumptions produce yet other types of addition of points, cf. STUDY, Wiener Berichten 91, 111 (1885).

[143] 77. After Nos. 254 and 262 one anticipates the introduction of a name for the products of two and three displacements corresponding to the names already introduced in Definitions 249, 257, and 265 for the products of two, three, and four points, say in the form:

We call the product \([ab]\) of two displacements a \textit{surface area}, the area of the parallelogram \(ab\) its content and the position of this parallelogram its position.

Further: We call the product \([abc]\) of three displacements \(a, b, c\) a \textit{volume} and the content of the parallelepiped its content.

In fact, in subsequent Numbers (cf. 330, 346, and 347) the expressions surface area and volume are often used for these products, but without these names being specifically introduced as technical terms. To be sure, on account of their similarity to the expressions surface element and volume element they are not especially suitable. On this account R. MEHMK has introduced (first in a lecture at the Stettin Polytechnic, Summer 1881) a new name, “the field” \{das Feld\}, for the product of two displacements, and he has been joined by G. MAHLER and F. KRAFT. From the other side the expression “the cell” \{das Fach\} has been proposed for the product of three displacements.

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[144] 78. “One could have defined … ”: This GRASSMANN had originally done. In the language of the \(A_1\) the content of the triangle \(ABC\) is designated the “extension” of the outer product \([ABC]\). On the other hand, in his Notice of the \(A_1\) published in Grunerts Archiv, GRASSMANN said explicitly: “The product \(ABC\) denotes the triangle with vertices \(A, B, C\), understood as (etc.).”

[146] 79. The Definition given in No. 71 of the concept of the \textit{elementary} linear evolution is not rigorously followed here and later (cf. No. 505); for the requirement that the magnitude whose multiple is to be added be a \textit{neighbor} of the incremented magnitude, is dropped.

[150] 80. By “the infinitely distant line element” is of course meant a line element resulting from the multiplication of two infinitely distant points, and thus the product of two displacements.

[160] 81. Regarding the Proof, refer to Editorial Note 34.

[163] 82. In the Proof it should have been mentioned that the homogeneous equation of \(n\)th degree between \(x_1, x_2, x_3\) includes a power of \(x_1\), say \(x_1^m\), as a factor; thus in the “ordinary coordinates” one gets an algebraic curve of \((n - m)\)th order, and only if one appends to this curve the infinitely distant straight line counted \(m\) times, does one obtain the structure of \(n\)th order represented by the equation \(\Phi_{n,x} = 0\). The Proof of No. 311 provides an occasion for a corresponding remark. Cf. also the Remark to No. 329.

[166] 83. It should have been noted that according to No. 320 the product \([paBc_1D]\) = \(q\) certainly does not vanish, and that on the same grounds \([qe]\) also does not vanish.
84. “Conversely, etc.” Thus the last equation says that \((g) L k = 0\), and since \( (g) C \) manifestly vanishes, \((g) \equiv [L_k C]\). On the other hand \((h) \equiv [L_k C]\), whence \((g) \equiv (h)\), and, since \((g)\) and \((h)\) are simple points, \((g) = (h)\), whence \(g = h\) follows immediately.

85. This Remark is already unclear in its second paragraph, and the fourth can only be preserved by main force despite the interpolations. Perhaps it would have been better to suppress the Remark completely except for its first and last paragraphs, for the type of normal shadow on points, lines and planes described in the Remark can only be realized if one replaces the concept of the supplement previously introduced by a completely different one.

Thus it would be necessary, besides the system of original units \(a, b, c, d\) consisting of a simple point \(a\) and three mutually orthogonal displacements \(b, c, d\), to adopt, for each individual finitely distant point \(x\) in space, yet another system of units that specify the supplements of the point \(x\) itself and the line and surface elements going through it, and indeed the special system of units belonging to the point \(x\) must, besides the unit displacements \(b, c, d\) established for all points of space, include as first unit the simple point congruent with the point \(x\). The supplements referred to this variable system of units — for the moment let them be denoted by \(\mathfrak{J}\) — then to be sure yield, when applied to line and surface elements, the fields and displacements orthogonal to these structures, and the normal shadow belonging to this type of supplement actually does acquire the meaning given in the text. On the other hand, for the new symbol \(\mathfrak{J}\) even the fundamental equation of the supplement,

\[
\mathfrak{J}[AB] = [\mathfrak{J}A \cdot \mathfrak{J}B],
\]

is no longer valid, so that one would also be required first to develop the calculus for the symbol \(\mathfrak{J}\) completely anew.

The difficulty that arises if one wants to apply the concepts of the supplement and normal to the points of Euclidean space rests on the following circumstance: Grassmann’s calculus, insofar as it makes use of these concepts, is tailored to the group consisting of all rotations of a Euclidean space about a point (cf. Editorial Note 74). However in the geometry of Euclidean space there is treated not just this group, but rather the more extensive group of all motions, which include, besides rotations, parallel displacement and twists. Now since both groups transform finitely distant points, that is displacements in Euclidean space, in the same way, Grassmann’s calculus applies in its full extent to the displacements in Euclidean space. If however one wants to apply it as well to the finitely distant points of Euclidean space, one must remodel it, and that would be achieved in the text if the concepts “supplement” and “normal” were always applied only to displacements and products of displacements.

86. The idea introduced here is carried out in Grassmann’s article on quaternions, Math. Ann. 12, 384–386 (1877). Regarding the use of the exterior angles of the spherical triangle instead of the interior angles, see also Grassmann’s presentation in his Lehrbuch der Trigonometrie für höhere Schulen, Enslin, Berlin (1865), pp. 100–115. However Möbius was the first to make serious use of the exterior angles of the triangle instead of the interior in spherical trigonometry (in the Analytische Sphärisk, 1846, and in the Grundformeln der sphärischen Trigonometrie, 1860).
87. For if one writes the first equation in the form $-\alpha a = \beta b + \gamma c + \cdots$, then the point $a$ endowed with the coefficient $-\alpha$ appears as the summation point of the multiple points $\beta b, \gamma c, \cdots$. But according to 222 the numerical coefficient of the summation point is equal to the sum of the coefficients of the summand points, that is

$$-\alpha = \beta + \gamma + \cdots,$$

whence $\alpha + \beta + \gamma + \cdots = 0$.

88. In fact the equation

$$2[(s - s')|r| + \mu = 0$$

with constant $s$ and $r$ and variable $s'$ represents a plane orthogonal to $r$. This plane goes through the point

$$s + \frac{\mu r}{2\rho^2} = p;$$

thus from equation (2) it follows that

$$2(s - p) + \mu \frac{r}{\rho^2} = 0$$

and from this upon inner multiplication with $r$, since $r^2 = \rho^2$, there results the equation

$$2[(s - p)|r| + \mu = 0,$$

which expresses precisely that the plane (1) goes through the point $p$.

89. (Cf. also No. 286.) A sum of line elements that cannot be reduced to a single line element HYDE, in agreement with the usage of BALL, calls a screw, and the representation of such a sum as the sum of a line element and a field orthogonal to it the normal form of a screw.*

If one combines HYDE’s technical word screw with the above chosen (Editorial Note 77) abbreviated names field and cell and adds the expressions rod {Stab} and sheet {Blatt} proposed by the undersigned** for the concepts “line element” and “surface element”, one obtains the following scheme of abbreviated names for the application of the fundamental concepts of extension theory to geometry:

- Displacement {Strecke};
- Field {Feld} (product of two displacements, surface area);
- Cell {Fach} (product of three displacements, volume);
- Point;
- Rod {Stab} (product of two points, line element);
- Screw {Schraube} (sum of rods);
- Sheet {Blatt} (product of three points, surface element), to which one could perhaps add
  - Block (product of four points, volume element).

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90. For the expression “interchangeable openings”, cf. Note 105 (to No. 485).

91. In the original edition this special type of affinity was denoted “equality” (Gleichheit) according to the prior usage of MÖBIUS (Barycentrische Kalkül (Werke, Bd. I, p. 194ff)).

92. {In this Note Hermann GRASSMANN Junior applies the general theory of geometric relations developed in Nos. 377–390 to collineations in space. This lengthy note — occupying over 24 pages of the Teubner Edition — is omitted here as being only of technical interest. Tr.}

93. The editors of the Teubner Edition replaced the material in braces with the following:

It will now be shown that such an assembly continues to be such an assembly under a certain special type of circular evolution of the magnitudes appearing in it, and indeed such that the number of reals among the $n$ magnitudes of the one assembly is the same as that in the other. This special type of circular evolution is to transform two magnitudes $a_1$ and $a_2$, when both are real or both simple imaginaries, into two other magnitudes $b_1$ and $b_2$, where

$$(h) \quad b_1 = xa_1 + ya_2, \quad b_2 = xa_2 - ya_1,$$

where $x$ and $y$ are both real, and the sum of their squares is one, thus

$$x^2 + y^2 = 1.$$ 

On the other hand if one of the two magnitudes $a_1$ and $a_2$ is real, the other imaginary, then the two magnitudes $a_1$ and $a_2$ are to transform into the magnitudes

$$b_1 = xa_1 + yia_2, \quad b_2 = xa_2 - yia_1,$$

where $i = \sqrt{-1}$, $x$ and $y$ are real, and $x^2 - y^2 = 1$, that is, $x^2 + (yi)^2 = 1$. One can therefore also say: Equations $(h)$ each represent a circular evolution of $a_1$ and $a_2$ if $x$ is always real, but $y$ is only imaginary, and indeed simple imaginary, if one of the magnitudes $a_1$ and $a_2$ is real and the other is simple imaginary.

94. Let $e_1, \ldots, e_n$ be the original units and

$$Qe_\kappa = \sum_{\nu} \alpha_{\kappa\nu} e_{\nu}, \quad (\kappa = 1, \ldots, n),$$

where the $\alpha_{\kappa\nu}$ are to signify real numbers. If one then sets $a = \sum \kappa e_\kappa$ and $b = \sum y_j e_j$, the expression $[Qa][b]$ takes the form

$$[Qa][b] = \sum_{\kappa;\nu; j} \alpha_{\kappa\nu} x_\kappa y_\nu |e_\kappa| |e_j| = \sum_{\kappa; j} \alpha_{\kappa j} x_\kappa y_j,$$

and the requirement that, for arbitrary $a$ and $b$, $[Qa][b] = [Qb][a]$ always, can therefore be replaced by the equations $\alpha_{\kappa j} = \alpha_{j\kappa} (\kappa, j = 1, \ldots, n)$. Again, these last say that $[Qa][b]$ is the first polar of $b$ with respect to the quadratic form $[Qa][a] = \sum \alpha_{\kappa j} x_\kappa x_j$.

The other requirement, that $[Qa][a]$ is to be different from zero for arbitrary $a$, amounts to the requirement that the quadratic form $\sum \alpha_{\kappa j} x_\kappa x_j$ can only vanish for real $x$ if all the $x$ are zero, and that it is therefore a positive definite or negative definite form. This assumption is utilized several times by GRASSMANN in the proof of No. 391, but would, as will be shown, have the consequence that the principal
numbers of the quotient \( Q \) be all positive or all negative, and must therefore, if one would retain the theorem 391 in its present general form, be replaced by another, namely that the determinant of the quadratic form \( \sum \alpha_{\kappa j} x_\kappa x_j \) or, what amounts to the same thing, the power of the fraction \( Q \) is to be different from zero.

We will show later how the proof of the theorem 391 proceeds if one replaces Grassmann’s assumption by this last. First however we will clarify for ourselves what properly is the sense and aim of the theorem. Thus we will assume at once that the power of the fraction \( Q \), that is the determinant of \( \alpha_{\kappa j} \), is different from zero.

Besides \( Q \), the quadratic form \( \sum \alpha_{\kappa j} x_\kappa x_j \) stands in a relation with yet another quotient, that is with the quotient \( P \), which is defined by the equations

\[
P e_\kappa = \sum_{\nu}^{1...n} \alpha_{\kappa \nu} e_\nu \quad (\kappa = 1, \ldots, n),
\]

and whose denominators are of \((n - 1)\)th order, while its numerators, as with \( Q \), are of first order.

This quotient \( P \) has a simple geometric meaning. Thus if we think of \( e_1, \ldots, e_n \) in an \( n \)-fold extended Euclidean space as \( n \) mutually orthogonal displacements of unit length through a point \( O \) and correspondingly interpret \( x_1, \ldots, x_n \) as rectangular coordinates in this space, with \( O \) as origin, then \( P \) coordinates with each displacement going through \( O \) the \((n - 1)\)-fold extended plane that includes all conjugate diameters of this displacement with respect to the \( \infty^1 \) manifolds of second degree \( \sum \alpha_{\kappa j} x_\kappa x_j = \text{const.} \)

The relation between the fraction \( P \) and the quadratic form \( \sum \alpha_{\kappa j} x_\kappa x_j \) depends on the choice of the units \( e_1, \ldots, e_n \). Thus let \( c_1, \ldots, c_n \) be \( n \) magnitudes of first order whose combinatorial product is equal to one; then

\[
c_\kappa = \sum_{\nu}^{1...n} \lambda_{\kappa \nu} e_\nu \quad (\kappa = 1, \ldots, n),
\]

where the determinant of the \( \lambda_{\kappa \nu} \) is equal to one, and conversely let

\[
e_\nu = \sum_{j}^{1...n} \Lambda_{j \nu} c_j \quad (\nu = 1, \ldots, n),
\]

and finally let \( I \) be symbol for the supplement belonging to the system of units \( c_1, \ldots, c_n \) (see No. 110). Then necessarily

\[
I e_\kappa = \sum_{\nu}^{1...n} \rho_{\kappa \nu} e_\nu,
\]

or, if one multiplies this equation by \( \sum \Lambda_{j \mu} e_j = e_\mu, \Lambda_{k \mu} = \rho_{k \mu} \), whence

\[
I c_\kappa = \sum_{\nu}^{1...n} \Lambda_{k \nu} e_\nu.
\]

*If one interprets \( x_1, \ldots, x_n \) as homogeneous coordinates in an \( R_{n-1} \), then \( P \) transforms each point \( x_1, \ldots, x_n \) into its \((n - 2)\)-fold extended polar plane with respect to the manifold \( \sum \alpha_{\kappa j} x_\kappa x_j = 0 \), whereas \( Q \) transforms the point \( x_1, \ldots, x_n \) into the pole of the same plane with respect to the manifold \( \sum x_\nu^2 = 0 \).
Thus upon solution it follows that
\[ |e_\nu = \sum_j \lambda_{j\nu} Ic_j. \]

Now one finds
\[ P_{c\kappa} = \sum_\nu \lambda_{\kappa \nu} Pe_\nu = \sum_{\mu \nu} \lambda_{\kappa \nu} \alpha_{\nu \mu} |e_\mu \]
\[ = \sum_{\mu \nu \tau} \lambda_{\kappa \nu} \lambda_{\tau \mu} \alpha_{\nu \mu} Ic_\tau \]

On the other hand, \( a = \sum x_\nu e_\nu = \sum x'_\nu c_\kappa \), whence \( x_\nu = \sum \lambda_{\kappa \nu} x'_\nu \), and thus the quadratic form \( \sum \alpha_{\kappa \nu} x_\kappa x_\nu \) is transformed, upon introduction of the new units \( c_1, \ldots, c_n \), into a form \( \sum \alpha'_{\kappa \nu} x'_\kappa x'_\nu \), where
\[ \alpha'_{\kappa \nu} = \sum_{\mu \nu} \lambda_{\kappa \nu} \lambda_{\tau \mu} \alpha_{\nu \mu}, \]
and if one compares this with the previously found expression for \( P_{c\kappa} \), one obtains (*)
\[ P_{c\kappa} = \sum_\nu \alpha'_{\kappa \nu} Ic_\nu. \]

But these equations say that the relation between the quotients \( P \) and the quadratic form \( \sum \alpha_{\kappa \nu} x_\kappa x_\nu \) continues to hold upon the introduction of new units.

It is otherwise with the relation between the quotient \( Q \) and our quadratic form. Thus according to the definitions of \( Q \) and \( P \), \( |QC_{\kappa} = PC_{\kappa} \). Now if the relation between the quotient \( Q \) and the form \( \sum \alpha_{\kappa \nu} x_\kappa x_\nu \) is to continue to hold upon the transformation to the units \( c_1, \ldots, c_n \), then necessarily
\[ QC_{\kappa} = \sum_\nu \alpha'_{\kappa \nu} c_\nu, \]
whence
\[ |QC_{\kappa} = \sum_\nu \alpha'_{\kappa \nu} |c_\nu = \sum_\nu \alpha'_{\kappa \nu} Ic_\nu. \]

But since according to No. 384 the determinant of the \( \alpha'_{\kappa \nu} \) is equal to the power of \( Q \) and thus certainly does not vanish, the last equation can only hold if \( Ic_\nu = |c_\nu \), in other words (cf. No. 167 and Editorial Note 62) if \( c_1, \ldots, c_n \) form a complete normal system of numerical value one, or what is the same thing, if the transformation \( x_\nu = \sum \lambda_{\kappa \nu} x''_\kappa \) is orthogonal (cf. Editorial Note 57).

Following these preparations we can now express the content of the theorem 391 in a form independent of the symbols of the Ausdehnungslehre.

First it is claimed in the theorem that one can always specify \( n \) magnitudes \( c_1, \ldots, c_n \) that stand in no numerical relation and for which \( [QC_{\kappa} |c_j] = 0 \), provided \( \kappa \neq j \). If we now imagine, as can be done without loss of generality, that the \( c_1, \ldots, c_n \),

One can also express this as follows: The quotient \( Q \) is understood as a simultaneous covariant of the two quadratic forms \( X = \sum \alpha_{\kappa \nu} x_\kappa x_\nu \) and \( U = \sum u''_\nu \), under the contragradient variables \( x \) and \( u \). In fact, if one interprets the \( x \) as point and the \( u \) as plane coordinates, the covariant
\[ \frac{1}{2} \sum \frac{\partial X}{\partial x_j} \frac{\partial U}{\partial u_j} = \sum \alpha_{\kappa \nu} x_\kappa u_j \]
set equal to zero represents precisely the linear substitution whose symbol is the quotient \( Q \). In this connection cf. Gundelfinger, Vorlesungen aus der analytischen Geometrie der Kegelschnitte, edited by Dingeldey, Leipzig, Teubner, 1895, p. 70ff.
$c_n$ are chosen so that their combinatorial product is equal to one, then equation (\(\ast\)) holds for the $c_\kappa$, and since according to what went before, $|Qc_\kappa = Pc_\kappa$ and according to No. 144, $|Qc_\kappa|c_j] = [c_j|Qc_\kappa]$, 
\[|Qc_\kappa|c_j] = [c_j \cdot Pc_\kappa] = \sum_\nu \alpha'_\kappa \nu [c_j Ic_\nu] = \alpha'_\kappa j.\]

But since the expressions $|Qc_\kappa|c_j] all vanish for $\kappa \neq j$, there follows from (\(\ast\)) that 
\[Pc_\kappa = \alpha'_\kappa Ic_\kappa,\]
and the quadratic form $\sum \alpha'_\kappa j x_\kappa x_j$, upon the transformation to the units $c_1, \ldots, c_n$, obtains the form $\sum \alpha'_\kappa j x_\kappa x_j^2$. Here, finally, none of the magnitudes $\alpha'_\kappa$ can vanish, for according to No. 92, 
\[|Pc_\kappa = |Qc_\kappa = (-1)^{n-1} Qc_\kappa,\]
whence 
\[|Qc_1 \cdot Qc_2 \cdots Qc_n] = (-1)^{n(n-1)} |Pc_1 \cdots Pc_n] \quad \text{[98]} \]
\[= \alpha'_1 \cdots \alpha'_n |Ic_1 \cdots c_n] \quad \text{[110]} \]
but this expression cannot vanish, since it represents the power of $Q$.

Thus if one specifies $c_1, \ldots, c_n$ as above, one basically solves the problem of reducing the quadratic form $\sum \alpha'_\kappa j x_\kappa x_j$, to a sum of $n$ squares by a real homogeneous substitution of determinant one.* The number of positives among these $n$ squares is therefore equal to the number of positives among the $n$ nonzero real numbers $[Qc_\kappa|c_n] = \alpha'_\kappa$.

In our theorem it is claimed in addition that there is always a normal system $e_1, \ldots, e_n$ such that $Qe_r = \rho_r e_r$, where the $\rho_r$ are real and among them as many are positive as there are among the magnitudes $[Qc_\kappa|c_n] = \alpha'_\kappa$. If, as is always feasible, we choose the normal system $e_1, \ldots, e_n$ so that its numerical value is equal to one, then from what has been said earlier it follows that under the transformation to the new units $e_1, \ldots, e_n$ the quadratic form $\sum \alpha'_\kappa j x_\kappa x_j$ takes the form $\sum \rho_r x_r^2$.**

Thereby the specification of such a normal system $e_1, \ldots, e_n$ basically solves the problem, to reduce the quadratic form $\sum \alpha'_\kappa j x_\kappa x_j$, to a sum of squares by a real substitution under which the form $\sum x_r^2$ remains invariant, that is, by an orthogonal substitution. But one can also say that here the problem is solved, to specify the principal axes of the $\infty^1$ manifolds of second degree $\sum \alpha'_\kappa j x_\kappa x_j = \text{const}$ of $R_n$, and at the same time is proved the well-known theorem that the equation of $n$th degree, on which the definition of these principal axes depends, has purely real roots (cf. p. 226, line 9, where after “second order” there ought properly to have been added “with center”).

Finally, the fact that among the $\rho_r$ the principal numbers of the quotient $Q$ of the $\rho_r$ are precisely as many positive as there are the magnitudes $[Qc_\kappa|c_n] = \alpha'_\kappa$, is obviously equivalent to Sylvester’s law of inertia of quadratic forms (cf.

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*The transformation to the coordinates $x'$ is naturally equivalent to relating the $\infty^1$ manifolds $\sum \alpha'_\kappa j x_\kappa x_j = \text{const}$ to a system of conjugate diameters, whence the expression “conjugate assembly” on p. 223, line 2.

**If one sets $a = \sum x_r e_r$, then $[Qa|a] = \sum \rho_r x_r^2$, an expression that comes out positive for every arbitrary $a$ only if the $\rho_r$ are all positive or all negative. This confirms what was said in the second paragraph of this Note.
But as is well known, one can utilize this law of inertia to ascertain how many real roots of an equation lie between two given limits (cf. HERMITE, Comptes Rendus 36, 294 (1853)), and thereby the connection with STURM's theorem is cleared up (cf. p. 226, lines 7–8).

It still remains for us to show how the proof of No. 391 proceeds when one replaces the assumption that \( |Qa|a| = 0 \) is different from zero for every arbitrary \( a \), with the other one, that the power of \( Q \), and thus the determinant of the \( \alpha_{\kappa,j} \), is not to vanish.

First we seek \( n \) magnitudes \( c_1, \ldots, c_n \) that stand in no numerical relation, and for which \( |Qc_\kappa|c_j| = 0 \) so long as \( \kappa \neq j \). If there are \( n \) such magnitudes, then according to the above the \( n \) expressions \( |Qc_\kappa|c_\kappa| \) are all different from zero, and thus we must from the beginning make sure that this condition is satisfied with the choice of the \( c_\kappa \).

We will assume that we have already found \( m \) magnitudes \( c_1, \ldots, c_m \) that stand in no numerical relation and satisfy the equations \( |Qc_\mu|c_\nu| = 0 \) for \( \mu \neq \nu \), while all \( |Qc_\mu|c_\mu| \neq 0 \). We claim that a magnitude \( c_{m+1} \) not derivable from \( c_1, \ldots, c_m \) can then always be specified that satisfies the \( m \) equations \( |Qc_\mu|c_{m+1}| = 0 \) (\( \mu = 1, \ldots, m \)) and thus also the \( m \) equations \( |Qc_{m+1}|c_\mu| = 0 \), while \( |Qc_{m+1}|c_{m+1}| \neq 0 \).

In order to prove our claim, we set \( Qc_\mu = k_\mu (\mu = 1, \ldots, m) \); then \( k_1, \ldots, k_m \) also certainly stand in no numerical relation, since otherwise the power of \( Q \) could not be different from zero. In addition we specify \( n - m \) magnitudes of first order \( b_{m+1}, \ldots, b_n \) so that they stand in no numerical relation and satisfy the equations

\[
|k_\mu b_{m+j}| = 0 (\mu = 1, \ldots, m; j = 1, \ldots, n - m).
\]

According to Nos. 163 and 159 this is always possible, for in the domain of \( k_1, \ldots, k_m \) we need only take any normal system of \( m \)th order \( k_1', \ldots, k_m' \) and to supplement this to a complete normal system \( k_1', \ldots, k_m', b_{m+1}, \ldots, b_n \), and then the \( b_{m+1}, \ldots, b_n \) are magnitudes with the desired property and indeed are normal to one another. Between \( c_1, \ldots, c_m \) and the magnitudes \( b_{m+1}, \ldots, b_n \) just found there can exist no numerical relation

\[
\gamma_1 c_1 + \cdots + \gamma_m c_m + \beta_{m+1} b_{m+1} + \cdots + \beta_n b_n = 0,
\]

for otherwise one would have

\[
\sum_\mu \gamma_\mu |k_\nu|c_\mu| + \sum_j \beta_{m+j} |k_\nu|b_{m+j} = \gamma_\nu |k_\nu|b_\nu| = 0
\]

for \( \nu = 1, \ldots, m \), whence \( \gamma_1, \ldots, \gamma_m \) would all have to vanish, and thus among the \( b_{m+1}, \ldots, b_n \) alone there must be a numerical relation, which is not the case. Further, the equation \( |Qb|b| = 0 \) cannot be satisfied for any magnitude, since otherwise

\[
|Qb|b| = \sum_{\kappa\mu}^{1..n-m} \lambda_{\kappa\mu}\lambda_{m+j}|Qb_{m+\kappa}|b_{m+j}| = 0
\]

for all values of \( \lambda_{\kappa\mu} \), but then \( |Qb_{m+\kappa}|b_{m+j}| = 0 \) for \( \kappa, j = 1, \ldots, n - m \), whence each of the magnitudes \( Qb_{m+\kappa} \) would be normal to all the magnitudes \( b_{m+1}, \ldots, b_n \), with the consequence that all the \( Qb_{m+j} \) would belong to the domain of the
magnitudes $k'_1, \ldots, k'_m$ or, what is the same thing, to the domain of the magnitudes $k_1, \ldots, k_m$:

$$Qb_{m+j} = \sum_{\mu} \delta_{\mu j} k_j,$$

from which it again follows that the power of $Q$ would be zero. We can therefore always choose $c_{m+1} = \sum \lambda_{m+j} b_{m+j}$ so that $[Qc_{m+1}|c_{m+1}] \neq 0$, and then the $c_{m+1}$ will stand in no numerical relation with $c_1, \ldots, c_m$ and in addition satisfy the equations $[Qc_\mu|c_{m+1}] = [Qc_{m+1}|c_{\mu}]$ for $\mu = 1, \ldots, m$.

Our claim is therefore proven. Now since the assumption made above is certainly permitted for $m = 1$, since one can always choose $c_1$ so that $[Qc_1|c_1] \neq 0$, it is obviously proven that it is always possible to specify $n$ magnitudes $c_1, \ldots, c_n$ with the desired property. We imagine them specified and besides for the sake of simplicity that $c_1, \ldots, c_n$ are chosen so that their combinatorial product is one. If as before* we then set $[Qc_\nu|c_\nu] = \alpha'_\nu$, the power of our fraction $Q$ is equal to the product $\alpha'_1 \cdots \alpha'_m$.

This replaces the development on p. 222, line 3 from the top to line 11 from the bottom. The rest of GRASSMANN's proof requires no other changes, and requires comment only on two other points.

P. 224, lines 11–20. Still missing here is the proof that the equation $x^2 + y^2 = 1$ can always be satisfied, and thus that $\gamma$ cannot equal $i$. In order to show this, one observes that for $\gamma = i$ either $(r + r')^2$ or $(r - r')^2$ must be zero, or, since $r$ and $r'$ are real, either $r + r'$ or $r - r'$ equals zero. This is however impossible, since each of the magnitudes $r$ and $r'$ results from one of the magnitudes $c_1, \ldots, c_n$ by multiplication by a real number, and there is certainly no numerical relation between the $c_1, \ldots, c_n$.

P. 225, lines 6–7. "Now since there must be a minimum ... ". According to our notation $[Qc_\kappa|c_\kappa] = \alpha'_\kappa$ the magnitudes $a_1, \ldots, a_n$ have the values

$$a_\kappa = c_\kappa : \sqrt{\alpha'_\kappa} \quad (\kappa = 1, \ldots, n),$$

whence the product $\alpha_1 \cdots \alpha_n$ of their numerical values is equal to the product of the numerical values of $c_1, \ldots, c_n$ divided by $\sqrt{\alpha'_1 \cdots \alpha'_n}$, where the sign under the root is chosen so that the root is real, and the root itself is taken to be positive. But now (cf. Editorial Note 69) the product of the numerical values of $c_1, \ldots, c_n$ is certainly not less than the numerical value of the product $[c_1 \cdots c_n]$, which is equal to one, and on the other hand the product $\alpha'_1 \cdots \alpha'_n$ is equal to the power of $Q$, and thus always has the same numerical value, whence it yields a lower limit for the product $\alpha_1 \cdots \alpha_n$ defined by the power of $Q$.

[231] 95. E. MüLLER, in his article "Die Kugelgeometrie nach den Principien der GRASSMANNschen Ausdehnungslehre" (Monatshefte für Mathematik und Physik, Jahrgang 3 u. 4, 1892 u. 1893), observes that it is more appropriate to interpret the orthogonal circle of three circles as the supplementary domain of the domain of those circles.

[235] 96. Compare this with the presentation in the $A_1$, §154ff.

[236] 97. The concept of the syncyclic relation of circles is noteworthy in that it was formulated by GRASSMANN on the basis of a general principle of transformation.

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*On p. 222, line 10 from the bottom, GRASSMANN sets $[Qc_r|c_r] = \alpha_r$, but later, on p. 224f, uses the same notation $\alpha_r$ in a completely different context.
On the other hand its practical significance is limited, since, under the transformation from an assembly of circles to a syncyclically related assembly, tangent circles do not in general transform into tangent circles.

[244] 98. This nomenclature is not very happily chosen; instead of “$f(q)$ vanishes with $q$” one ought to say: “$f(q)$ is infinitely small with $q$”.

[263] 99. According to this Definition every true series is unconditionally convergent, but not conversely is every unconditionally convergent series true.

[265] 100. One misses here the proof that every power series, so long as it is true, is a continuous function of its argument,* and that the termwise differentiated series is really the differential quotient of the series. It cannot be our problem to fill this and similar gaps in GRASSMANN’s establishment of differential calculus and the theory of functions, but it may be mentioned that GRASSMANN’s proofs of these two theorems are found in the Nachlaß, and thus that GRASSMANN himself had perceived the necessity of proofs for them.

[268] 101. Here it must be noted that the left side also remains zero for infinite $n$, since

$$\lim_{n \to \infty} n(\Theta - 1) = 2\pi i.$$  

[278] 102. Thus if one replaces $y$ by $y + x_1 \tau$, where $x_1$ is an arbitrary magnitude of first order and $\tau$ is an arbitrary number, then one recognizes at once that

$$f^{(n)}(0)x_1 y^{n-1} = n!a_n x_1 y^{n-1},$$  

and by continuing this procedure one finally obtains

$$f^{(n)}(0)x_1 \cdots x_n = n!a_n x_1 \cdots x_n,$$

whereupon No. 357 can be applied directly.

[283] 103. If $x$ is an extensive magnitude derivable from $n > 1$ units, the integral of $f(x)dx$ has no definite sense in and of itself; rather, the path of integration must be prescribed. The path of integration here is so chosen by GRASSMANN that, if one interprets $x_1, \ldots, x_n$ as rectangular coordinates of an $R_n$, it coincides with the straight line from the origin of coordinates to the point $x$. In this way $d^{-1} f(x)dx$ gets a definite value, which on account of $t = \sqrt{x^2}$ and $e = x : \sqrt{x^2}$ can again be represented as a function of $x$ alone. — The requirement of complete integrability of the differential $f(x)dx$ is equivalent to the requirement that the value of the integral of $f(x)dx$ depends only on the initial and final points, but not on the shape of the path of integration.

[284] 104. Here in the original edition, besides No. 433, No. 431c was cited, a citation that does not fit at all, for the expression $\frac{1}{a}$ is not an open expression, into whose opening a magnitude is to enter, but conversely that $a$ is to enter in the opening $l$. One must therefore instead reason thus: Since $a$ is a constant, it is unimportant whether it is to enter the opening before or after the differentiation.

[284] 105. The editors of the Teubner Edition replaced the material in braces with the following:

Since it is necessary here to establish a definite coordination between the individual openings of the function $f(x)$ and its filling magnitudes $a$ and $dx$, the open

*In the Remark following No. 461, GRASSMANN assumes this theorem as self-evident.
expression \( f(x) \) — we will call it an expression with “\( bound \)” openings — appears in contrast to the previously considered open expressions with “free” openings, with which a linking of openings to individual filling magnitudes does not obtain.* The difference of the two types of open expression is immediately apparent, in that an expression with \( n \) bound openings under multiplication with the product of \( n \) filling magnitudes always yields only a single term, whereas an expression with \( n \) free openings under such a multiplication yields a group of \( n! \) terms, whose arithmetic mean is to be taken.

If one sticks to the above chosen notation, in which the openings are distinguished from one another by indices, while their linkage with the filling magnitudes is marked by the denominators appended to the filling magnitudes, it remains completely arbitrary whether one places the filling magnitudes before or after the open expression, since for each of them the place it is to obtain upon its entry into the open expression is exactly prescribed. Still, in the following another, more convenient method of notation will find its application. Thus for fixing the coordination between the openings and their filling magnitudes it suffices if one bestows on the openings, perhaps again by appending indices, a certain \textit{rank ordering}, and places the filling factors, but without appending denominators, after the open expression, with the condition that the first listed filling magnitude is to enter into the first opening, the second into the second, and so on.

Whether the openings of an open expression are to be interpreted as free or bound usually follows from the way the expression originates. An example may clarify this.

Let \( f(x) \) be an expression with one or more openings, and

\[
x = x_1e_1 + x_2e_2 + \cdots,
\]

where the magnitudes \( e_1, e_2, \ldots \) form an elementary normal system. Then (according to 438 and 382) the differential quotient

\[
f'(x) = [l|e_1] \frac{d}{dx_1}f(x) + [l|e_2] \frac{d}{dx_2}f(x) + \cdots.
\]

It therefore includes one more opening than the function \( f(x) \) itself. Further (according to 435)

\[
f'(x)dx = dx f(x).
\]

It is therefore self-evident that the opening \( l \) coming into the expression \( f'(x) \) by differentiation must be assigned to the filling magnitude \( dx \); and, since the magnitude \( dx \) is usually placed immediately after the differential quotient, the coordination between the opening \( l \) and the filling magnitude \( dx \) is formally established by the condition that \( l \) is to be interpreted as the \textit{first} opening of \( f'(x) \).

Of particular importance for the following is the case where an expression with bound openings has the property that the same resultant always follows, whether one introduces the filling magnitudes into the openings in the order described or in another; then its openings are called \textit{interchangeable}. Thus for example in an expression \( A_{l_1} \) with two bound openings \( l \) and \( l_1 \), the first of which is \( l \), these openings are \textit{interchangeable} if for every two arbitrary filling magnitudes \( a \) and \( b \) the equation

\[A_{l_1}ab = A_{l_1}ba\]

---

*In the text the open expressions of the latter type were until now less significantly called expressions with “interchangeable” openings.
holds, that is, if for arbitrary values of $a$ and $b$

\[ A_{ab} = A_{ba}. \]

In this case, then, multiplication of the expression $A_{\text{II}_1}$ by the product of the filling magnitudes $a$, $b$ yields precisely the same product as if the expression possessed two free openings. For, if one denotes the expression that results from the open expression $A_{\text{II}_1}$ upon suppression of the linkage of its openings by $A_{\text{II}}$, then

\[ A_{\text{II}}ab = \frac{1}{2}\{A_{ab} + A_{ba}\}, \]

that is (according to *)

\[ = A_{ab} \]
\[ = A_{\text{II}_1}ab. \]

According to the general concept of the open expression (No. 357), which is also to be retained in this form for expressions with bound openings, one can in this case set

\[ A_{\text{II}_1} = A_{\text{II}}; \]

that is: Interchangeable bound openings can always be replaced by free openings.* Thus if, in the formula of the above theorem (that is, in 485), the two openings of the function $f(x)$ are to be interchangeable, the distinction between its openings is superfluous.

For the rest, it is often advantageous in an expression with interchangeable bound openings to retain the linkages of its openings, since the multiplication of such an expression proceeds more simply, both conceptually and formally, than with an expression having free openings. The importance of the concept of interchangeable bound openings is demonstrated in the following theorem (that is, No. 486).

[291] 106. The proof of this theorem has been found in Grassmann's Nachlaß, but it is too long and the whole theorem is of too little practical importance to reproduce here.

[295] 107. The extremely fruitful idea of reducing the integration of a partial differential equation of first order with an unknown function to the more general problem of the integration of an equation of the form

\[ (*) \quad X_1(x_1, \ldots, x_n)dx_1 + \cdots + X_n(x_1, \ldots, x_n)dx_n = 0 \]

is found first in the celebrated article of Pfaff: "Methodus generalis, aequationes differentiarum partium ... primi ordinis, inter quotcumque variables integrandi," Abhandlungen der Berliner Akademie 1815–16, pp. 76–136. In general it is therefore now customary to call an equation such as (*) a Pfaffian equation.—The two articles by Jacobi referred to in the text have the following titles: "Ueber die Pfaffsche Methode, eine gewöhnliche lineare Differentialgleichung zwischen $2n$ Variablen durch ein System von $n$ Gleichungen zu integrieren" and "Ueber die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung . . .

*On the other hand, between an expression $A_{\text{II}_1}$ with two non-interchangeable bound openings and the corresponding expression $A_{\text{II}}$ with free openings there is the relation

\[ A_{\text{II}} = \frac{1}{2}\{A_{\text{II}_1} + A_{\text{II}_1}\}, \]

a relation that one can easily transpose to expressions with arbitrarily many openings.
auf die Integration eines einzigen Systemes gewöhnlicher Differentialgleichungen.”
They were published in the years 1827 and 1837.

[296] 108. If one imagines the given partial differential equation of second order solved by \( t : t = F(x, y, z, p, q, r, s) \), then one has to set:

\[
\begin{align*}
    u &= e_1 x + e_2 y + e_3 z + e_4 p + e_5 q + e_6 r + e_7 s, \\
    U_1 &= [l|e_3] - p[l|e_1] - q[l|e_2], \\
    U_2 &= [l|e_4] - r[l|e_1] - s[l|e_2], \\
    U_3 &= [l|e_5] - s[l|e_1] - F[l|e_2], \\
    U &= e_1 U_1 + e_2 U_2 + e_3 U_3,
\end{align*}
\]

and in this way one obtains the equation

\[ U du = 0, \]

where the product \( U du \) is to be understood such that \( du \) is to enter in the openings of \( U \).

[297] 109. In the original edition this read just “where \( c_1, \ldots, c_n \) are constants,” but there can be no doubt that Grassmann had thought of them as arbitrary constants. Thus in the Proof he obviously assumes that the equations \( u_1 = c_1, \ldots, u_n = c_n \) represent an integrating assembly, whatever values the constants \( c_1, \ldots, c_n \) may have; otherwise he could not have concluded that the expression \( X dx \) is identically zero as soon as one expressed \( n \) of the differentials \( dx, \) by the other \( m - n \).

[298] 110. This theorem is not correct in the generality expressed here. In fact the very beginning of the Proof is open to criticism.

Thus from No. 502 it only follows that, under the assumptions of No. 503, \( X dx = 0 \) cannot be integrated by an assembly of \( n' < n \) equations that contain \( n' \) arbitrary constants and which are solvable for these constants. On the other hand it is perfectly conceivable that an integrating assembly of \( n' < n \) equations exists, from which an equation free of constants can be derived. This case occurs, for example, if in the expression \( X dx = X_1 dx_1 + \cdots + X_m dx_m \), the functions \( X_1, \ldots, X_{m-1} \) all vanish for \( x_m = 0 \), while \( X_m \) is not identically zero for \( x_m = 0 \), for then \( x_m = 0 \) is obviously an integral equation of \( X dx = 0 \).

One cannot say that Grassmann had never thought of the possibility of the occurrence of such an integrating assembly, for in No. 503 he remarks explicitly that \( U_1 = 0, \ldots, U_n = 0 \) is an integrating assembly: It therefore by no means escaped him that in certain cases one obtains an integrating assembly by setting to zero all the coefficients of the differential expression under consideration. He had however not noticed that in this way, even under the assumptions made in No. 503, one can sometimes find integrating assemblies of less than \( n \) independent equations, and indeed that this is especially implicit in another circumstance that he had overlooked, namely in the circumstance that an integrating assembly can be lost when one replaces the original equation \( X dx = 0 \) by the equation

\[ U_1 du_1 + \cdots + U_n du_n = 0. \]

This possibility however always exists when some of the magnitudes \( u_1, \ldots, u_n, U_1, \ldots, U_n \) assume infinitely large values for all systems of values \( x_1, \ldots, x_m \) that satisfy the integrating assembly under consideration.
The circumstance just mentioned, which Grassmann overlooked, has the consequence that the proof of No. 503 itself is not perfectly free of objections, if one completely excludes from consideration the possibly existing case of integrating assemblies that contain less than \( n \) mutually independent equations. Thus if \( v_1 = 0, \ldots, v_n = 0 \) is an integrating assembly with \( n \) independent equations, one still cannot know whether the functions \( u_1, \ldots, u_n \) ultimately remain for the system of values \( x_1, \ldots, x_m \) that satisfy the equations \( v_1 = 0, \ldots, v_n = 0 \); it is therefore by no means certain that the system of equations \( v_1 = 0, \ldots, v_n = 0 \) can be replaced by an assembly of equations between \( u_1, \ldots, u_n \) and \( m - n \) of the \( x \), as is assumed on p. 298, lines 7–1 from the bottom.

Fundamentally, therefore, Grassmann’s proof is only applicable if from the beginning one is restricted to integrating assemblies of the form \( v_1 = c_1, \ldots, v_n = c_n \), where \( c_1, \ldots, c_n \) are arbitrary constants. If one modifies the treatment of the proof in this way, it will be completely rigorous and will show that every integrating assembly of the given form can be cast into the form of the set \((b)\) on p. 298, where \( r \) is one of the numbers \( 1, \ldots, n \) and where the arbitrary constants \( c_1, \ldots, c_n \) only appear in the functions \( \phi_1, \ldots, \phi_r \). One can then still observe that equations \((b)\) represent an integrating assembly, whatever functions one may choose for \( \phi_1, \ldots, \phi_r \), and that the case \( r = 0 \) also yields an integrating assembly.

Despite this exception we have had to take to the theorem 503 and its proof, the theorem and proof are still noteworthy. First, it certainly represents a certain advance over the what Jacobi had said in the articles mentioned on p. 299, although naturally Grassmann’s claim “that other than the assembly of equations \((b)\) there is no other assembly of integrating equations” requires a qualification. Second, however, it is noteworthy that with the proof of No. 503 there is implicitly solved a problem that later played a role as an “auxiliary problem” in the investigations of Lie on partial differential equations and contact transformations, namely the problem of defining all the assemblies of at most \( n \) equations between the \( 2n \) independent variables \( u_1, \ldots, u_n, U_1, \ldots, U_n \), by virtue of which the equation

\[
U_1 du_1 + \cdots + U_n du_n = 0
\]

is integrated. The considerations in No. 503 show in fact that such an integrating assembly contains at least \( n \) equations, and that, if it contains exactly \( n \) equations, it can maintain the form \((b)\), where \( r \) is one of the numbers \( 0, 1, \ldots, n \). On various occasions Lie had commented on this number, cf. Archiv für Mathematik og Naturvidenskab 2, p. 341, Kristiania 1877, and Abh. d. Ges. d. Wiss. zu Leipzig, math.-phys. Klasse XIV, No. XII, p. 539 (1888). In this last Lie also mentioned “that Grassmann’s proof is incorrect and his theorem does not hold generally.”

[300] 111. If \( L \) has exactly \( n \) openings and upon filling these openings is a numerical magnitude, then one can write the expression \([La_1 \cdots a_n]\) as follows:

\[
[La_1 \cdots a_n] = \frac{1}{n!} \sum \pm \phi(i_1 \cdots i_n).
\]

Here for \( i_1, \ldots, i_n \) one has to set all interchanges of the numbers \( 1, 2, \ldots, n \) one by one and to choose the plus or minus sign according as the particular interchange is even or odd; finally, \( \phi(i_1, \ldots, i_n) \) denotes the numerical magnitude that one obtains if one lets the magnitudes \( a_{i_1}, a_{i_2}, \ldots, a_{i_n} \) enter sequentially into the first, second, \ldots, \( n \)th openings of \( L \), whence (cf. the Remark following 428)

\[
La_{i_1}a_{i_2} \cdots a_{i_n} = \phi(i_1 \cdots i_n).
\]
In particular, if \( L \) is a product of \( n \) expressions, each with one opening, which upon filling their openings are all numerical magnitudes, then the function \( \phi \) has the form
\[
\phi(i_1 \cdots i_n) = \phi_1(i_1) \cdots \phi_n(i_n),
\]
and the expression \([La_1 \cdots a_n]\) is an ordinary determinant, which to be sure is divided by \( n! \).

Grassmann’s symbol \([La_1 \cdots a_n]\) is therefore a generalization of the concept of the determinant, and indeed is in essence precisely the generalization that Cayley was already aiming at in 1848, in his investigation of the Jacobian symbol* (1, 2, \ldots, 2n) (cf. the short article “Sur les déterminants gauches”, Crelles Journal 38, pp. 93–96, and The Collected Mathematical Papers of Cayley, Vol. I, pp. 410–413).

[304] 112. The proof of this theorem is not completely satisfactory, since it takes no regard of the numerical factor which, according to No. 504, must be appended to the development of every relative open product.

[305] 113. It should have been remarked here that the procedure in the text also yields, without further ado, to the constraint
\[
\left[ \left( \frac{d}{dx} X \right)^{n+1} \right] = 0,
\]
but that this, as it turns out later, is a consequence of the equation
\[
\left[ X \left( \frac{d}{dx} X \right)^n \right] = 0.
\]

[305] 114. If not only \( x \), but also \( Xdx \) is an extensive magnitude, then the equation \( Xdx = 0 \) is equivalent to a system of Pfaffian equations:

\[
(1) \quad \sum_{\mu=1}^{m} X_{\kappa \mu} dx_\mu = 0 \quad (\kappa = 1, \ldots, h).
\]

Thus if we set \( \sum x_\mu e_\mu = x \) and
\[
X_\mu = \sum_{\mu=1}^{m} X_{\kappa \mu} [l|e_\mu] \quad (\kappa = 1, \ldots, h),
\]
then we can write the system (1) in the form

\[
(2) \quad X_1 dx = 0, \ldots, X_h dx = 0,
\]
where \( X_1 dx, \ldots, X_h dx \) are numerical magnitudes; thus if we set
\[
e_1 X_1 + \cdots + e_h X_h = X,
\]
then the system (2) is equivalent to the equation \( Xdx = 0 \).

Now should the system (1) or (2) be reducible to a system in which \( n < m \) differentials appear, then it must be possible to define the magnitudes \( u_\nu \) and \( U_{\kappa \nu} \) as functions of \( x_1, \ldots, x_m \) or, what amounts to the same thing, as functions of \( x \), in such a way that

\[
(3) \quad X_\kappa dx = U_{\kappa 1} du_1 + \cdots + U_{\kappa n} du_n \quad (\kappa = 1, \ldots, h).
\]

*Later Cayley introduced the name “Pfaffian” for this symbol; we have made no use of this name, since with Lie we reserve the name “Pfaffian expression” exclusively for expressions of the form \( \sum X_\nu dx_\nu \).
But then it follows that

$$X = \sum_{\nu=1}^{n} U_{\kappa\nu} \frac{d}{dx} u_{\nu}, \quad \frac{d}{dx} X_\nu = \sum_{\nu=1}^{n} \left( \frac{d}{dx} U_{\kappa\nu} \frac{d}{dx} u_{\nu} + U_{\kappa\nu} \frac{d^2}{dx^2} u_{\nu} \right) \quad (\kappa = 1, \ldots, h),$$

and if one handles these equations according to the pattern of No. 511, one obtains as the necessary condition for the possibility of that reduction the following:

$$\left[ X_1^{\varepsilon_1} X_2^{\varepsilon_2} \cdots X_h^{\varepsilon_h} \left( \frac{d}{dx} X_1 \right)^{r_1} \cdots \left( \frac{d}{dx} X_h \right)^{r_h} \right] = 0,$$

where each $\varepsilon_i$ has one of the values 0 or 1, and each $r_i$ one of the values 0, 1, $n+1$, and one has to choose $\varepsilon_1, \ldots, \varepsilon_h, r_1, \ldots, r_h$ in all possible ways such that

$$\varepsilon_1 + \cdots + \varepsilon_h + r_1 + \cdots + r_h = n + 1.$$  

These are obviously the conditions that GRASSMANN had in mind; possibly he had even reduced it to equations for the magnitude $X$ alone:

$$\left[ X^\mu \left( \frac{d}{dx} X \right)^{n+1-\mu} \right] = 0 \quad (\mu = 0, 1, \ldots, h),$$

where the expression in brackets is interpreted as an algebraic product in the sense of Nos. 364–371.

Equation of condition (5), whose appearance seems to have been completely unnoticed previously, deserves a closer investigation; in particular it would be of interest to know whether it is not only necessary, but also sufficient, in cases where $h > 1$.

[307] 115. With JACOBI the expression (2, 3, \ldots, $2n+1$) was at first defined in a somewhat different way (see the Remark following No. 510).

[309] 116. It should have been said right here that in the case $\lambda = 0$ the problem is solved, provided it is possible to define $\delta x$ so that it satisfies the equation

$$\left[ \frac{d}{dx} X \cdot c \cdot \delta x \right] = 0$$

for every value of the magnitude $c$ and in addition the equation $X \delta x = 0$. Thereby the representation in No. 516 would essentially have been achieved.

[310] 117. These numerical equations read

$$\lambda \cdot X e_\nu = 2 \left[ \frac{d}{dx} X \cdot e_\nu \cdot \delta x \right] = 2 \sum_{\kappa=1}^{2n} \left[ \frac{d}{dx} X \cdot e_\nu \cdot e_\kappa \right] \delta x_\kappa.$$

Under the assumptions made here one can, according to the procedure leading up to Theorem 1 of Editorial Note 112, write down the solution of this equation directly, and at the same time survey how the solution comes to its form. In GRASSMANN's presentation, on the other hand, everything depends on an artifice that is suited only to impede one's understanding. The application of this artifice has in addition the drawback that GRASSMANN must still first prove in particular that the value of $\delta x$ found actually satisfies the given equations.
118. The editors of the Teubner Edition replaced the material in braces with the following: we can, without changing the meaning of the product

$$X \left( \frac{d}{dx} X \right)^{n-1},$$

(according to 504) introduce within the brackets yet another opening \( l \) as the first factor of the same family as the other openings. Then

119. Compare with Editorial Note 112. That the method used in No. 515 “makes clear itself” one can scarcely grant; but one must admit in any case that Grassmann’s symbols lead directly to the result of No. 515.

120. Compare this with Editorial Notes 112 and 116.

121. Compare this with Theorem 2 of Editorial Note 112.

122. If one inserts for \( \delta x \) its value \( \delta x = \sum e_\mu \delta x_\mu \), the equations \( G_s = 0 \) take the form

$$G_s = 2 \sum_{\mu=1}^{m} \left[ \frac{d}{dx} X \cdot e_s \cdot e_\mu \right] \delta x_\mu - \lambda X e_s = 0 \quad (s = 1, \ldots, m),$$

and the theorem proved here simply expresses that, in the matrix belonging to these \( m \) linear homogeneous equations, all \((2n + 1)\)-row determinants vanish, provided equation \((b)\) is satisfied. In the proof it is tacitly assumed that the expression

$$\left[ \left( \frac{d}{dx} X \right)^n e_1 \cdots e_{2n} \right] \neq 0,$$

for only under this assumption does the equation \( \sum \alpha_a G_a = 0 \), derived on p. 320 represent a numerical equation in which \( G_m \) actually appears. Only later (on the same page) does Grassmann take note of this assumption.

123. It would have been better if this constraint had been written in the form

$$\left[ X \left( \frac{d}{dx} X \right)^{n-1} e_1 e_2 \cdots e_{2n} \right] \neq 0,$$

124. It is obviously assumed that, according to p. 320, lines 15–16, one has already set \( \delta x_{2n+1}, \ldots, \delta x_m \) equal to zero in the equations \( G_1 = 0, \ldots. \)

125. It is also assumed here that

$$\left[ X \left( \frac{d}{dx} X \right)^{n-1} e_1 e_2 \cdots e_{2n} \right] \neq 0.$$

Obviously this assumption is no restriction of generality, since from the beginning it is assumed that

$$\left[ \left( \frac{d}{dx} X \right)^{n-1} \right]$$

does not vanish.

126. This Corollary was written in order to explain why Grassmann did not mention the case where \( m = 2n - 1 \). It is completely wrong to conclude from the absence of any mention of this case that Grassmann had excluded it, and it
is not really comprehensible how FORSYTH\textsuperscript{*} can claim: “It is assumed implicitly that, if the coefficients of an equation satisfy no characteristic condition, then the number of variables is even; so GRASSMANN practically considers only the even classes of unconditioned equations.” Yet in No. 512 GRASSMANN says explicitly that for \( m < 2n + 1 \), and thus in particular for \( m = 2n - 1 \), no equation of condition at all enters, and thus it is from this completely clear that the equation
\[
X_1 dx_1 + \cdots + X_{2n-1} dx_{2n-1} = 0
\]
can always be integrated by an assembly of \( n \) equations, now whether
\[
\left[ X \left( \frac{d}{dx} X \right)^{n-1} \right]
\]
is equal to zero or different from zero.

**Grassmann’s investigations into Pfaff’s problem**

We would like at least to make GRASSMANN’s heretofore so little noticed investigations into PFaff’s problem reasonably accessible to those that are unwilling to assimilate GRASSMANN’s calculus. To this end we will now attempt to present the contents of Nos. 502–527, insofar as it relates to PFaff’s problem, in the language of ordinary analysis. For this we will make use of JACOBI’s symbol \((1, 2, \ldots, n)\), and in fact in the way that CAYLEY has presented it. With the help of this symbol we are in a position to replace all GRASSMANN’s calculations and transformations with complete equivalents.

Otherwise we by no means intend to follow GRASSMANN step by step — this would lead us too far afield — rather, we will only reproduce his train of thought as completely as possible, in order to show what GRASSMANN had actually accomplished.

In No. 502 GRASSMANN first showed that the equation
\[
X_1 dx_1 + \cdots + X_m dx_m = 0
\]
can be integrated by an assembly of \( n \) equations
\[
u_1(x_1, \ldots, x_m) = \text{const}, \ldots, u_n(x_1, \ldots, x_m) = \text{const}
\]
if and only if the expression \( X dx \) is reducible to the form
\[
\sum_{\mu=1}^{m} X_{\mu} dx_{\mu} = \sum_{\nu=1}^{n} U_{\nu}(x_1, \ldots, x_n) du_{\nu}
\]
with only \( n \) differentials.\textsuperscript{**} The proof is perfectly comprehensible.

If \( \sum X_{\mu} dx_{\mu} \) can be cast into the form (3), the equations
\[
\begin{aligned}
X_{\mu} &= \sum_{\nu=1}^{n} U_{\nu} \frac{\partial u_{\nu}}{\partial x_{\mu}}, \\
\frac{\partial X_{\mu}}{\partial x_{\kappa}} &= \sum_{\nu=1}^{n} \frac{\partial U_{\nu}}{\partial x_{\kappa}} \frac{\partial u_{\nu}}{\partial x_{\mu}} - \sum_{\nu=1}^{n} U_{\nu} \frac{\partial^{2} u_{\nu}}{\partial x_{\mu} \partial x_{\kappa}}
\end{aligned}
\]
\textsuperscript{*} Theory of Differential Equations, Part I, Cambridge 1890, p. 83.

\textsuperscript{**} On No. 503 see Editorial Note 110.
must exist. In order to eliminate the \( U \) and \( \upsilon \) from these, GRASSMANN forms the expression

\[
\sum_{i=1}^{n} \pm X_{\mu_1}^i \frac{\partial X_{\mu_2}}{\partial x_{\mu_3}} \cdots \frac{\partial X_{\mu_{2n}}}{\partial x_{\mu_{2n+1}}}
\]

where \( \mu_1, \ldots, \mu_{n+1} \) are any \( 2n + 1 \) of the numbers \( 1, 2, \ldots, m \), and where the sum is to be formed in such a way that one permutes \( \mu_1, \ldots, \mu_{n+1} \) in all possible ways, giving each even permutation the plus sign, each odd permutation the minus sign. With this method of formation of expression (5) all the second differential quotients of \( \upsilon \), drop out, and there remains

\[
\sum_{\nu_1, \ldots, \nu_{n+1}} U_{\nu_1} \frac{\partial u_\nu_1}{\partial x_{\mu_1}} \frac{\partial u_\nu_2}{\partial x_{\mu_2}} \frac{\partial U_\nu_2}{\partial x_{\mu_3}} \cdots \frac{\partial u_\nu_{n+1}}{\partial x_{\mu_2n}} \frac{\partial U_\nu_{n+1}}{\partial x_{\mu_{2n+1}}},
\]

where the inner sum is formed exactly as with (5). Now since among the \( n + 1 \) indices \( \nu_1, \ldots, \nu_{n+1} \) there are always at least two that are equal, the inner sum vanishes identically, and it therefore follows that all expressions of the form (5) must have the value zero if an equation of the form (3) is to exist. Naturally this condition is only in force if \( m \geq 2n + 1 \).

This is the content of No. 511. The derivation of the necessary conditions for the existence of (3) is completely GRASSMANN’s own, and is decidedly most noteworthy, especially as it can be applied to systems of Pfaffian equations (cf. Editorial Note 114).

In No. 512 GRASSMANN shows how one can write the equations of condition found in No. 511 with the help of JACOBI’s symbols (1, 2, \ldots, 2n). Thus since with JACOBI he sets

\[
\frac{\partial X_{\mu}}{\partial x_{\kappa}} - \frac{\partial X_{\kappa}}{\partial x_{\mu}} = (\mu, \kappa)
\]

and utilizes that symbol,\(^*\) he brings the equation that results from setting expression (5) to zero to the form

\[
\sum_{i=1}^{n} X_{\mu_1} (\mu_1, \mu_2, \ldots, \mu_{2n+1}) = 0,
\]

where now the sum consists of all the expressions that one obtains from the one as written by once, twice, \ldots, \( 2n + 1 \) times cyclically interchanging the indices \( \mu_1, \mu_2, \ldots, \mu_{n+1} \).

We do not occupy ourselves with establishing this result, and we will therefore just note that, with the help of an artifice due to CAYLEY,\(^**\) equation (7) can be written in a very elegant form, which will serve us well later on. Thus if with CAYLEY we set

\[
(0, \mu) = X_{\mu}, \quad (\mu, 0) = -X_{\mu},
\]

\(^*\)This symbol is a complete function of \( n \)th degree of the expression \((\mu, \kappa)\) and is defined by the recursion formula

\[
(1, 2, \ldots, 2n) = \sum (1, 2)(3, 4, \ldots, 2n),
\]

where the sum on the right consists of all expressions that one obtains if, in the expression as written, one cyclically interchanges the indices 2, 3, \ldots, \( 2n \) once, twice, \ldots, \( 2n - 1 \) times (Cf. the development in Editorial Note 112.) It then easily follows that the expression \((\mu_1, \ldots, \mu_{2n})\) changes its sign under interchange of two indices and also under cyclic interchange of all \( 2n \) indices, and that it vanishes if two of the indices are equal. Refer to the article by CAYLEY cited in Editorial Note 111.

\(^**\)Cf. CAYLEY, Crelle Journal 57, 275 (1860), Mathematical Papers IV, p. 361.
then (7) assumes the form  

\[(7') \quad (0, \mu_1, \mu_2, \ldots, \mu_{n+1}) = 0. \]

We now turn from this short detour back to No. 512. Properly one has to form all equations (7) or (7') that one obtains when one sets for \(\mu_1, \ldots, \mu_{2n+1}\), any \(2n + 1\) of the numbers 1, 2, \ldots, \(m\). But GRASSMANN proves that under the assumption \(X_1 \neq 0\), the existence of those equations (7) or (7') in which one of the numbers \(\mu_1, \ldots, \mu_{2n+1}\) has the value one entails that of all the other equations.

If we translate GRASSMANN’s proof into the notation chosen here, it just comes to the following: the equation  

\[ (0, 0, 1, \mu_1, \mu_2, \ldots, \mu_{2n+1}) = 0 \]

is an identity, since JACOBI’s symbol on the left side contains two equal indices. But if one develops this equation according to the rules holding for JACOBI’s symbols, one obtains (since the first term vanishes)  

\[ (0, 1)(\mu_1, \mu_2, \ldots, \mu_{2n+1}, 0) + \sum_{\kappa=1}^{2n+1} (0, \mu_\kappa)(\mu_{\kappa+1}, \ldots, \mu_{2n+1}, 0, 1, \mu_1, \ldots, \mu_{\kappa-1}) = 0, \]

and this equation is equivalent to the following:  

\[ X_1(0, \mu_1, \ldots, \mu_{2n+1}) = \sum_{\kappa=1}^{2n+1} (-1)^{\kappa-1}X_\kappa(0, 1, \mu_1, \ldots, \mu_{\kappa-1}, \mu_{\kappa+1}, \ldots, \mu_{2n+1}), \]

from which the theorem to be proved follows immediately.

No. 514 considers the problem of reducing the equation  

\[ X_1dx_1 + \cdots + X_mdx_m = 0, \]

by the introduction of new variables, to an equation in which only \(m - 1\) variables appear. Analytically this problem amounts to bringing the expression \(\sum X_\mu dx_\mu\), by the introduction of the new variables \(y_1, \ldots, y_{m-1}, t\), to the form  

\[ \sum_{\mu=1}^{m} X_\mu dx_\mu = N \sum_{\nu=1}^{m-1} Y_\nu(y_1, \ldots, y_{m-1})dy_\nu, \]

where \(t\) only appears in the factor \(N\). The whole problem and the procedure used originated with PFAFF,* so the content of No. 514 is in essence just a translation of PFAFF’s ideas into GRASSMANN’s language.

The problem would be solved if one has satisfied the equations  

\[ \sum_{\mu=1}^{m} X_\mu \frac{\partial x_\mu}{\partial t} = 0, \quad \frac{\partial}{\partial t} \left( \frac{1}{N} \sum_{\mu=1}^{m} X_\mu \frac{\partial x_\mu}{\partial y_\nu} \right) = 0 \quad (\nu = 1, \ldots, m - 1). \]

Now if one sets  

\[ \frac{1}{N} \frac{\partial N}{\partial t} = \lambda \]

one obtains, upon combining with equations (8), the following:  

\[ \lambda \sum_{\mu=1}^{m} X_\mu \frac{\partial x_\mu}{\partial y_\nu} = \sum_{\kappa=1}^{m} \left( \frac{\partial X_\mu}{\partial x_\kappa} - \frac{\partial X_\kappa}{\partial x_\mu} \right) \frac{\partial x_\kappa}{\partial t} \frac{\partial x_\mu}{\partial y_\nu} \quad (\nu = 1, \ldots, m - 1), \]

* Cf. the article mentioned in Editorial Note 107.
which is obviously satisfied identically, if one can satisfy the equations

\[ \lambda(0, \mu) = \sum_{\kappa=1}^{m} (\mu, \kappa) \frac{\partial x_{\kappa}}{\partial t} \quad (\mu = 1, \ldots, m), \]

where \((\mu, \kappa)\) and \((0, \kappa)\) have the meaning given in (6) and (6').

Now conversely, if it is possible to define the \(x_{\kappa}\) as functions of \(t\) and \(y_1, \ldots, y_{m-1}\) in such a way that (12) is satisfied and that \(\lambda\) does not vanish, then obviously

\[ \sum_{\mu=1}^{m} (0, \mu) \frac{\partial x_{\mu}}{\partial t} = \sum_{\mu=1}^{m} X_{\mu} \frac{\partial x_{\mu}}{\partial t} = 0, \]

and at the same time relations (11) obtain. But from (13) and (11) there follows, upon using (10), equations (9), whence the problem is solved.

On the other hand, if equations (12) are so constituted that the vanishing of \(\lambda\) follows from them, then one must define the \(x_{\kappa}\) so that besides the equations

\[ 0 = \sum_{\kappa=1}^{m} (\mu, \kappa) \frac{\partial x_{\kappa}}{\partial t} \quad (\mu = 1, \ldots, m), \]

they also satisfy the equation

\[ \sum_{\mu=1}^{m} (0, \mu) \frac{\partial x_{\mu}}{\partial t} = 0. \]

Thus (11) would then also be satisfied if one set \(\lambda = 0\), and since in the case \(\lambda = 0\) equation (10) expresses that \(N\) is free of \(t\), it again follows from (11) and (13') or (13) that equations (9) obtain and the problem is solved.*

The case that \(m\) is even is in Nos. 515–517 completely disposed of under certain assumptions.

If \(m = 2n\), then it follows from (12) that

\[ \lambda \sum_{\kappa=1}^{m} (0, 1) (2, 3, \ldots, \mu - 1, \mu + 1, \ldots, 2n) \]

\[ = \sum_{\kappa=1}^{2n} \sum_{\kappa=1}^{m} (1, \kappa) (2, 3, \ldots, \mu - 1, \mu + 1, \ldots, 2n) \frac{\partial x_{\kappa}}{\partial t}, \]

where \(\sum_{(c)}\) signifies that one is to form the sum of all expressions that result from once, twice, \(, (2n-1)\) times cyclically interchanging the indices \(1, 2, \ldots, \mu - 1, \mu + 1, \mu + 2, \ldots, 2n\) in the expression as written. But the last equation can be

---

*To be sure the case \(\lambda = 0\) is only considered in No. 516, but it is better to mention it right here. Naturally it would be even more appropriate to say that the problem is solved if one has defined \(\lambda\) and \(x_1, \ldots, x_m\) as functions of \(y_1, \ldots, y_{m-1}\) and \(t\) such that the equations

\[ \lambda(0, \mu) + \sum_{\kappa=1}^{m} (\kappa, \mu) \frac{\partial x_{\kappa}}{\partial t} = 0 \quad (\mu = 1, \ldots, m) \]

are identically satisfied; but this would be a real deviation from GRASSMANN’s train of thought, and we only mention it here because one thereby recognizes that CAYLEY’s notation \(X_{\mu} = (0, \mu)\) is grounded in the nature of the subject.
written
\[
\lambda(0, 1, 2, \ldots, \mu - 1, \mu + 1, \ldots, 2n) \\
= - \sum_{\kappa=1}^{2n} (\kappa, 1, 2, \ldots, \mu - 1, \mu + 1, \ldots, 2n) \frac{\partial x_\kappa}{\partial t},
\]
and since here on the right all those terms vanish in which \(\kappa \neq \mu\), there results
\[
\lambda(0, 1, 2, \ldots, \mu - 1, \mu + 1, \ldots, 2n) \\
= (-1)^\mu (1, 2, \ldots, 2n) \frac{\partial x_\mu}{\partial t} (\mu = 1, 2, \ldots, 2n).
\]

(14)

On the other hand,
\[
(\kappa, 0, 1, 2, \ldots, 2n) = 0,
\]
whatever one may set \(\kappa\) to among the numbers 0, 1, 2, \ldots, 2n, but from this, if one develops the left side according to the recursion formula in the footnote on p. 379, it follows that
\[
(\kappa, 0)(1, 2, \ldots, 2n) + \sum_{\mu=1}^{2n} (\kappa, \mu)(\mu + 1, \ldots, 2n, 0, 1, \ldots, \mu - 1) = 0,
\]
or
\[
\sum_{\mu=1}^{2n} (-1)^\mu (\kappa, \mu)(0, 1, \ldots, \mu - 1, \mu + 1, \ldots, 2n) = (0, \kappa)(1, 2, \ldots, 2n)
\]

(15)

\((\kappa = 0, 1, 2, \ldots, 2n)\).

If therefore not all \(2n\) expressions
\[
(0, 1, \ldots, \mu - 1, \mu + 1, \ldots, 2n) \quad (\mu = 1, \ldots, 2n)
\]
are equal to zero and if \((1, 2, \ldots, 2n) \neq 0\) as well, then equations (12) are satisfied by the values of the differential quotients of \(x_1, \ldots, x_{2n}\) following from (14), provided \(\lambda\) does not vanish, and in fact one can, if say
\[
(0, 1, 2, \ldots, 2n - 1)
\]
is different from zero, set \(x_{2n} = t\), and thus obtain \(\lambda\) and the differential quotients of \(x_1, \ldots, x_{2n-1}\) with respect to \(t\), represented as functions of \(x_1, \ldots, x_{2n-1}\) and \(x_{2n} = t\).

If on the other hand not all of the expressions (16) vanish, but \((1, 2, \ldots, 2n)\) is zero, and therefore on account of (14) so is \(\lambda\), then equations (15) show that (12') and (13') are identically satisfied if one sets
\[
\frac{\partial x_\mu}{\partial t} = \rho(-1)^\mu (0, 1, \ldots, \mu - 1, \mu + 1, \ldots, 2n),
\]
meaning by \(\rho\) an arbitrary magnitude. Thus if, in particular, expression (17) is different from zero, one can again set \(x_{2n} = t\) and obtain \(\rho\) and the differential quotients of \(x_1, \ldots, x_{2n-1}\) with respect to \(t\) expressed by \(x_1, \ldots, x_{2n-1}\) and \(x_{2n} = t\).

Thus if \(m = 2n\) and not all the expressions (16) are zero, then one always obtains a system of equations
\[
\frac{\partial x_\mu}{\partial t} = \xi_\mu(x_1, \ldots, x_{m-1}, t) \quad (\mu = 1, \ldots, m - 1),
\]

(19)
which together with \( x_m = t \) satisfy either equations (12) with nonvanishing \( \lambda \) or equations (12') and (13'); if the latter case occurs, one has to set \( \lambda = 0 \).

Differential equations (19) are now integrated on the basis of the initial conditions* \( x_\nu = y_\nu \) for \( t = 0 \) \((\nu = 1, \ldots, m - 1)\). If in addition one calculates \( N \) from equation (10), then \( x_1, \ldots, x_m \), as functions of \( y_1, \ldots, y_{m-1} \) and \( t \), satisfy equations (9) and thus there exists an equation of the form (8). Finally, if in (8) one makes the substitution \( t = x_{2n} = 0 \), one recognizes at once that, upon the transition to the new variables \( y_1, \ldots, y_{m-1} \), the equation \( \sum X_\nu dx_\nu = 0 \) takes the form

\[
(20) \quad \sum_{\nu=1}^{m-1} X_\nu(y_1, \ldots, y_{m-1}, 0) dy_\nu = 0.
\]

In addition Grassmann explicitly takes note that, in the case where \((1, 2, \ldots, 2n)\) and thus also \( \lambda \) vanish, the multiplier \( N \) is free of \( t \), and therefore that, should not all the expressions (16) be zero, the equation \((1, 2, \ldots, 2n) = 0\) is necessary and sufficient for the Pfaffian expression \( \sum X_\mu dx_\mu \) to be reduced, by the introduction of new variables \( y_1, \ldots, y_{m-1}, t \), to an expression containing only \( m - 1 \) variables.

The content of Nos. 515–517 is for the most part just a broadening and completion of Jacobi’s ideas. For Jacobi had already taken notice of the special properties of equations of the form (12), and had also showed that by the introduction of the initial values one can find the new form (20) of the Pfaffian equation \( \sum X_\mu dx_\mu = 0 \) (cf. Crelle 17, p. 157ff., Werke, Bd. 4, p. 121ff).

Grassmann prepares for the general solution of the problem posed in No. 514 (see p. 412f) in No. 518 by proving the theorem

“If all expressions of the form

\[
(0, \mu_1, \mu_2, \ldots, \mu_{2n+1})
\]

vanish, then so also do all expressions of the form

\[
(\mu_1, \mu_2, \ldots, \mu_{2n+2})
\]

meaning by \( \mu_1, \mu_2, \ldots \) any of the numbers 1, 2, \ldots, \( m \).

Grassmann’s proof of this theorem** emerges from the identity (15), in which one has only to replace \( n \) by \( n + 1 \). Thus if \( i \) is one of the numbers 1, \ldots, \( 2n + 2 \), then it is of course true that

\[
(\mu_1, 0, \mu_1, \ldots, \mu_{2n+2}) = 0 = (\mu_1, 0)(\mu_1, \ldots, \mu_{2n+2}),
\]

whence if not all the expressions \((\mu_1, 0), \ldots, (\mu_{2n+2}, 0)\) vanish, it follows at once that \((\mu_1, \ldots, \mu_{2n+2}) = 0\). But if these expressions are all zero, then one cannot come to this conclusion; rather, one must form the expression

\[
(\mu_{2n+3}, 0, \mu_1, \ldots, \mu_{2n+2}),
\]

which under the assumptions of the theorem reduces to

\[
(\mu_{2n+3}, 0)(\mu_1, \ldots, \mu_{2n+2}),
\]

*Here it is tacitly assumed that the functions \( \xi_1, \ldots, \xi_{m-1} \) remain finite and continuous for \( t = 0 \), an assumption of which Grassmann himself took note later on (in the Remark following No. 522). It should also be noted that in No. 494 Grassmann had not actually proved the existence of the solutions that he makes use of here.

**For the method of proof used here this theorem appears as the immediate broadening of a theorem proved in No. 512 (cf. p. 380, equation (7')ff).
which on the other hand can be written in the form

\[(\mu_1, 0, \mu_2, \ldots, \mu_{2n+3}),\]

and consequently is also equal to

\[(\mu_1, 0)(\mu_2, \ldots, \mu_{2n+3}).\]

Now if all the magnitudes \((\mu, 0), \ldots, (\mu_{2n+2}, 0)\) vanish, but not \((\mu_{2n+3}, 0)\), it now also follows that \((\mu_1, \ldots, \mu_{2n+2}) = 0\).

In No. 519 Grassmann shows that to each Pfaffian equation \(X_1dx_1 + \cdots + X_mdx_m = 0\) belongs a completely definite number \(n\) such that all expressions of the form

\[(0, \mu_1, \mu_2, \ldots, \mu_{2n+1})\]

vanish, but not all expressions of the form

\[(0, \mu_1, \mu_2, \ldots, \mu_{2n-1}).\]

Thus if all expressions of the form (23) are equal to zero, then as was just shown so also do all expressions \((\mu_1, \mu_2, \ldots, \mu_{2n+1})\) and thus also all expressions (21). On the other hand for \(2n + 1 > m\) every expression of the form (21) certainly has the value zero, since it then contains two equal indices, whereas for \(n = 1\) it is certain that not all expressions (23), that is, not all magnitudes \(X_1, \ldots, X_m\), vanish.

Consequently between the limits \(1 \leq n \leq \frac{1}{2}(m + 1)\) there is given a completely definite value of \(n\) that possesses the required property. Grassmann supposes this value as chosen for \(n\).

Now the problem posed in No. 514 is already settled for the case where \(m\) is even and \(n = \frac{1}{2}m\) (cf. 515–517 and p. 381f). Thus there remains only the two cases \(m = 2n - 1\) and \(m > 2n\). Of these, Grassmann considers only the second in No. 519, and on very good grounds indeed, for in the case \(m = 2n - 1\) the problem posed is generally not solvable. To be sure Grassmann did not mention this explicitly, but there is no doubt that he was completely clear in his mind about it.

Thus let \(m > 2n\). According to what has gone before, everything comes down to satisfying the equations

\[(12) \quad \lambda(0, \mu) = \sum_{\kappa=1}^{m} (\mu, \kappa) \frac{\partial x_{\kappa}}{\partial t} \quad (\mu = 1, \ldots, m),\]

and if this should only be possible for \(\lambda = 0\), in addition the equation

\[(13') \quad \sum_{\mu=1}^{m} (0, \mu) \frac{\partial x_{\mu}}{\partial t} = 0.\]

But Grassmann shows that these requirements can be reduced to the simpler case already solved previously, where \(m = 2n\).

Under the given assumptions all expressions of the form (21) equal zero, but not all of the form (23). Further we will at first assume as well that not all expressions of the form \((\mu_1, \mu_2, \ldots, \mu_{2n})\) vanish, and in fact that, say,

\[(24) \quad (1, 2, 3, \ldots, 2n)\]

may be different from zero. Then it can be shown, as Grassmann proves, that all the equations (12) are a consequence of the first \(2n\) of them.
Thus if, with Grassmann, we set
\[ G_\mu = \lambda(0, \mu) + \sum_{\kappa=1}^{m} (\kappa, \mu) \frac{\partial x_\kappa}{\partial t}, \]
and, if by \( \sum^{(c)} \) we mean the sum of all expressions that one obtains if one once, twice, \ldots, \((2n+1)\) times cyclically interchanges the \(2n+1\) indices \(1, 2, \ldots, 2n, 2n+\nu\), it follows that
\[
\sum^{(c)} G_1(2,3,\ldots,2n,2n+\nu) = \lambda \sum^{(c)} (0,1)(2,3,\ldots,2n,2n+\nu)
+ \sum_{\kappa=1}^{m} \frac{\partial x_\kappa}{\partial t} \sum^{(c)} (\kappa,1)(2,3,\ldots,2n,2n+\nu),
\]
but here the right side can be written in the form
\[
\lambda(0,1,2,\ldots,2n,2n+\nu) + \sum_{\kappa=1}^{m} \frac{\partial x_\kappa}{\partial t} (\kappa,1,2,\ldots,2n,2n+\nu)
\]
and this expression vanishes identically, since by assumption all expressions (21) and consequently (see the previous page) all expressions (22) also vanish. Therefore
\[
\sum^{(c)} G_1(2,3,\ldots,2n,2n+\nu)
\]
is identically zero, and since here the factor of \(G_{2n+\nu}\) has the value \((1,2,\ldots,2n)\) and thus is different from zero, it is herewith proved that the equations \(G_{2n+1} = 0, \ldots, G_m = 0\) are all consequences of the equations \(G_1 = 0, \ldots, G_{2n} = 0\).

It therefore follows, at least provided \((1,2,\ldots,2n) \neq 0\), that \(m-2n\) of the magnitudes \(\frac{\partial x_\nu}{\partial t}\) can be assumed completely arbitrarily, and in fact with Grassmann we set
\[
(25) \quad \frac{\partial x_{2n+1}}{\partial t} = \cdots = \frac{\partial x_{m}}{\partial t} = 0.
\]
But thereby, upon dropping the superfluous equations \(G_{2n+1} = 0, \ldots, G_m = 0\), equations (12) are reduced to the following:
\[
(12'') \quad \lambda(0,\nu) = \sum_{\kappa=1}^{2n} (\nu,\kappa) \frac{\partial x_\kappa}{\partial t} \quad (\nu = 1,\ldots,2n),
\]
and as with equation (14) we obtain
\[
(26) \quad \lambda(0,1,2,\ldots,\nu-1,\nu+1,\ldots,2n) = (-1)\nu(1,2,\ldots,2n) \frac{\partial x_\nu}{\partial t} \quad (\nu = 1,2,\ldots,2n),
\]
where the factors of \(\lambda\) on the left side certainly do not all vanish, for otherwise (as on p. 383) \((1,2,\ldots,2n)\) would also have to be zero.

Therewith it is shown that, provided \((1,2,\ldots,2n) \neq 0\), equations (12) can be satisfied identically, unless \(\lambda\) vanishes.

It still remains to settle the case where all expressions \((\mu_1,\mu_2,\ldots,\mu_{2n})\) vanish.

For this we can assume, without loss of generality, that not all of the \(2n\) expressions
\[
(0,1,2,\ldots,\nu-1,\nu+1,\ldots,2n) \quad (\nu = 1,2,\ldots,2n)
\]
vanish. Thus if we make the substitutions (25) in equations (12), then, as follows from (26), the resulting equations can certainly not be satisfied unless \( \lambda \) vanishes. On the other hand, as on p. 381, we can arrange that equations (12) are satisfied for \( \lambda = 0 \), and that in addition (13') is satisfied.

Thus if we set
\[
\frac{\partial x_\nu}{\partial t} = \rho(-1)(0, 1, \ldots, \nu - 1, \nu + 1, \ldots, 2n) \quad (\nu = 1, \ldots, 2n),
\]
then for \( \kappa = 0, 1, 2, \ldots, m \),
\[
\sum_{\nu=1}^{2n}(\kappa, \nu)\frac{\partial x_\nu}{\partial t} = \rho\sum_{\nu=1}^{2n}(\kappa, \nu)(\nu + 1, \ldots, 2n, 0, 1, \ldots, \nu - 1)
\]
\[= \rho(\kappa, 0, 1, \ldots, 2n) - \rho(\kappa, 0)(1, 2, \ldots, 2n),\]
whence under the given assumptions is equal to zero. Therefore, however, our assertion is proved.*

Now, in order to settle our problem completely, we will assume that \((0, 1, 2, \ldots, 2n - 1) \neq 0\), as we obviously may. Then we can set \( x_{2n} = t \) and under all circumstances obtain for \( x_1, \ldots, x_{2n-1} \) differential equations of the form
\[
\frac{\partial x_\nu}{\partial t} = \xi_\nu(x_1, \ldots, x_{2n-1}, t) \quad (\nu = 1, \ldots, 2n - 1),
\]
while \( x_{2n+1}, \ldots, x_m \) must satisfy equations (25). If we integrate equations (28) on the basis of the initial conditions \( x_\nu = y_\nu \) for \( t = 0 \) and in addition set \( x_{2n+1} = y_{2n+1}, \ldots, x_m = y_m \), we obtain \( x_1, \ldots, x_m \) represented as functions of \( y_1, \ldots, y_{2n-1}, t, y_{2n+1}, \ldots, y_m \), and it follows exactly as on p. 383 that in the new variables the equation \( \sum X_\mu dx_\mu = 0 \) takes the form
\[
\sum_{\nu=1}^{2n-1} X_\nu(y_1, \ldots, y_{2n-1}, 0, y_{2n+1}, \ldots, y_m)dy_\nu
\]
\[+ \sum_{\kappa=1}^{m-2n} X_{2n+\kappa}(y_1, \ldots, y_{2n-1}, 0, y_{2n+1}, \ldots, y_m)dy_{2n+\kappa} = 0.
\]

This is the content of No. 519. It goes considerably beyond what was achieved by JACOBI, and shows that GRASSMANN was fully conversant with the theory of systems of equations.

In Nos. 520 and 521 is analyzed which cases the considerations thus far permit the PFAFFian equation \( X_1dx_1 + \cdots + X_mdx_m = 0 \) to reduce to an equation between just \( m - 1 \) variables. The reduction considered is then always possible if all expressions of the form
\[
(0, \mu_1, \mu_2, \ldots, \mu_{2n'+1})
\]
(*In the previous considerations, which are only a translation of GRASSMANN’s, there is hidden a theorem on the skew-symmetric determinant
\[
\Delta = \det(0, \mu)1, \mu \cdots (m, \mu) \quad (\mu = 0, 1, \ldots, m),
\]
namely the theorem that the vanishing of all expressions of the form \( (0, \mu_1, \ldots, \mu_{2n'+1}) \) entails the vanishing of all \((2n+1)\)-row subdeterminants of \( \Delta \). Since every expression \( (0, \mu_1, \ldots, \mu_{2n'+1}) \) is the square root of a \((2n+2)\)-row subdeterminant of \( \Delta \) (cf. Editorial Note 112), and in fact of a subdeterminant that contains exactly \( 2n+2 \) elements of the diagonal of \( \Delta \), it is clear that we have to do here with one of the well-known FROBENIUS theorems on skew-symmetric determinants (cf. Crelle 82, p. 242, Theorem V, 1877).
vanish and in addition \( m \geq 2n' \), for the number\(^*\) defined on p. 384 satisfies \( n \leq n' \) and thus also \( m \geq 2n \). For \( m > 2n \) the reduction is carried out in No. 519, and for \( m = 2n \) in Nos. 515–517.

In particular, the reduction is always possible if \( m \) is even, \( = 2n' \), for then each of expressions (29) contain two equal indices and is therefore certainly zero.

In No. 522 is proved the important, and before GRASSMANN unknown, theorem that the equation \( X_1dx_1 + \cdots + X_mdx_m = 0 \) is always reducible to an equation between just \( 2n' - 1 \) variables, provided all the expressions (29) vanish.

Thus if \( m \geq 2n' \), one can according to what has gone before always introduce such new variables \( y_1, \ldots, y_{m-1}, t \) that the equation \( \sum X_\mu dx_\mu = 0 \) takes the form

\[
(30) \quad \sum_{\mu=1}^{m-1} X_\nu(y_1, \ldots, y_{m-1}, 0)dy = 0.
\]

But now by assumption all expressions (29) are equal to zero, and therefore also remain zero if in them one sets \( x_1 = y_1, \ldots, x_{m-1} = y_{m-1}, x_m = 0 \). Thus if \( m - 1 \geq 2n' \), then the conditions are satisfied under which equation (30) can be reduced to an equation between just \( m - 2 \) variables. When one repeats this procedure sufficiently often one finally obtains the equation \( \sum X_\mu dx_\mu = 0 \) reduced to one between just \( 2n' - 1 \) variables.

From this point, in Nos. 524 and 525 the proof that the condition derived in No. 511 is not just necessary but also sufficient is treated, and thus the Pfaffian expression \( X_1dx_1 + \cdots + X_mdx_m \) can always be brought to the form \( U_1du_1 + \cdots + U_{n'}du_{n'} \) with just \( n' \) differentials if and only if all expressions (29) vanish. This theorem too was new at GRASSMANN's time.

According to No. 511 the reduction considered is certainly only possible if all expressions (29) vanish. If this condition is satisfied, one first brings the equation \( \sum X_\mu dx_\mu = 0 \) to the form of an equation between just \( 2n' - 2 \) variables, which according to the above is always possible. In this equation, according to the process of PFANN, one of the variables, let it be \( u_1 \), is set to a constant, and then the resulting equation between \( 2n' - 2 \) variables is reduced to one between \( 2n' - 3 \). One of these \( 2n' - 3 \) variables is again set to a constant, call it \( u_2 \), and so forth. If one applies this procedure \( r \) times, one has set \( r \) functions \( u_1, \ldots, u_r \) of the original variables to constants and still has an equation between \( 2n' - 2r - 1 \) variables. Thus if one makes \( r = n' - 1 \) and also sets the last remaining variable \( u_{n'} \) to a constant, then obviously \( u_1 = \text{const}, \ldots, u_{n'} = \text{const} \) is an assembly that integrates the equation \( \sum X_\mu dx_\mu = 0 \). According to No. 502 one must therefore have

\[
\sum_{\mu=1}^{m} X_\mu dx_\mu = \sum_{\nu=1}^{n'} U_\nu du_\nu,
\]

and the theorem is thereby proved. At the same time it is clear that the definition of each \( u_\kappa \) requires the integration of a certain ordinary differential equation that one can formulate without having previously defined \( u_1, u_2, \ldots, u_{n-1} \).

(Nos. 525–529). From the theorems just proved it follows that a Pfaffian expression in \( 2n' \) variables is reducible to one with only \( n' \) differentials under all

\(^*\)It is a defect of GRASSMANN’s presentation that the letter \( n \) is used with different meanings. We therefore denote only the number defined on p. 384 by \( n \) and otherwise write \( n' \) for \( n \).

\(^{**}\)It is true that GRASSMANN did not say this explicitly, but it was without doubt known to him, as follows from his remarks in No. 523.
circumstances, and from this it follows again that the equation \( \sum X_{\mu} dx_{\mu} = 0 \) is never reducible to an equation between fewer than \( 2n - 1 \) variables, where by \( n \) is meant the number defined on p. 384. Thus even if the equation could only be reduced to an equation between \( 2n - 2 \) variables, one could reduce the expression \( \sum X_{\mu} dx_{\mu} \) to an expression with just \( n - 1 \) differentials, and thus according to No. 511 all expressions

\[ (0, \mu_1, \ldots, \mu_{2n-1}) \]

would equal zero, which conflicts with the definition of the number \( n \).

Finally, at this point the integration of the equation \( \sum X_{\mu} dx_{\mu} = 0 \) can be carried out, at least if one is limited to integrating assemblies of the form \( u_1 = \text{const}, \ldots, u_{n'} = \text{const} \). Thus if \( n \) is the number defined on p. 384, then there is no integrating assembly of this type, for which \( n' < n \); on the other hand there is an integrating assembly for which \( n' = n \), and according to No. 503 one can find all these integrating assemblies, provided one has cast the expression \( \sum X_{\mu} dx_{\mu} \) to the form \( U_1 du_1 + \cdots + U_n du_n \).

Now that we have become familiar with Grassmann’s investigations into Pfaff’s problem, we will briefly review what Grassmann has added to the contributions of his predecessors.

Grassmann’s merit consists, first, in that—as we can express it today—he has completely developed the invariant theory of an arbitrary Pfaffian equation to a certain degree. Thus he shows that to each given Pfaffian equation \( X_1 dx_1 + \cdots + X_m dx_m = 0 \) there belongs a certain whole number, which is a characteristic for the equation. This whole number lies between the limits 1 and \( \frac{1}{2}(m + 1) \), including the limits, and can be found without integration. If it has the value \( n \), the equation \( \sum X_{\mu} dx_{\mu} = 0 \) can, by the introduction of new variables, be reduced to a Pfaffian equation in \( 2n - 1 \) variables, but not to one in less than \( 2n - 1 \), and in addition the left side of the equation \( \sum X_{\mu} dx_{\mu} = 0 \), that is the Pfaffian expression \( \sum X_{\mu} dx_{\mu} \), can be reduced to an expression \( U_1 du_1 + \cdots + U_n du_n \) with exactly \( n \) differentials, but not to one with less than \( n \) differentials. Herewith it is actually proved that every Pfaffian equation \( \sum X_{\mu} dx_{\mu} \), whose characteristic value has the value \( n \), can take the normal form

\[ y_1 dy_2 + y_3 dy_4 + \cdots + y_{2n-3} dy_{2n-2} + dy_{2n-1} = 0, \]

where \( y_1, \ldots, y_{2n-1} \) denote mutually independent variables; in other words, it is proved that the number \( n \) is the only invariant of the Pfaffian equation \( \sum X_{\mu} dx_{\mu} = 0 \). It is true that Grassmann did not express his result in this form, and in addition the so-called normal form is not found in his work, but nevertheless one can justly say that he has given the criteria from which one can recognize into what normal form a given Pfaffian equation can be cast.

For the Pfaffian expression \( \sum X_{\mu} dx_{\mu} \) the question remains to be settled whether the form \( U_1 du_1 + \cdots + U_n du_n \), into which it can be cast, is capable of yet a further simplification or not, and thus whether the expression \( \sum U_{\nu} dw_\nu \) can be reduced to an expression with \( n \) differentials but only \( 2n - 1 \) variables, or not; for the reduction to an expression with only \( 2n - 2 \) variables is certainly impossible, since otherwise the number of differentials could be reduced to \( n - 1 \). For this

*That Grassmann considered only integrating assemblies of this form is shown in the Remark following No. 503.
question too Grassmann found the answer: the reduction to an expression in
$2n - 1$ variables is possible if and only if, for each given expression
$\sum X_{\mu} dx_{\mu}$ all expressions of the form $(\mu_1, \mu_2, \ldots, \mu_{2n})$ vanish, meaning by $\mu_1, \ldots, \mu_{2n}$ numbers from the series $1, 2, \ldots, m$. If not all of these expressions vanish, then $\sum X_{\mu} dx_{\mu}$ can take the form

$$y_1 dy_2 + \cdots + y_{2n-1} dy_{2n},$$

while if all vanish, $\sum X_{\mu} dx_{\mu}$ can take the form

$$\rho(y_1, \ldots, y_{2n-1})(y_1 dy_2 + y_3 dy_4 + \cdots + y_{2n-3} dy_{2n-2} + dy_{2n-1}),$$

where both times the $y$ denote mutually independent variables. There is therefore only lacking the demonstration that the last of these expressions can always be reduced to one of the same form in which $\rho$ has the value 1; but even without this simplification, whose possibility only Clebsch had recognized, what Grassmann had achieved for the theory of the invariants of a Pfaffian expression is still most noteworthy, for with him had basically been found the criteria by which one can recognize to which of the two possible normal forms a given Pfaffian expression is reducible, and exactly with respect to the correctness and completeness of the criteria Clebsch lagged considerably behind Grassmann.

Whichever actual formulation of the normal form of a given Pfaffian equation may be feasible, Grassmann had indeed shown that it can be achieved by the integration of a series of ordinary differential equations and that all these systems can be given without integration, but one can, after the prior work of Jacobi, discover in this no special contribution. The question of whether the order of the required integrations can be reduced, Grassmann had not considered; in this connection one must therefore be supplied with the almost contemporaneous investigations of Natani and Clebsch, although even these did not bring the question to a conclusion, which was reserved for the following decade.

On the other hand one cannot undervalue what Grassmann has achieved for systems of equations of the form (12), and in connection with this one must also mention his characteristic notation, which is exactly contemporaneous with that of Jacobi–Cayley, and in fact is superior, insofar as Grassmann's symbols always directly call to mind the Pfaffian expression for which it was formed, whereas in the symbol $(1, 2, \ldots, 2n)$, as such, no relation to Pfaff's problem can be recognized. On this account Grassmann's notation is applicable to systems of Pfaffian equations without further ado, which is not the case for that of Jacobi–Cayley.

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*To be sure he had only spoken explicitly of the case $m = 2n$, see the Remark following No. 416.

**See his two articles "Ueber das Pfaffsche Problem", Crelle 60, pp. 193–251 (1862; the article is dated 28 Sept. 1860), 61, pp. 146–179 (1863, dated 25 Jan. 1861), as well as a short preliminary note of March 1861, 59, pp. 190–192 (1861).

***Lie was the first to note this, and at the same time stated that Grassmann's main accomplishments for the theory of the Pfaffian problem are to be sought in the formulation of the correct criteria. Cf. the Obituary of Grassmann, Math. Ann. 14, p. 28 (1879).

****See his article (dated January 1860) "Ueber totale und partielle Differentialgleichungen", Crelle 58, pp. 301–328, which only appeared in 1861 and therefore certainly had no influence on Grassmann, just as little as the works of Clebsch.
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Supplementary Notes

Part I. The Elementary Conjunctions of Extensive Magnitudes

Chapter 1. Addition, subtraction, multiples and fractions of extensive magnitudes

In his “definitions”, Numbers 1 and 2, Grassmann admits the operations of a real vector space. “Magnitude” is taken as a primitive concept, as we should take “element” of a vector space, or “vector” — in the abstract, not the geometric, sense.

Definition Number 3 is a little fuzzy. Any real number is “derivable” from any other non-zero real number, so $\pi$ is as good a unit as 1. One has to stretch a little beyond his context for “1”. To be precise, one would say something like: “The numbers have a unit 1 that is distinguished by its multiplicative properties. I call this number the absolute unit.” Really, any vector $\neq 0$ is a unit.

Definition Number 4 is that of a “linearly independent” set of magnitudes.

In Definition Number 5 he insists that a basis (or at least a linearly independent set) be chosen, and defines an extensive magnitude to be a linear combination of some such set. From the remarks to Number 3, every magnitude is an extensive magnitude, and vice versa.

Of course, this observation changes if one works in the full Grassmann algebra (exterior algebra) $\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V)$. His reference to “the original units” is imprecise, for reasons cited in connection with Number 3. What he is saying here about “first order” is the extensive magnitudes of first order are to be those that are linear combinations of a fixed linearly independent set of magnitudes. Thus the concept of “first order” defines a vector space (in the abstract).

Then addition and subtraction are defined “coefficientwise” in terms of this basis, as is multiplication by numbers. He then derives the fundamental properties of these operations, summed up in Number 8. (Nowadays, we take an approximately minimal subset of these properties as axioms for a vector space, and prove the existence of bases.)

There follow, in Numbers 16–20, basic properties of linear combinations and linear (in)dependence, including the “exchange property” (Numbers 19–20).

Numbers 21–24 discuss bases and dimension. His remark after Number 24 notices the comment about Number 3 noted above. Numbers 25, 26 give what we would write $\dim(U \cap V) + \dim(U + V) = \dim U + \dim V$. Proofs are essentially those given in modern texts.
Numbers 27–32, with the exceptions of Numbers 30–32 (that really go nowhere) are just basic properties of linearly independent sets and the operations in vector spaces.

His “shadow” in Number 33 is more carefully defined than one might expect from his times. It is, in modern terms, the “projection on the subspace \( \langle a_1, \ldots, a_m \rangle \) relative to the decomposition \( \langle a_1, \ldots, a_n \rangle = \langle a_1, \ldots, a_m \rangle \oplus \langle a_{m+1}, \ldots, a_n \rangle \). [Here \( \langle u_1, \ldots, u_r \rangle \) is used for the vector subspace of linear combinations of \( u_1, \ldots, u_r \).]

Numbers 34–36 are just further properties of operations in vector spaces.

Chapter 2. The product structure in general

It is here that Grassmann launches into what is currently called “multilinear algebra”. He is already breaking ground in Chapter 1, but a number of his contemporaries would probably have said “oh, yes” to that, even if they hadn’t thought about it in that way before. More would have been mystified by what comes next.

In order to view Definition 37 in its proper generality one should probably speak about a bilinear pairing \( U \times V \to W \), where \( U, V, W \) are vector spaces. If \( \{e_1, \ldots, e_n\} \) is a basis for \( U \), \( \{f_1, \ldots, f_m\} \) is a basis for \( V \), and if \( [w] \) denotes the image of the pair \((u, v) \in U \times V\), then all products \([w] \) are determined by the \([e_i f_j] \) as in his formula, where he takes \( V = U \). It is clear that he wants to allow (indeed, ultimately to insist) that the magnitudes in \( W \) be of different kind than those in \( V \).

Numbers 38–42 then just express the bilinearity of such a product.

The notion of Number 43 amounts to “specialization” or “substitution” or “homomorphism”. Here one notes from his remark that he is not necessarily assuming products to be associative. The way he has introduced products, really as a map \( V \times V \to W \), he is not yet equipped to speak about products of more than two factors, so a good context for the remark, and for Number 45 below, has not been established. Thus to make sense of Number 45 (with conventions of Number 7), he would need something like this: “Suppose we have bilinear mappings (products) \( V \times V \to W_1, W_1 \times V \to W_2, W_2 \times V \to W_3, \ldots, W_{m-1} \times V \to W_m \). Then \( (a) \) holds, where \([v_1, v_2, \ldots, v_m] \) means \([\ldots [[[v_1 v_2] v_3] v_4] \ldots v_n] \).”

The rest of §2 expresses the “multilinearity” of such products.

By “unit products” in Number 48 he seems to mean products \([e_i e_j] \) (or more generally \([e_{i_1} \cdots e_{i_m}] \)) where the \( e_i \) are his “original units”. It seems that Definition 50 amounts to saying that the “defining equations” are bilinear, or multilinear. It’s really stronger than that, and by a lot. For instance one might have a product which is bilinear with \([e_1 e_1] = 0 = [e_2 e_2] = [e_2 e_1], \) while \([e_1 e_2] \neq 0. \) But this would not satisfy Number 50 as he uses it. A modern proof might go a little faster, if one gave a proper formulation to the notion of linear product structure, which might go something like this: There is a bilinear mapping \( V \times V \to U \), where \( U \) is a subspace of \( V \otimes V \) and our product has a factorization \( V \times V \to V \otimes V \to W \), where the image of \( U \) in \( W \) is zero. If the mapping \( V \times V \to U \) sends \((v_1, v_2) \) into \( v_1 \otimes v_2 \), then \([v_1 v_2] = 0. \) If \((v_1, v_2) \to 0, \) then \([v_1, v_2] \approx v_1 \otimes v_2, \) i.e. there is no defining equation. Now his definition requires something more, really that \( U \) be stable under the action of all linear substitutions from \( V \). The symmetric and skew-symmetric elements of \( V \otimes V \) form subspaces satisfying this condition, and
$V \otimes V = S^2(V) \oplus \Lambda^2(V)$. Each of these summands is irreducible for the action of the group $\text{GL}(V)$ of all invertible linear transformations of $V$, and they are not isomorphic as $\text{GL}(V)$ modules (different dimensions). It follows from general algebraic principles that $U = S^2(V)$ or $U = \Lambda^2(V)$, so that [ ] is skew-symmetric or commutative, respectively (“combinatorial” or “algebraic” in his terminology).

One may view Number 51 as providing his rationale for concentrating on products satisfying the conditions of Chapter 3. It is almost certainly his deepest theorem up to this point.

Chapter 3. Combinatorial product

In Numbers 52 and 53 he starts with a “system of units”— i.e. a basis for a vector space, and considers arbitrary products (formally) of these, always following the association $[ab \cdots z] = [[\cdots [ab]c] \cdots y]z$, subject to the conditions that interchanging the last two (unit) factors changes the sign. Then he extends this by multilinearity. We’d say that if $V$ is a vector space with the original system of units as basis, then $\Lambda^j(V)$ (the $j$th exterior power) is the vector space linearly generated by the products involving precisely $j$ of these units, subject to his relations. From this, he is able to speak of products of “elementary factors” and derive Number 58. Extended to multilinearity it says that if $B \in \Lambda^n(V)$, $C \in \Lambda^s(V)$, then $[BC] \in \Lambda^{n+s}(V)$ is equal to $(-1)^{rs}[CB]$. At this point he has only defined these “products” of higher order by “rearranging parentheses” in products of elementary factors. It is not until later, culminating in the associative law of Number 80, that he really justifies speaking in the terms used here.

Number 63 shows that the coordinates of the units in a product of $n$ factors are the $n$-rowed minors of the matrix of coefficients. Then, up to Number 68, he goes over to the multilinear point of view and derives properties.

To §2: Number 69 is the basis for saying $\dim(\Lambda^r(V)) = \binom{n}{r}$ if $\dim V = n$, and Number 70 for saying that two “pure $m$-vectors” $a_1 \wedge \cdots \wedge a_m$ and $b_1 \wedge \cdots \wedge b_m$ are scalar multiples one of the other if and only if the vector subspaces $\langle a_1, \ldots, a_m \rangle$ and $\langle b_1, \ldots, b_m \rangle$ of $V$ are equal. Thus the $m$-dimensional subspaces of $V$ are a part (actually a subvariety) of the projective space of lines in $\Lambda^n(V)$; this means of geometrizing the set of such subspaces is one of the aspects of Grassmann’s work that has contributed most to modern geometry. The next sections deal with the effects of linear transformation of the factors on a combinatorial product, starting with “elementary [ . . . ]” or “linear evolutions”, or (after Dieudonné, La Géométrie des Groupes Classiques) “transvections”. Number 76 is a result of the fact (and Grassmann’s arguments offer a proof of this) that the “linear evolutions” generate the special linear group $V \rightarrow V$ of determinant 1).

In §3, he proceeds from exterior products of “units” in $\Lambda^m(V)$ and $\Lambda^{n-m}(V)$ to get units (or 0) in $\Lambda^n(V)$ (possibly with sign change), and by bilinearity to get the exterior products $\Lambda^m(V) \times \Lambda^{n-m}(V) \rightarrow \Lambda^n(V)$. (Here “$n$” has nothing to do with $\dim V$, although $\Lambda^n(V) = 0$ if $n > \dim V$.)

Number 84 contains the germ of the idea of giving equations that tell which elements of $\Lambda^m(V)$ are pure $m$-vectors, i.e. pure outer products, or alternatively, represent $m$-dimensional subspaces of $V$. Such equations define what are called “Grassmann varieties” (e.g. Hodge and Pedoe, Methods of Algebraic Geometry, vol. 2).
In §4, we see this: The “C” of Number 87 represents (as Grassmann notes) the intersection of the subspaces represented by A and by B.

His notion of “supplement” is defined in terms of a fixed basis for the “principal domain”, taken in a fixed order: \([e_1 \cdots e_n] = 1\). This last “= 1” is not what we’d do today; rather, we’d say, “fix the basis \(e_1 \wedge \cdots \wedge e_n\) for \(\Lambda^n(V)\). Then if \(E\) is a unit of \(\Lambda^j(V)\), the supplement \(E'\) of \(E\) is the unit in \(\Lambda^{n-j}(V)\) such that \([EE'] = (E \wedge E')\), in modern notation \(= e_1 \wedge \cdots \wedge e_n\). In general if \(A \in \Lambda^j(V)\), \(B \in \Lambda^{n-j}(V)\), \([AB]\) is of the form \(\lambda(A, B)e_1 \cdots e_n\), where \(\lambda(A, B)\) is a number.” Then the formula of Number 89 reads \(|E = \lambda(E, E')E'|\).

In Number 90, Grassmann extends the map “\(\iota\)” of units in \(\Lambda^j(V)\) to units of \(\Lambda^{n-j}(V)\) by linearity to a linear map \(\Lambda^j(V) \to \Lambda^{n-j}(V)\).

To §5: To give the regressive product a coordinate-free setting, one really needs the notion of dual vector space to \(V\), \(V^* = \text{all linear maps } V \to \mathbb{R}\). i.e. all linear functionals on \(V\). Then \(\dim V^* = n\), and one can form all \(\Lambda^j(V^*)\). There is a natural identification of \(\Lambda^j(V)\) with the dual space \(\Lambda^j(V)^*\) of \(\Lambda^j(V)\). Having chosen (as above, for example) a basic element for \(\Lambda^n(V)\), one has a pairing as \(\lambda\) above that makes \(\Lambda^{n-j}(V)\) be isomorphic to \(\Lambda^j(V)^*\), so to \(\Lambda^j(V^*)\). Now suppose \(r + s > n\). Then \(\Lambda^r(V) \simeq \Lambda^{n-r}(V^*)\), \(\Lambda^s(V) \simeq \Lambda^n-s(V^*)\), and the outer product of elements of \(\Lambda^{n-r}(V^*)\) and \(\Lambda^n-s(V^*)\) is \(\Lambda^{2n-r-s}(V^*)\), with \(2n - r - s < n\), so corresponds to an element of \(\Lambda^{n-(2n-r-s)}(V)\), i.e. to an element of \(\Lambda^{r+s-n}(V)\). This is the “regressive product” of the two original elements. (If \(r + s = n\), Grassmann’s unfortunate identification \([e_1 \cdots e_n] = 1\)” makes for some confusion.)

Normally, nowadays we have little use for the “regressive product”. Grassmann makes it fit into a more general product on \(\Lambda(V)\) (the “relative product”) and it has a nice geometric meaning. If \(A \in \Lambda^r(V), B \in \Lambda^s(V)\) represent subspaces of respective dimensions \(r, s\) (say \(U\) and \(W\)), then there are subspaces \(U^\perp, W^\perp \in V^*\), of respective dimensions \(n-r, n-s\) (\(U^\perp = \{f \in V^* | f(U) = 0\}\), and \(U^\perp + W^\perp\) is represented by an element of \(\Lambda^{2n-r-s}(V^*)\) if \(U^\perp \cap W^\perp = \{0\}\). Then \(U^\perp + W^\perp = (U \cap W)^\perp\), so \((U^\perp + W^\perp)^\perp = U \cap W\) back in \(V\). This subspace corresponds to the regressive product of \(A\) and \(B\). For example, if \(A = [e_1 \cdots e_r], B = [e_{n-s+1} \cdots e_n], r + s > n\), then \(U = \langle e_1, \ldots, e_r\rangle, V = \langle e_{n-s+1}, \ldots, e_r\rangle, U \cap V = \langle e_{n-s+1}, \ldots, e_r\rangle, U^\perp = \langle e_{r+1}, \ldots, e^*_r\rangle, V^\perp = \langle e_{s-r}, \ldots, e^*_n\rangle\), where the \(e^*_i\) are the “dual basis” to \(e_1, \ldots, e_n\), \(U^\perp + V^\perp = \langle e^*_1, \ldots, e^*_r\rangle, (U^\perp + V^\perp)^\perp = \langle e_{n-s+1}, \ldots, e_r\rangle\), which is \(U \cap V\), and is what the requisite product gives: \(|A = [e_{r+1} \cdots e_n], |B = \pm[e_1 \cdots e_{n-s}]|, |A, B| = \pm[e_1 \cdots e_{n-s} e_{r+1} \cdots e_n], ||A, B| = \pm [e_{n-s+1} \cdots e_r].\) (From this point of view, \([e_1 \cdots e_s]\) corresponds to that element of \(\Lambda^{n-r}(V)^*\) that is \(\pm 1\) on \([e_{r+1} \cdots e_n]\) and \(0\) on other “units” if its basis, so to \(\pm[e_{r+1} \cdots e_n]\), \([e_{n-s+1} \cdots e_n]\) to \(\pm[e^*_1 \cdots e^*_s]\), and, in turn, \([e^*_1 \cdots e^*_s] \wedge [e^*_1 \cdots e^*_s] = \pm[e^*_1 \cdots e^*_s e_{r+1} \cdots e^*_n].\)

Now \([e_{n-s+1} \cdots e_r]\) in \(\Lambda^{s-r}(V)\) corresponds to the element of \(\Lambda^{2n-s-r}(V)^*\) that is \(\pm 1\) on \([e_1 \cdots e_{n-s} e_{r+1} \cdots e_n]\) and \(0\) on other units, thus \(\pm[e^*_1 \cdots e^*_s \cdots e^*_n]\). Thus we again have \([|A, B| = \pm [e_{n-s+1} \cdots e_r]|\). (The signs are determined in a consistent way.) The point is that it is better to think of “\(\iota\)” as a mapping \(\Lambda^j(V) \to \Lambda^{n-j}(V)^*\) (or from \(\Lambda^j(V)^*\) to \(\Lambda^{n-j}(V)\)) than without the dual spaces; of course the concept of dual space was not there for Grassmann.
In Number 110 what he is saying is that one can replace the original basis \(e_1, \ldots, e_n\) by any basis resulting from this by a linear transformation of determinant 1, and all his multiplication laws are unchanged.

Perhaps part of the failure of the mixture of progressive and regressive products to catch on lies in the failure of the associative law in general, as noted in Number 125.

In Number 127, he makes sure, by his exclusions, that \(\{A_1, \ldots, A_u\}\) and \(\{A_{u+1}, \ldots, A_v\}\) have no nontrivial linear combinations in common. Thus the map \(C \mapsto C_1\) is a well-defined “projection”. That it is linear is the content of Number 130.

In Number 129, the denominator \([BC]\) is really in \(\Lambda^n(B + C)\), and is non-zero; he treats it as a number by his identification \(\langle e_1 \cdots e_n \rangle = 1\).

Number 133 is just preparing ground for his presentation of “Cramer’s rule” in Number 134. There he derives the rule using exterior multiplication in place of the more customary determinantal rules (in Solution 1) for the case where the matrix of coefficients is invertible. The non-invertible case can’t be regarded as a “solution” until he tells how to select a maximal linearly independent set of columns from the coefficient matrix. This can of course be done by “Gaussian elimination”, but that is a more efficient say of solving the original system, anyway. “Solution 2” amounts to trying to reduce to the “homogeneous” case.

In Number 136 he arrives at the resultant of two polynomials (the quantity \([u_1 u_2 \cdots u_{n_1+n_2}]\) of p. 90). It is on this discovery that he staked his priority claim against Cauchy.

Chapter 4. Inner product

This chapter is quite interesting; from a modern viewpoint what Grassmann does is to show that his identification \(\langle e_1 \cdots e_n \rangle = 1\) and his “relative product” give a structure of inner product space (with a positive definite inner product and with the units of a given degree as orthonormal basis) to each \(\Lambda^d(V)\), where \(V\) is the “principal domain”. Once one has the inner product on \(V\), one can reconstruct everything he does in the chapter. He does concentrate on \(V\), the principal domain, in Numbers 188ff. He does not seem to have noticed that one must prove the Cauchy–Schwarz inequality in order for \(\preceq AB\) of Number 195 to make sense, that is that \(|[A|B]| \leq \alpha \beta\), in his notation (cf. Editorial Note 68). In that case all his inner products can then be expressed as one would contract tensors, using the inner product in \(V\). [This is not to detract from his achievement, but rather to point out how much he has achieved.]

The concept of “circular evolution” would now be called a “plane rotation or reflection”, depending on the \(\pm\) sign [Number 154]. What he shows then is, in effect, that such transformations generate the orthogonal group of \(V\) (relative to its inner product), so that he can show what we’d call “independence of the choice of orthonormal basis for \(V\)” (cf. Number 168). Then “normal shadows” of \(\S 3\) are orthogonal projections, in case the degrees are equal (cf. Remark after Number 164).
Chapter 5. Applications to Geometry

§1 is a summary of the notions of 3-dimensional vector geometry, with careful distinctions between points and “displacements” (or “vectors”, or “translations”). In §2, “points at infinity”, “lines at infinity”, etc. are introduced—in a way that was probably current at the time, perhaps from Möbius—so as to complete the affine space to a projective space.

The bulk of Chapter 5 consists of expressing equations for various geometric loci in Grassmann-algebra terms. There are no remarkable derivations or impressive geometrical results here.

Part II. The Theory of Functions

Chapter 1. Functions in general

Other publications on the history of 19th century mathematics, especially analysis, have reviewed the evolution of the notion of “function”—in Grassmann’s case, he leaps at once to the abstract one, although in fact all his functions are pretty classical ones.

§1 amounts to connecting systems of \( n \) real-valued functions, each of (the same) \( m \) real variables to mappings \( f : \mathbb{R}^m \to \mathbb{R}^n \), with therefore vector-valued functions of vector variables.

§2 deals with symmetric functions of vector variables, as obtained by “polarization” (the “arithmetic mean” of Number 353) from arbitrary functions formed from pure products (whether “combinatorial”, or “algebraic”, or more general).

In §3, one would say nowadays that Grassmann is working in the “symmetric algebra” based on the underlying vector space ("principal domain", “Hauptgebiet”) \( V \). His “algebraic quotient” of Number 374 may then be viewed as an element of the “field of fractions” of the “integral domain”, which is the symmetric algebra of \( V \). This should not be confused with the “quotient” of Number 377, which is really a linear transformation \( V \to V \).

§4: Here \( Q \) is the unique linear transformation sending \( a_j \) to \( b_j \) for each \( j \), \( 1 \leq j \leq n \). He adds them, multiplies by scalars, and (in Number 381) shows that the so-called “matrix units” relative to a given basis \( \{e_1, \ldots, e_n\} \) for \( V \) form a basis (the “\( E_{r,s} \)”) for the linear mappings \( V \to V \). The “power” \( [q^n] \), with his identification of \( \Lambda^n(V) \) with numbers, is just the determinant of the linear transformation \( Q \). Although he does not seem ready to compose linear transformations \( V \to V \), so as to obtain a connection with products of matrices, what he says in Number 384 is very close to saying that “the determinant of the composite of two linear transformations is the product of the determinants.”

We’d call the vectors \( a_1, \ldots, a_m \) of Number 386 a “basis for the image of \( Q \).”

Numbers 388 and 389 deal with “eigenvalues”, especially the case of distinct eigenvalues.

Number 390 could be expected to culminate in a version of the “Jordan normal form” for a (complex) matrix. It seems that Grassmann almost gets there, the
othesis being that he apparently doesn’t allow for more than one “Jordan block”, i.e. matrix of the form

\[
\begin{bmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
0 & 0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda
\end{bmatrix}
\]

Number 391 has as its final conclusion what we’d call the “spectral theorem for real symmetric linear transformations” (or matrices), under the additional hypothesis \([Qa][a] \neq 0\) for \(a \neq 0\) (which forces the matrix to be positive definite or negative definite, an observation he seems to miss (but Editorial Note 94 picks up)). However, his conclusions from (c) on are reached by fallacious reasoning; what he says amounts to saying that the relations (b) are preserved under every orthogonal transformation of the space spanned by \(c_1, \ldots, c_n\), and that is false unless \(Q\) is a scalar. In particular, the relation “[\(Qb_2][a_2]\] = 0” on p. 223 would force \(x = 0\), so would constrain the kind of orthogonal transformation one could use. Editorial Note 93 does not pick up the error.

Nowadays one proves Number 391 either by maximizing the quadratic form \([Qx][x]\) on the unit sphere, or by appealing to the fundamental theorem of algebra after extending \([Qx][x]\) to a Hermitian form on \(V + iV\). Then one gets an eigenvector for \(Q\) (necessarily one in \(V\), with real eigenvalue); \(Q\) stabilizes the orthogonal complement, and one completes the proof by induction on dimension \(n\). The requirement of Number 391 or that of Editorial Note 94 is not needed.

Regarding the Remark after Number 396. What can be simpler than this “proof by coordinates”? Choose the coordinate system so that center of circle is at \((0,0)\), desired point on positive axis, say at \((x,0)\). If circle has radius \(r\), equation of circle is \(x^2 + y^2 - r^2 = 0\) and “double-distance” is \((x - r)(x + r) = x^2 - r^2 = f(x,0)\). (If \((x,0)\) is inside the circle, \(x^2 - r^2 < 0\), but “double-distance” is product of directed distances.)

Number 402 just is the notion of defining a vector space by generators and (linear) relations. Here “linear” is not the same as “linear product”—see end of Chapter 2, Part I. The notion of “linear product” was put in modern terms in these notes there.

Number 405: The correspondence is that of projective space. That is, in each case one has coordinates \((\alpha, \beta, \gamma, \delta)\), not all 0, where for \(\lambda \neq 0\), \((\lambda \alpha, \lambda \beta, \lambda \gamma, \lambda \delta)\) defines the same “point” or “circle”.

Number 410: Here Grassmann comes back to the idea that he is working over a Euclidean vector space \(V\) with a distinguished ordered basis of unit vectors \(e_1, \ldots, e_n\). Then one has the corresponding basis (ordered monomials in \(e_1, \ldots, e_n\), without repetitions) for the exterior algebra \(\Lambda(V)\) and that (ordered monomials, allowing repetitions) for the symmetric algebra of \(V\).

Number 411: The earlier “matrix units” are now generalized to the corresponding ones for linear transformations of one vector space to another (in terms of fixed bases).

Number 418 is what we call the “triangle inequality”.
Number 420 is the definition of “lim_{q \to a^+} f(q) = p”; here of course Grassmann is getting ready for the notion of continuity, as well as that of “derivative”.

Number 427: We’d write \( C = \lim_{x \to a} \frac{f(x)}{f_1(x)} \).

Chapter 2. Differential calculus

In Number 431, he is working in the “symmetric algebra” of his “principal domain” or “Hauptgebiet”, and \( A \) is either a monomial or a homogeneous element (of degree \( n \)).

Number 437 and some later assertions are forms of the “chain rule”. All is well, because he assumes continuity of the derivatives and partial derivatives concerned.

Number 438 presents the “derivative as linear transformation”, the modern way of teaching “vector calculus” (e.g. Spivak, Calculus on Manifolds, Theorem 2–1).

Chapter 3. Infinite series

Number 454: The notion of “true” series seems motivated by the behavior of the power series for an analytic function within its circle of convergence. It says that the series of norms is dominated by a convergent geometric series. Much of the rest of Chapter 3 is devoted to “Cauchy’s integral theorem”, using a discrete version of integration around a circle, with application, in Number 466, to deduce the analyticity of twice continuously differentiable functions of a complex variable. The last sections of Chapter 3 extend these results to multiple variables.

Chapter 4. Integral calculus

Number 505: Apparently one is thinking of \( L \) here as a multilinear function of \( n \) vector variables, with scalar values. Then the notation “[\( La_1 \cdots a_n \)]” stands for what one gets by completely skew-symmetrizing the quantity \( L(a_1, \ldots, a_n) \). The result is an alternating \( n \)-linear function \( \Lambda(L) \) on the vector space \( V \), or a linear function on \( \Lambda^n(V) \). But the space of such functions is 1-dimensional, so \( \Lambda(L) \) acts on \( \Lambda^n(V) \) by a fixed scalar \( \lambda \), and \( [La_1 \cdots a_n] = \lambda[a_1 \cdots a_n] \) (cf. Editorial Note 111). The following Numbers, 508 and 509, seem to be a kind of “expansion by minors” in this setting. It is not clear how this connects with the last paragraph of Editorial Note 111; Cayley’s “Pfaffian” is associated with a skew-symmetric matrix of \( 2n \) rows and columns, and is a polynomial of degree \( n \) in the entries; the square of this polynomial is the determinant (see Remark, p. 313).

The Editorial Notes on pp. 378–389 go systematically through the last sections of Grassmann, interpreting in more standard notation. One might remark that a 20th century treatment of the problem is to be found in C. Caratheodory, Calculus of Variations and Partial Differential Equations of First Order, 2nd English Edition, Chelsea, 1982 (Chapter 8).
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Hermann Grassmann

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