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Volume 21

Essays in the History of Lie Groups and Algebraic Groups

Armand Borel

American Mathematical Society
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10 9 8 7 6 5 4 3 2 1 06 05 04 03 02 01

Contents

Introduction ix

Terminology for Classical Groups and Notation xi

Photo Credits xiii

Chapter I. Overview 1
§1. Lie's theory 1
§2. Lie algebras 5
§3. Globalizations 6
References for Chapter I 8

Chapter II. Full Reducibility and Invariants for $\mathbf{SL}_2(\mathbb{C})$ 9
§1. Full reducibility, 1890–96 9
§2. Averaging. The invariant theorem 11
§3. Algebraic proofs of full reducibility 16
§4. Appendix: More on some proofs of full reducibility 18
References for Chapter II 26

Chapter III. Hermann Weyl and Lie Groups 29
§1. First contacts with Lie groups 29
§2. Representations of semisimple Lie groups and Lie algebras 31
§3. Impact on É. Cartan 35
§4. The Peter-Weyl theorem. Harmonic analysis 37
§5. Group theory and quantum mechanics 38
§6. Representations and invariants of classical groups 40
§7. Two later developments 44
Notes 46
References for Chapter III 54

Chapter IV. Élie Cartan, Symmetric Spaces and Lie Groups 59
A. Building Up the Theory 60
§1. The spaces $\mathcal{E}$. Local theory 60
§2. Spaces $\mathcal{E}$ and semisimple groups. Global theory 64
§3. An exposition of Lie group theory from the global point of view 79
B. Further Developments 80
§4. Complete orthogonal systems on homogeneous spaces of compact Lie groups 80

§5. Differential forms and algebraic topology 84
§6. Bounded symmetric domains 88
References for Chapter IV 90

Chapter V. Linear Algebraic Groups in the 19th Century 93
§1. S. Lie, E. Study, and projective representations 93
§2. E. Study, Gordan series and linear representations of $\mathbf{SL}_3$ 97
§3. Emile Picard 99
§4. Ludwig Maurer 102
§5. Élie Cartan 114
§6. Karl Carda 115
References for Chapter V 117

Chapter VI. Linear Algebraic Groups in the 20th Century 119
§1. Linear algebraic groups in characteristic zero. Replicas 119
§2. Groups over algebraically closed ground fields I 119
§3. Groups over an algebraically closed ground field II 124
§4. Rationality properties 126
§5. Algebraic groups and geometry. Tits systems and Tits buildings 131
§6. Abstract automorphisms 134
§7. Merger 142
References for Chapter VI 144

Chapter VII. The Work of Chevalley in Lie Groups and Algebraic Groups 147
§1. Lie groups, 1941–1946 147
§2. Linear algebraic groups, 1943–1951 150
§3. Lie groups, 1948–1955 152
§4. Linear algebraic groups, 1954 155
§5. Algebraic groups, 1955–1961 156
References for Chapter VII 162

Chapter VIII. Algebraic Groups and Galois Theory in the Work of Ellis R. Kolchin 165
§1. The Picard-Vessiot theory 165
§2. Linear algebraic groups 169
§3. Generalization of the Picard-Vessiot theory 170
§4. Galois theory of strongly normal extensions 173
§5. Foundational work on algebraic sets and groups 176
References for Chapter VIII 179

Name Index 181

Subject Index 183

Introduction

This book consists of essays, some published previously, on topics belonging mainly to the first century of the history of Lie groups and algebraic groups. Partly written upon request, for various purposes, they do not aim at giving a comprehensive and exhaustive exposition. In order to put them in context, the first chapter attempts to sketch how they fit in the overall picture and complements them by some discussion of items not, or only briefly, touched upon elsewhere.

The "finite and continuous groups" of Sophus Lie were in fact analytic local groups of analytic transformations, as recalled in I, §1, but we shall deal almost exclusively with global aspects of Lie groups and algebraic groups. In retrospect, one can say that the passage from the local to the global was carried out in two ways, the transcendental or differential geometric one, highlighted by the contributions of A. Hurwitz, I. Schur, H. Weyl and É. Cartan, and the algebraico-geometric one, initiated in the 19th century mainly by L. Maurer, revived in the nineteen forties by C. Chevalley and E.R. Kolchin, and then developed by many others. Chapters II, III and IV pertain to the former, Chapters V, VI, VII and VIII to the latter.

Chapter II is the most elementary, devoted mostly to various proofs of the full reducibility of linear representations of $\mathbf{SL}_2(\mathbb{C})$.

Chapter III is more generally concerned with the work of Hermann Weyl on Lie groups, Lie algebras and invariant theory. It describes in particular his synthesis of Cartan's infinitesimal approach with the transcendental point of view initiated by A. Hurwitz (discussed in Chapter II), further developed by I. Schur, and its influence on É. Cartan, who was at the time noticing a remarkable connection between the theory of real simple Lie algebras and the study of a new class of Riemannian manifolds, later called symmetric spaces. This led Cartan to the work described in Chapter IV: first the building up of a theory of semisimple Lie groups and Riemannian symmetric spaces, in which both are remarkably intertwined, and then further developments which were all to have a far-reaching influence: generalization of the Peter-Weyl theorem to compact symmetric spaces, introduction of differential forms in algebraic topology, and bounded symmetric domains.

The remaining chapters are devoted to algebraic groups. Even though it was not a broadly recognized field in the 19th century, several, largely independent, contributions fit well under that heading, the most systematic being those of L. Maurer. They are surveyed in Chapter V.

The topic then fell into oblivion for almost half a century, and was taken up again in the 1940s, first by C. Chevalley and E.R. Kolchin. Their motivations were completely different: Chevalley wanted to develop and generalize the theory initiated by L. Maurer, while Kolchin's interest was mainly in the Picard-Vessiot Galois theory of homogeneous linear differential equations. Although he did not

really need it, Kolchin started a theory over algebraically closed groundfields of arbitrary characteristic (while Chevalley was at first essentially bound to characteristic zero). After this pioneering work, the theory underwent spectacular developments in several directions. Chapter VI attempts to give an idea of the main steps. One can distinguish a first phase over algebraically closed groundfields, culminating with Chevalley's classification of algebraic simple groups (§§2, 3) and then the study of rationality properties over arbitrary fields (§4). Relationships with geometry, which in a way were there from the beginning, took a new prominence in the framework of Tits systems and Tits buildings (§5). From the thirties on, a rather persistent theme has been to what extent Lie group or, later, algebraic group properties can be read off the abstract group structure. Concretely, to what extent are abstract automorphisms described by Lie group or algebraic group automorphisms and field automorphisms? This is also a topic in which algebraic groups and geometry mix. It is discussed in §6. I have also included, as Chapters VII and VIII, two articles published earlier on C. Chevalley and E.R. Kolchin (slightly revised).

Chapter VI does not aim at completeness. The field of linear algebraic groups is still very active, and it was not my intention to cover the most recent developments. This, to me, is anyhow the purview of another type of publication. Without being strict about it, I have limited myself to work done (or at any rate well under way) during the first century of the theory (1873-1973). Following this rule, I have limited myself to some brief indications in VI, §7 on some very fruitful relations between the transcendental and algebraico-geometric points of view, woefully short in view of their growing importance.

When the editorial board for this series kindly suggested I contribute a volume to it, I felt that on one hand the papers underlying II, III, VII, VIII, with some minor modifications, belonged to it, and that, on the other hand, I could not contemplate rehashing them to fit them into a seamless narrative. I was even less tempted to do so in view of the forthcoming book by T. Hawkins: "The emergence of the theory of Lie groups", Springer 2000, which, among other things, includes a systematic, thorough exposition of much of the material in Chapters I, II, III. Thus from the start, as hinted by its title, this book was intended to be a rather heterogeneous collection of essays. It consists of four "old" (i.e. essentially published earlier) chapters and four "new" ones. This entails some overlap, especially between the old and the new, which I have not tried to suppress, preferring to let the old chapters keep the degree of autonomy initially intended. I still hope the book gives a good idea of the development of the topics it covers.

I am very grateful to T. Hawkins and T.A. Springer for corrections to, and remarks or questions on, earlier drafts, which led to a number of improvements and additions. As usual, deserved thanks are due to E. Gustafsson, who tirelessly typed into impeccable TEX rather unappealing typescripts, emanating from an old-fashioned typewriter, about to reach the status of an endangered species.

Finally, I am glad to thank the editorial staff of the AMS for suggestions and help in various aspects of the production of this book.

Terminology for Classical Groups and Notation

I shall use mostly present-day notation, as indicated below, without further reference. Here, I also indicate the terminology used by Lie.

$$k \text{ is a commutative field}, \quad k^* = k - \{0\}$$

1. $\mathbf{GL}_n(k)$ is the group of $n \times n$ invertible matrices with coefficients in k, and $\mathbf{SL}_n(k)$ its subgroup of matrices of determinant one.

Lie considered these groups mainly for $k = \mathbb{C}$, and I shall sometimes drop the $(\mathbb{C})$ in that case in Chapters I and V. These groups are now called respectively the general and special linear groups, but this is not Lie's terminology. For him, they are the general and special homogeneous linear groups. He reserves the term general linear group for our affine group $\mathrm{Aff}(k^n)$, i.e. the group of linear, not necessarily homogeneous, transformations of n-dimensional space. It maps canonically onto $\mathbf{GL}_n(k)$, and the inverse image of $\mathbf{SL}_n(k)$ is, for Lie, the special linear group (in n variables).

If V is a finite dimensional vector space over k and no basis of V is specified, we let $GL(V)$ be the group of (homogeneous) linear transformations of V.

2. The group $\mathbf{PGL}_n(k)$ is, by definition, the quotient $\mathbf{GL}_n(k)/k^*$ of $\mathbf{GL}_n(k)$ by its center, the group of dilations $x \mapsto k.x$ $(k \in k^*)$. As usual, it may be identified with the group $\mathrm{Aut}(\mathbf{P}_{n-1}(k))$ of projective transformations of $(n-1)$-dimensional projective space $\mathbf{P}_{n-1}(k)$. For $k = \mathbb{C}$, it is called by Lie the general projective group.

3. For k of characteristic not two, $\mathbf{O}_n(k)$ is the subgroup of $\mathbf{GL}_n(k)$ leaving the unit quadratic form $\sum_i x_i^2$ invariant, and $\mathbf{SO}_n(k)$ is its subgroup of elements of determinant one.

Let $k = \mathbb{C}$. Lie usually considers the image $\mathbf{PO}_n(k)$ of $\mathbf{O}_n(k)$ in $\mathbf{PGL}_n(k)$, viewed as the group of projective transformations leaving the standard non-degenerate hyperquadric invariant. For $n = 3$, it is the group of projective transformations leaving a non-degenerate conic invariant and was for some time called the "conic section group" (Kegelschnitt Gruppe).

If F is a non-degenerate quadratic form on k^n, then $O(F)$ denotes the subgroup of $\mathbf{GL}_n(k)$ leaving F invariant, and $SO(F)$ the subgroup of elements of determinant one in $O(F)$.

4. The symplectic group $\mathbf{Sp}_{2n}(k)$ is the subgroup of $\mathbf{GL}_{2n}(k)$ leaving invariant the standard non-degenerate antisymmetric bilinear form

$$(x, y) = \sum_1^n (x_i.y_{n+i} - x_{n+i} \cdot y_i). \tag{1}$$

This terminology was introduced by H. Weyl. For $k = \mathbb{C}$, Lie calls the image $\mathbf{PSp}_{2n}(k)$ of $\mathbf{Sp}_{2n}(k)$ in $\mathrm{Aut}(\mathbf{P}_{2n-1}(k))$ the group of a non-degenerate linear complex.

Similarly, if J is a non-degenerate antisymmetric bilinear form on k^{2n}, the subgroup of $\mathbf{GL}_{2n}(k)$ leaving J invariant will be denoted $Sp(J)$. Since J can always be put in the form (1) by a linear transformation, $Sp(J)$ is conjugate to $\mathbf{Sp}_{2n}(k)$ within $\mathbf{GL}_{2n}(k)$.

5. As usual,

$$\mathbf{U}_n = \{X \in \mathbf{GL}_n(\mathbb{C}), X.{}^t\bar{X} = 1\}, \qquad \mathbf{SU}_n = \{X \in \mathbf{SL}_n(\mathbb{C}), X.{}^t\bar{X} = 1\}$$

are the *unitary* and *special unitary* groups in n variables.

6. Let me also recall some standard notation in group theory, also to be used often without further reference.

Let G be a group, A a subset. Then the normalizer $\mathcal{N}A$ or $\mathcal{N}_G A$ and centralizer $\mathcal{Z}A$ or $\mathcal{Z}_G A$ of A in G are defined by

$$\mathcal{N}_G A = \mathcal{N}A = \{g \in G | g.A.g^{-1} = A\},$$
$$\mathcal{Z}_G A = \mathcal{Z}A = \{g \in G | g.a = a.g\ (a \in A)\}.$$

The inner automorphism $x \mapsto g.x.g^{-1}$ $(x \in G)$ is denoted by i_g, and its effect on x is sometimes written ${}^g x$.

If G is a Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{a}$ is a subset of $\mathfrak{g}$, then similarly

$$\mathfrak{z}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \{X \in \mathfrak{g} |\ [X, \mathfrak{a}] = 0\}$$

and

$$\mathcal{N}(\mathfrak{a}) = \mathcal{N}_G(\mathfrak{a}) = \{g \in G | \mathrm{Ad}\, g(\mathfrak{a}) = \mathfrak{a}\},$$
$$\mathcal{Z}\mathfrak{a} = \mathcal{Z}_G(\mathfrak{a}) = \{g \in G | .\mathrm{Ad}\, g(X) = X\ (X \in \mathfrak{a})\},$$

where Ad refers to the adjoint representation, which associates to g the differential at the identity of i_g.

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Name Index

Araki, S., IV, 2.5.3[13)] (78)
Aronhold, S., V, 2,1, 4.17 (97, 112)
Artin, M., III, 13 (44)

Barsotti, I., VI, 2.5 (123)
Bergmann, S., IV, 6.1 (88)
Bochner, S., III, 8 (38)
Bohr, H., III, 8 (38)
Borel, A., VI, 3.2; 4.4, 4.6; VII, 5[5)],6 (124, 128, 129, 154)
Bourbaki, N., II, 10 (17)
Brauer, R., II, 10; III, 9, 12; IV, 5.3; V, 4.17; VII, 5 (17, 39, 43, 87, 113, 153)
Brown, K., VI, 7.3 (143)
Bruhat, F., VI, 2.3, 7.3; VII, 14[9)] (121, 143, 161)
Buium, A., VIII, 10.7 (178)
Burnside, W., II, 7 (15)

Capelli, A., I, 2.2; III, 11 (6, 42)
Carda, K., V, §6 (115–116)
Cassidy, P.J., VIII, 10.7 (178)
Cartan, É., I, 2.1, 2.4, 3.3; II, 3, 7, 12; III, 1, 2, 4, 6, 7; IV; V, §5; VI, 4.8, 6.3; VII, 4, 11 (5–7, 10, 15, 20; III–IV almost *passim*; 114, 130, 136, 150, 157)
Cartan, H., IV, 6.1; VII, 5 (87, 153)
Casimir, H.B.G., II, 8; III, 9 (16, 39)
Chevalley, C., I, 2.1; II, 9; VI, 2.3, 2.5, 3.2, 3.3, 3.5, 4.4, 6.5; VII (5, 17; VI–VII almost *passim*)
Chow, W.L., VI, 6.9 (141)
Clebsch, A., V, 2.4 (99)
de Concini, C., III, 13; IV, 4.5 (44, 84)

Demazure, M., VII, 13 (160)
Deruyts, J., II, 14 (24)
Dickson, L., VII, 11 (157)

Ehresmann, C., IV, 5.4; VI, 2.3; VII, 1 (87, 121, 147)
Engel, F.,I, §1; II, 2; III, 4[8)]; V, §1, 4.6-7, 5.2 (1, 10, 49, 94–96, 106–107, 114)

Fano, G., II, 3, 13 (11, 21–24)
Freudenthal, H., III, 8; VI, 6.5, 6.8 (38, 137–141)
Frobenius, G., V, 4.0, 4.7 (102, 107)

Gantmacher, F., IV, 2.5.3[13)] (78)
Garrett, P., VI, 7.3 (143)
Gordan, P., II, 6; V, 1.3, 2.1–2.3, 4.15, 4.17 (14, 96–98, 111–113)
Goto, M., VI, 1.2, 2.3 (119, 121)
Green, J.A., II, 14 (24)
Grosshans, F., V, 4.17 (114)
Grothendieck, A., VI, 4.4 (127)

Haboush, W., II, 11; III, 13 (18, 44)
Hadžiev, Dž., V, 4.17 (113)
Harish-Chandra I, 2.1; II, 9; III, 8; IV, 6.3[21)]; VI, 2.3; VII, 7 (5, 17, 38, 88, 121, 155)
Helgason, S., IV, 4.5 (84)
Hilbert, D., II, 6; V, 4.17 (14, 112)
Hopf, H., IV, 2.5.2[11)]; VI, 3.2; VII, 5 (76, 124, 153)
Howe, R., III, 14 (45)
Humt, G., IV, 2.4.2[6)] (69)
Hurwitz, A., I, 3.3; II, 5, 6; III, 3, 4, 10 (7, 12–15, 33–34, 40)

Iwahori, N., VI, 7.3 (143)
Iwasawa, K., VII, 1 (147)

Jacobson, N., III, 9; VI, 4.4 (39, 128)

Killing, W., I, 2.1; III, 4; V, 5.2 (5, 34, 114)
Klein, F., I, 1.3; II, 4; VI, 5.1 (4, 11, 131)
Kolchin, E.R., VI, 2.1, 2.6, 3.2; VIII (120, 123, 124; VIII *passim*)
Koszul, J.-L., VII, 5 (153)

Lang, S., VI, 4.3 (127)
Lardy, P., IV, 2.5.3[13)] (78)
Lichnerowicz, A., IV, 6.3[20)] (88)
Lie, S., I, §1; V, 1.1, 1.4; VI, 5.1; VIII, 1 (1–4, 93, 96, 131, 165)
Loewy, A., II, 4 (11)

Malcev, A., VII, 1 (147)
Maschke, H., II, 4 (11-12)
Matsumoto, H., VI, 7.3(143)
Matsushima, Y., VI, 1.2 (119)
Maurer, L., V, §4; VI, 1.1, 2.3; VII, 4, 10[6)]; VIII, 3 (102–113, 119, 121, 150, 156, 169)
Moore, E.H., II, 4 (11)

Morozov, V.V., VI, 2.3; VII, 12 (121, 158)
Mostow, G.D., IV, 3.3[14]; VII, 12 (79, 158)
Mumford, D., II, 11; III, 13 (18, 44)

Nagata, M., III, 13; V, 4.17 (44, 113)
von Neumann, J., III, 8; IV, 3.3; VI, 6.5 (38, 79, 137)

Ono, T., VII, 11 (158)

Peter, F., III, 7 (37)
Piatetski-Shapiro, I., IV, 6.4[22] (89)
Picard, E., V, §3, 5.1; VII, 4, 10; VIII, 2 (99–102, 114, 150, 156, 166)
Plancherel, M., III, 1 (31)
Poincaré, H., I, 2.2; II, 11; IV, 5.1 (6, 18, 84)
Pontrjagin, L.S., IV, 5.3 (87)
Procesi, C., I, 2.2; III, 13; IV, 4.5 (6, 44, 84)

Racah, G., II, 9 (17)
Raševskiĭ, P.K., II, 10 (17)
de Rham, G., IV, 5.2[17] (84)
Rinow, W., IV, 2.5.2[11] (76)
Ritt, J.F., VIII, 3 (168)
Ronan, M., VI, 7.3 (143)
Rosenlicht, M., VI, 2.5, 4.4, 4.5 (123, 127, 129)

Samelson, H., VI, 3.2; VII, 5 (124, 153)
Satake, I., VI, 4.6, 4.8 (129, 130)
Schafer, R.D., VII, 11 (157)
Schiffer, M., II, 6 (15)
Schmid, W., I, 2.2 (6)
Schmidt, E., III, 7 (37)
Schouten, J.A., IV, intr, 1.5 (59, 63)
Schreier, O., VI, 6.1 (134)
Schubert, H., IV, 5.4[19] (87)
Schur, F., IV, 3.2 (79)
Schur, I., I, 3.3, II, 7; III, 2, 3, 4, 10, 12 (7, 15, 32–35, 40, 43)
Schwarz, H.A., V, 6.2 (116)
Serre, J-P., VI, 4.3 (127)
Seshadri, C.S., II, 11; III, 14; V, 4.17 (18, 44–45, 113)
Springer, T.A., IV, 4.5; VI, 4.4 (84, 128)
Steinberg, R., V, 4.17; VI, 6.8 (113, 141)
Stiefel, E., VI, 3.2 (124)
Study, E., I, 3.2; III, 1, 11; V, §1, 4.17 (7, 31, 42, 93–99, 112)
Sylvester, J.J., V, 4.17 (112)

Tits, J., VI, 4.6, 4.8, 5.1, 7.3; VII, 14[9] 129–131, 143, 161)
Tuan, H.F., VI, 1.2; VII, 4 (119, 150)

Verma, D., VII, 14[9] (161)
Vust, T., IV, 4.5 (84)

van der Waerden, B.L., II, 9; III, 9; VI, 6.1, 6.3, 6.5 (17, 39, 134–137, 139)
Weil, A., III, 14; IV, 5.0; VI, 4.8; VII, 1, 5[5] (45, 84, 130, 147, 154)
Weisfeiler, B., VI, 6.10 (142)
Weitzenböck, R., V, 4.17 (112–113)
Weyl, H., I, 2.2, 2.4, 3.3; III; V, 2.4, 4.17; VII, 3, 5, 11 (6, 7; III *passim*; 99, 113, 149, 153, 157)
Whitehead, J.H.C., II, 10; VII, 5 (17, 153)

Yamabe, H., VI, 6.5 (138)
Yen Chih Ta, VII, 5 (154)

Subject Index

abelian variety VI, 2.5 (122)
absolutely simple group VI, 6.5 (137)
algebraic group VI ff.
  linear V, 3.2, 4.1 (101–103)
algebraic torus V, 4.11; VI, 3.2 (108, 124)
angular parameter IV, 2,2, 2.4.2 (65, 69)
anisotropic quadratic form VI, 6.6 (139)
apartment VI, 5.2 (132)

Barsotti-Chevalley theorem VI, 2.6; VIII, 10.5 (123, 177)
base change VI, 6.8 (140)
Borel subalgebra VI, 2.2 (120)
Borel subgroups VII, 12 (158)
Bruhat decomposition VI, 2.3 (121–122)

Campbell-Hausdorff formula I, 2.3 (6)
Cartan
  decomposition
    of Lie group IV, 2.4.4, 2.4.5 (71–72)
    of semisimple Lie algebra IV, 1.4 (62)
  polyhedron IV, 2.2 (65)
  subalgebra IV, 2.4.2; VII, 2, 9 (69, 148, 155)
  subgroup VI, 3.2; VII, 9 (124, 156)
Casimir operator II, 9 (17)
Cayley's operator III, 11 (42)
Clebsch-Gordan series V, 2.4 (99)
complex of flags VI, 5.3 (132)
Coxeter complex VI, 5.4 (133)

diagram VI, 3.3 (125)
differential field
  of constants VIII, 5.1 (170)
  ordinary VIII, 5.1 (170)
  partial VIII, 5.1 (170)
  universal VIII, 8.1 (173)
dual Weyl module VI, 2.4 (122)

extension
  of a differential field VIII, 5.2 (170)
  Picard-Vessiot VIII, 5.3 (171)
  strongly normal VIII, 8.2 (174)
extremal involution VI, 6.6 (139)
extremal pair of involutions VI, 6.6 (139)

finite and continuous group
  in Cartan's sense IV, 3.1 (79)
  In Lie's sense I, 1.3 (4)
flat IV, 2.4.2; VI, 7.1 (68, 142)
FTPG VI, 6.1 (134)

Galois group
  of a P.V.-extension VIII, 5.4 (171)
  of a strongly normal extension VIII, 8.3 (174)
Gleichzusammensetzung I, 1.3 (4)

Haar measure III, 8 (38)
Heisenberg commutation relations III, 9 (38)
Helmholtz-Lie problem III, 1 (29)

invariant problem II, 1 (9)
isogeny
  purely inseparable of height one VI, 4.4 (128)
  special VI, 3.3 (126)
isotropic
  quadratic form VI, 6.6 (139)
  subspace VI, 5.3 (132)

Jacobi identity I, 1.2 (3)

k-closed VI, 4.1 (126)
k-form VI, 4.8 (130)
k-variety VI, 4.1 (126)
Killing form I, 2.1 (5)

Lie algebra
  absolutely simple (over $\mathbb{R}$) IV, 2.4 (67)
  of $\mathbf{SL}_2(\mathbb{C})$ II (9–25)
    irreducible representations of II, 2 (10)
    linear representations of II, 1 (9)
    proof of full reducibility of representations of
      by Cartan II, 12 (20–21)
      by Casimir II, 8 (16–17)
      by Fano II, 13 (21–24)
      by Schur II, 14 (24–25)
      by Study II, 2 (10)
  of $\mathbf{SO}_3$ II, 8 (16)
  semisimple I, 2.1 (5)
  unitary IV, 1.4 (62)

Lie-Kolchin theorem VIII, 4.1 (169)
linear representation II, 1 (9)
  full reducibility II (9–25)
  holomorphic II, 1 (10)
  of class one IV, 4.4 (83)
  of semisimple Lie algebras III, 3, 4 (33–34)
linear transformation
  irregular V, 4.8 (107)
  regular of first or second kind V, 4.8 (107)

Maurer-Cartan equations V, 4.7 (107)
mixed isotropy group IV, 2.2 (66)
moment II, 8 (16)
  of momentum II, 8 (16)

normal form (Study) V, 1.3 (96)
normal isomorphism (Cartan) IV, 1.4 (62)

one-parameter group I, 1.1 (1)
order (subvariety in $\mathbf{P}_n(\mathbb{C})$) V, 1.1[1)] (93)

parabolic subgroup VI, 2.2 (121)
parameter group I, 1.3 (4)
Peter-Weyl theorem III, 8 (38)
Picard-Vessiot theory VIII, 2 (167)
Poincaré-Birkhoff-Witt theorem I, 2.2 (6)
Poisson bracket I, 1.2 (3)
P.V. extension VIII, 5.3 (171)

rational normal curve II, 2 (10)
rational normal scroll II, 13.2 (24)
real form of a Lie algebra I, 2.4 (6)
regular group (Maurer) V, 4.7 (107)
Ricci form IV, 1.3 (62)
root VI, 2.2, 3.3, 3.4 (120, 125, 126)
root system
  relative VI, 4.6 (129)
  restricted IV, 2.4.1 (68)

Schubert cell VII, 13 (160)
space $\mathcal{E}$ IV, 1.1 (60)
split solvable group VI, 4.5 (128–129)
symmetric space IV (59)
  hermitian IV, 6.6 (89)
  Riemannian IV, 3.4 (79)
  semisimple IV, 1.5 (64)
symmetry
  global IV, 1.1 (61)
  local IV, 1.1 (61)
syzygy II, 1 (9)

Tits geometry VI, 5.1 (131)
Tits system VI, §5 (131–134)
transvection IV, 1.2 (61)

unitarian trick II, 5 (12–13)
universal enveloping algebra I, 2.2 (5–6)

weight VI, 2.2 (121)
  dominant VI, 2.2 (121)
Weyl group III, 4; IV, 2.2 (34, 65)
  relative VI, 4.6, 5.4 (129, 133)
  restricted IV, 2.4.1 (68)
Weyl's character formula III, 4(8) (35)
Weyl's degree formula III, 4(9) (35)

zonal function IV 4.1 (80)

Essays in the History of Lie Groups and Algebraic Groups

Armand Borel

Lie groups and algebraic groups are important in many major areas of mathematics and mathematical physics. We find them in diverse roles, notably as groups of automorphisms of geometric structures, as symmetries of differential systems, or as basic tools in the theory of automorphic forms. The author looks at their development, highlighting the evolution from the almost purely local theory at the start to the global theory that we know today. Starting from Lie's theory of local analytic transformation groups and early work on Lie algebras, he follows the process of globalization in its two main frameworks: differential geometry and topology on one hand, algebraic geometry on the other. Chapters II to IV are devoted to the former, Chapters V to VIII, to the latter.

The essays in the first part of the book survey various proofs of the full reducibility of linear representations of $\mathbf{SL}_2(\mathbb{C})$, the contributions of H. Weyl to representations and invariant theory for semisimple Lie groups, and conclude with a chapter on E. Cartan's theory of symmetric spaces and Lie groups in the large.

The second part of the book first outlines various contributions to linear algebraic groups in the 19th century, due mainly to E. Study, E. Picard, and above all, L. Maurer. After being abandoned for nearly fifty years, the theory was revived by C. Chevalley and E. Kolchin, and then further developed by many others. This is the focus of Chapter VI. The book concludes with two chapters on the work of Chevalley on Lie groups and Lie algebras and of Kolchin on algebraic groups and the Galois theory of differential fields, which put their contributions to algebraic groups in a broader context.

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