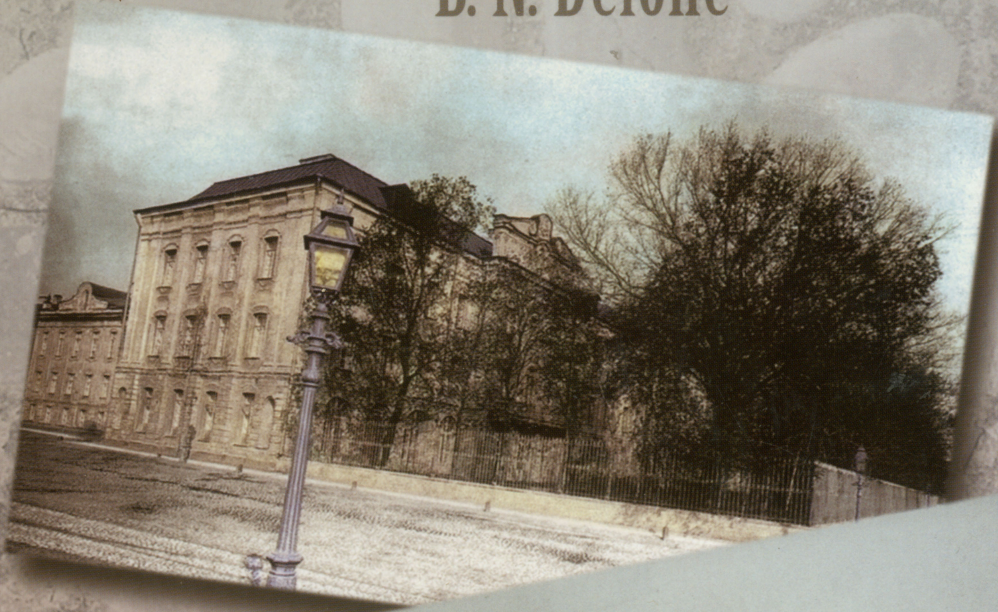
HISTORY OF MATHEMATICS • VOLUME 26

The St. Petersburg School of Number Theory

B. N. Delone



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Translated by Robert Burns

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Б. Н. ДЕЛОНЕ

ПЕТЕРБУРГСКАЯ ШКОЛА ТЕОРИИ ЧИСЕЛ

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Foreword to the English Edition

"A series of Russian mathematicians—Chebyshev, Korkin, Zolotarev, Markov, Voronoĭ, and others—have worked in the theory of numbers. One can become acquainted with the content of the classical work of these notable mathematicians in B. N. Delone's book "The St. Petersburg School of Number Theory"." So begins the Preface to the book "Elements of Number Theory" by I. M. Vinogradov. Vinogradov does not inform us that he himself is the subject of the longest and most detailed chapter of Delone's book. These mathematicians, whose life and work in number theory form the substance of this work, are indeed a very distinguished group. Their work is of the highest quality and of lasting significance.

Delone's book was published, in Russian, in 1947. The present volume is a translation of the original into English. The text is essentially unaltered, but footnotes are added to clarify the exposition and in some cases to give more up to date references. For each of the six mathematicians considered, a brief biography is provided followed by an exposition of several of their most important contributions to number theory. In all cases, the mathematical exposition is much longer and more detailed than the biography. In spite of that, we get a good sense of their professional lives. We also get to see how their individual work is interrelated with the work of the others.

It should be emphasized that this book is strictly about number theory. For example, Chebyshev and Markov made very important contributions to the theory of probability, but no mathematical discussion of their work on probability is presented. Another point to make to the potential reader is that the mathematical discussion is not superficial. Delone himself was a distinguished member of the St. Petersburg school of number theory, and was well versed in the work of his predecessors. His exposition is at times taxing, but the effort expended in following the argument is amply rewarded. This is beautiful material.

One final comment before providing an overview of the mathematical content on this book. In 1947, the Cold War was just getting under way in earnest. Perhaps because of this historical fact, Delone's treatment is marred somewhat by an excessive nationalistic pride. While this is grating on modern sensibilities, it should not interfere with one's enjoyment while reading this book. After all, the mathematical works of these authors are well worth being proud of, and the Soviet era Russians are not the only ones who have been guilty of this fault.

The book begins with a consideration of P. L. Chebyshev and his contributions to number theory. Chebyshev was a first rate mathematician who contributed important results in a variety of fields. His main work in number theory concerns the distribution of prime numbers. For a positive real number x, Chebyshev denotes by $\phi(x)$ the number of positive prime numbers which are less than or equal to x. In modern notation his function is called $\pi(x)$, but we will follow Chebyshev. Legendre

proposed the approximation

$$\phi(x) \approx \frac{x}{\ln(x) - 1.08366}.$$

This agrees well with tables of prime numbers, but in his first memoir on the subject, Chebyshev shows that it cannot be true. He shows that, in fact, the best approximation (in a certain precise sense) to $\phi(x)$ is

$$Li(x) = \int_2^x \frac{dt}{\ln(t)} \sim \frac{x}{\ln(x)}.$$

As usual, $f(t) \sim g(t)$ means that $f(t)/g(t)$ tends to 1 as t tends to infinity. In the proofs, Chebyshev uses properties of the zeta function, $\zeta(s)$, as defined by Euler. Namely, he uses the product formula and the fact that $(s-1)\zeta(s) \to 1$ as $s \to 1$ from above.

In his second memoir on the subject of prime numbers, Chebyshev goes much further and comes within striking distance of the prime number theorem as proposed by Gauss, namely,

$$\phi(x) \sim \int_2^x \frac{dt}{\ln(t)}.$$

In this work, Chebyshev introduces the auxilliary functions

$$\theta(x) = \sum_{p \le x} \ln(p) \quad \text{and} \quad \psi(x) = \sum_{p^m \le x} \ln(p),$$

which have played a key role in the literature on the distribution of primes ever since. Perhaps the main result of this paper is

$$.92129 \frac{x}{\ln(x)} < \phi(x) < 1.10555 \frac{x}{\ln(x)}.$$

A consequence of this is a proof of Bertrand's hypothesis, which states that for all $n > 1$, there is a prime number between n and $2n$.

Chebyshev's work on prime numbers was an absolutely major step forward in our knowledge of how prime numbers are distributed. In spite of this, it took almost fifty more years before the first proofs of the prime number theorem were given by J. Hadamard and, independently, by Ch.-J. de la Vallé Poussin.

We now pass on to the work of later members of the St. Petersburg circle. There is a problem in the theory of quadratic forms that provides a unifying thread in the work of Korkin, Zolotarev, Markov, and Voronoĭ, so it is a good idea to begin by explaining the problem. We will work in Euclidean space $\mathbb{R}^n$, which we can think of as being given by column vectors of real numbers with the usual inner product $(x, y) = \sum x_i y_i$. A quadratic form is a homogeneous polynomial of degree 2, usually written $Q(x) = \sum\sum a_{ij} x_i x_j$. We require the coefficients to be real numbers and the symmetry condition $a_{ij} = a_{ji}$. The quantity $D = \det(a_{ij})$ is called the discriminant of Q. If $Q(x) \ge 0$ for all x, with $Q(x) = 0$ if and only if $x = 0$, we say that Q is positive definite. Let $G = \mathbb{Z}^n \subset \mathbb{R}^n$ denote the integer lattice. Set $\mu(Q)$ equal to the minimum value of $Q(\gamma)$ as γ runs through the nonzero elements of G. C. Hermite was able to show that the quantity $\mu(Q)/\sqrt[n]{D}$ was bounded by some constant depending only on the dimension n. The smallest value of this constant is sometimes called Hermite's constant for dimension n, and the problem is to determine its value for each $n > 1$. The answer for $n = 2$ is $2/\sqrt{3}$. For $n = 3$, the answer is $\sqrt[3]{2}$. On the basis of this (and perhaps more), Hermite

conjectured that the answer in general is $2/\sqrt[n]{n+1}$. This was disproved by Korkin and Zolotarev in a short paper (1872) where they show that for $n = 4$ the answer is $\sqrt[4]{4}$. Delone gives a fairly complete overview of this paper and its successors.

The problem referred to in the last paragraph can be reformulated as a problem about the smallest vector in a lattice. This problem, in turn, is closely related to the problem of optimal sphere packing in $\mathbb{R}^n$. In order to orient the reader to other parts of Delone's exposition, it is worthwhile to briefly explain these connections. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of $\mathbb{R}^n$. A lattice, L, in $\mathbb{R}^n$ is the free abelian group generated by a basis, i.e.

$$L = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_n.$$

Among all the nonzero lattice vectors λ, it is easy to see that there will be one (not necessarily unique), λ_o, such that $\|\lambda_o\| = \sqrt{(\lambda_o, \lambda_o)}$ is least. If $\lambda_o = \sum n_i v_i$, then $(\lambda_o, \lambda_o) = \sum\sum(v_i, v_j)n_i n_j$, so that $\|\lambda_o\|^2$ is the least value of the quadratic form $Q(x) = \sum\sum(v_i, v_j)x_i x_j$ evaluated at nonzero elements of $\mathbb{Z}^n$.

Conversely, if a positive definite quadratic form $Q(x) = \sum\sum a_{ij}x_i x_j$ is given, set $A = (a_{ij})$. We can find a nonsingular matrix B such that $B^t(a_{ij})B = I$, the identity matrix. Let $C = B^{-1}$. Then $A = C^t C$. Let v_i be the ith column vector of C. From $C^t C = I$ it follows immediately that $(v_i, v_j) = a_{ij}$. Let L be the lattice generated by the vectors $\{v_1, v_2, \ldots, v_n\}$. Then, the values of $Q(x)$ at elements of $\mathbb{Z}^n$ coincides with the values $\|\lambda\|^2$ as λ varies through L. We have shown that the problem of finding the smallest value of $Q(x)$ at integer points coincides with the problem of finding vectors of least length in a lattice.

The lattice sphere packing problem has to do with placing nonoverlapping spheres of fixed radius at each lattice point and determining the proportion of space taken up by the spheres. The idea is to find the best possible lattice packing. For a fixed lattice, the maximal radius of the spheres to be used is clearly equal to half the length of the smallest nonzero element of the lattice. People also consider sphere packings of space for which the centers of the spheres do not necessarily fill out a lattice. This problem may seem special, but it has wide applications throughout mathematics. It is interesting to note that it is mentioned by Hilbert in his famous list of problems (1899). It is discussed toward the end of problem 18. The most comprehensive modern source on sphere packing is the 1988 book by J. H. Conway and N. J. A. Sloane [**1**]. A shorter, and perhaps more readable, treatment is given in the book of C. Zong [**11**]. Both books refer to the work of Korkin, Zolotarev, and Voronoĭ.

In another direction, it is interesting to note that the work of Korkin and Zolotarev has found applications to lattice reduction algorithms which are of practical importance in cryptography. A famous 1990 paper [**7**] of J. C. Lagarias, H. W. Lenstra, and C. P. Schnorr is the genesis of the LLL-BKZ algorithm. The last two letters in this acronym stand for Korkin and Zolotarev, a tribute to how their fundamental work continues to find new applications.

Delone spends a lot of time on another aspect of Zolotarev's work. In two papers, one published in 1874, the other in 1880, Zolotarev establishes a general theory of algebraic integers and prime decomposition. The work is somewhat later than Dedekind's famous Supplement XI to Dirichlet's book "Vorlesungen über Zahlentheorie", in which he establishes a general theory of algebraic integers based on the new notion of ideals as certain subsets of the ring of integers. Zolotarev's ideas are more in keeping with Kummer's treatment of the ring of cyclotomic integers using

the notion of ideal numbers. In his first paper on this subject, he assumes that the ring of integers is of the form $\mathbb{Z}[\xi]$. If $f(x) \in \mathbb{Z}[x]$ is the irreducible polynomial for ξ, the prime decomposition of a rational prime p is related to the decomposition of the polynomial $\bar{f}(x) \in \mathbb{Z}/p\mathbb{Z}\,[x]$. In general, the ring of integers in a number field is not of the form $\mathbb{Z}[\xi]$. Zolotarev overcomes this difficulty in his (posthumous) paper of 1880. It is a sad fact that in 1878, at the age of 31, Zolotarev was fatally injured in a train accident and died shortly thereafter.

Up to now, we have been concerned with positive definite quadratic forms. Markov discussed in detail the theory of indefinite binary quadratic forms. These behave quite differently, but Markov was able to give a rather complete theory which connects the theory of indefinite binary quadratic forms in a nontrivial way with the theory of continued fractions. As a by-product of his investigation he is able to give a procedure for finding all integer solutions to the diophantine equation $x^2+y^2+z^2 = 3xyz$. Today, this is called Markov's equation in his honor. Delone gives a careful discussion of Markov's work on binary quadratic forms. In addition he reformulates much of it in geometric terms. The reader will find this reformulation to be a valuable aid to understanding Markov's analysis.

Although it is not directly related to the work described in the present book, it is worth giving the statement of the Oppenheim conjecture, which concerns indefinite quadratic forms $Q(x)$ in three or more variables. This states that unless $Q(x)$ is proportional to a rational form, i.e. one with rational coefficients, the values of $Q(x)$ on the integer lattice $\mathbb{Z}^n$ are dense in $\mathbb{R}$. The Russian mathematician G. Margulis (now at Yale University) proved this conjecture in 1986. Although the statement of the conjecture is certainly in the spirit of the work of the classical mathematicians covered in the present book, the methods of proof are very different.

Voronoĭ made important contributions to many aspects of number theory. While still in graduate school he published a paper on Bernoulli numbers which simultaneously found new results about their arithmetic properties and gave a new method of deriving some of their known properties. These are based on some beautiful congruences which today bear Voronoĭ's name. A complete treatment is given in the text of Uspensky and Heaslet [**8**]. This approach is used in the book of Ireland and Rosen [**6**], which also shows how these congruences are related to the construction of p-adic L-functions which were first defined and explored by H. W. Leopoldt in 1975.

In another direction, Voronoĭ made an interesting contribution to analytic number theory. Let $\tau(n)$ denote the number of positive divisors of a positive integer n. Dirichlet was able to show that

$$\sum_{n,x} \tau(n) = n\ln(n) + (2\gamma - 1)n + O(\sqrt{n}).$$

Here, γ denotes Euler's constant. It has been a big problem over the years to improve the error term in this formula. Voronoĭ gave the first significant improvement by showing that the error term was $O(\sqrt[3]{n}\ln(n))$. One of the first achievements of I. M. Vinogradov, whom we will consider later, was to significantly generalize Voronoĭ's work. It is conjectured that the error term is $O(n^{1/4+\epsilon})$, which would follow from the Riemann hypothesis. On the other hand, it is known that $O(n^{1/4})$ won't do.

A further important contribution of Voronoĭ was his thorough investigation of cubic number fields. In all cases he was able to construct an explicit integral basis of

the ring of integers of such a field and to discuss in detail the prime decomposition of rational primes. Perhaps his main result is to give an efficient algorithm for finding a set of fundamental units. Here, he takes as his model the continued fraction algorithm for finding the fundamental unit in a real quadratic number field. Others have given algorithms for finding independent units, but Voronoĭ's is both more efficient and has the advantage of always leading to a set of fundamental units. This work is covered at length in the present book. A more elaborate presentation is given in the book "The theory of irrationalities of the third degree" by B. N. Delone and D. K. Faddeev [**3**].

The last topic concerning Voronoĭ's contributions to number theory which is covered by Delone is his work on quadratic forms and lattices. Voronoĭ introduced important new concepts, e.g., perfect quadratic forms and Voronoĭ cells of a lattice. We will not go into the notion of a perfect quadratic form in this introduction, but will briefly describe what is meant by a Voronoĭ cell of a lattice. Let $L \subset \mathbb{R}^n$ be a lattice. For each $\lambda \in L$ let $V(\lambda)$ be the set of points x in $\mathbb{R}^n$ such that x is closer to λ than to any other lattice point in L. It is not hard to see that $V(\lambda)$ is a convex polytope. It is called a *Voronoĭ cell* of the lattice. The Voronoĭ cells are disjoint and the union of their closures fills all of $\mathbb{R}^n$. This decomposition is discussed in detail in the book by Conway and Sloane [**1**]. Another good place to read about Voronoĭ cells is the charming book by Conway entitled "The sensual (quadratic) form" [**2**].

Finally, we come to I. M. Vinogradov. Before surveying the chapter on Vinogradov, it should be pointed out that the mathematical exposition was written by B. A. Venkov. In the Preface, Delone refers to Venkov as his coauthor and extends his gratitude for writing the exposition of the work of Vinogradov and two parts of the work of Voronoĭ (on sums of divisors and on perfect quadratic forms). Nevertheless, Venkov's name does not appear on the title page.

The first contribution of Vinogradov which is discussed is his generalization of Voronoĭ's work on the divisor problem. The sum

$$\sum_{n=1}^{N} \tau(n)$$

which is to be estimated, is easily seen to be the number of integral lattice points in the first quadrant under the curve $xy = N$ with x-coordinate between 1 and N. What Vinogradov sets out to do is to estimate the number of integral lattice points in such a region where the top curve $y = N/x$ is replaced by a much more general function $y = f(x)$. With some restrictions on $f(x)$, Vinogradov is able to show that the number of lattice points is equal to the area of the region plus a well controlled error term. The reader should consult the text for details. In the case of Dirichlet's divisor problem, Vinogradov gets an error term of order $\sqrt[3]{N}[\ln(N)]^2$, which is only slightly worse than Voronoĭ's. On the other hand, his result is much more widely applicable.

While this early work clearly shows promise, the next two accomplishments show that the early promise was more than fulfilled by the mature mathematician. Vinogradov made signal contributions to two of the most venerable problems of number theory, Waring's problem and the Goldbach conjecture.

In 1770 E. Waring conjectured that for every $n \geq 2$ there is a number r such that every positive integer is the sum of at most r nth powers, i.e.

$$N = x_1^n + x_2^n + \cdots + x_r^n$$

is solvable in positive integers for every $N \geq 0$. The prototype for such a result is Lagrange's theorem that every positive integer is the sum of four squares.

In 1909, Hilbert proved Waring's conjecture in full generality. This was a great step forward, but there remained the question of finding a good bound on r as a function of n. Hilbert's proof produced values which were much too large. In the early 1920s, Hardy and Littlewood, using their newly invented "circle method" produced the first reasonable bounds. Let us define $G(n)$ to be the smallest integer such that every N with at most finitely many exceptions is the sum of $G(n)$ nth powers. By allowing for a finite number of exceptions, the theory is made much easier.

It is not hard to see that $n+1$ is the lower bound for $G(n)$ (see Chapter 18 of the book "Introduction to number theory" by H. K. Hua [**4**]). The upper bound obtained by Hardy and Littlewood is

$$G(n) \leq (n-2)2^{n-1} + 5.$$

In 1937, Vinogradov was able to prove that for $n \geq 16$

$$G(n) < 6n(\ln(n) + 1).$$

This is a remarkable result. One goes from a bound which is exponential in n to one that is almost linear in n. Vinogradov was not content, and later (1959) he showed that for large n one has

$$G(n) < n(2\ln(n) + 4\ln\ln(n) + 2\ln\ln\ln(n) + 13).$$

This upper bound was superseded, thirty years later, by a result of T. D. Wooley which shows that

$$G(n) < n(\ln(n) + \ln\ln(n) + O(1)).$$

In 1995, Wooley further improved the $O(1)$ term. This seems to be the best result at present. In less precise but simpler terms, we now know that for all large n, $n+1 \leq G(n) < n^{1+\epsilon}$ for any $\epsilon > 0$.

The Goldbach conjecture states that every even number greater than four is the sum of two odd primes and that every odd number greater than seven is a sum of three odd primes. This famous conjecture was first stated in a letter from C. Goldbach to L. Euler in 1742. Once again, the first big step forward was due to Hardy and Littlewood. They showed that almost all even numbers are the sum of two odd primes. This means that $f(m)/m \to 0$ as $m \to \infty$, where $f(m)$ is the number of positive even numbers $\leq m$ which cannot be written as the sum of two odd primes. They also showed that every sufficiently big odd number is the sum of three odd primes provided that the generalized Riemann hypothesis is true. Unfortunately, to this day, neither the Riemann hypothesis nor the generalized Riemann hypothesis has been proved.

This was the situation in 1937 when Vinogradov proved, unconditionally, that every sufficiently large odd number is the sum of three odd primes. Yet another utterly remarkable result!

It is worth noting that neither Vinogradov nor anyone else has been able to resolve the Goldbach conjecture for even numbers. This remains inaccessible at the present time.

Vinogradov's success was due in large part to his ingenious methods for evaluating arithmetic sums, in particular, trigonometric sums. The last part of the chapter on his work is concerned with his estimates of Weyl sums (named after Hermann Weyl and introduced in Weyl's 1916 paper entitled "Über Gleichverteilung von Zahlen mod. Eins"). The sums in question look like

$$\sum_{n=1}^{N} e^{2\pi i f(n)},$$

where $f(x)$ is a polynomial with real coefficients and positive degree.

The reader who would like to see Vinogradov's method exposited in full generality should consult his book "The method of trigonometric sums in the theory of numbers" [**10**]. Another excellent resource is the recent book on analytic number theory by H. Iwaniec and E. Kowalski [**5**].

This brings us to the conclusion of this Foreword. Of course, the St. Petersburg school of number theory did not end in 1947 when the original of the present book was published. As Delone points out in his Preface (somewhat immodestly), "The next most important work of the St. Petersburg number-theoretic school, in the chronological order established above, belongs to the author of the present book, and to Venkov, Kuz'min, Tartakovskiĭ, and Linnik." While these are all estimable mathematicians, it is fair to say that if the volume were to continue beyond 1947, it would have to include Venkov and Linnik and perhaps the others. In any case, by composing the present volume, Delone did a great service to posterity. The new translation should have a wide readership among English speaking mathematicians with enough background to enjoy it.

Michael Rosen
Brown University

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2004 R. G. Burns

Preface

The work of Russian mathematicians in the theory of numbers constitutes a glorious contribution to Russian science. Although mention should perhaps be made in this connection of the brilliant work of the great Leonhard Euler, since he was a member of the Russian Academy of Sciences for a considerable portion of his life[1], in its proper sense the Russian school of number theory begins with Chebyshev. The St. Petersburg school was illumined by names such as that of Chebyshev himself, Korkin, Zolotarev, Markov and Voronoĭ[2]; and at the present time[3] the Soviet school can boast of several excellent number theorists, led by Academician Vinogradov.

The aim of the present book is to acquaint the lover of mathematics with the most important works of the above-named six preeminent members of the St. Petersburg school of number theory. For each of these six a short biography is given, followed by an exposition of two or three of the most significant of his number-theoretical contributions. Each such contribution is first expounded in the author's original terminology and notation, i.e., in the form of a summary, as it were, facilitating the reading of the original work itself, and this is then followed by a more or less broad commentary on it. Certain of the works in question, for instance those of Chebyshev on primes, have proved amenable to a relatively complete exposition, while others, more wide-ranging in nature, are dealt with much more briefly—for instance Zolotarev's dissertation on integral complex numbers.[4]

The next most important work of the St. Petersburg number-theoretic school, in the chronological order established above, belongs to the author of the present book, and to Venkov, Kuz'min, Tartakovskiĭ and Linnik. Outstanding contributions to number theory have also been made by Chebotarev, and by the Moscow mathematicians Khinchin, Shnirel'man and Gel'fond. However considerations of space have made it impossible to include expositions of the work of these authors, let alone the great many other lesser Russian contributions to number theory.

I wish to express my deepest gratitude to my coauthor, Professor Boris Alekseevich Venkov, for writing the exposition of the work of I. M. Vinogradov and of two of Voronoĭ's works—on the sum of the numbers of divisors and on complete forms. In addition I thank both him and Corresponding Member of the Academy P. O. Kuz'min for valuable advice concerning the compilation of material for this book.

1947 B. Delone

[1] 1725–1741 and 1766–1783. *Trans.*

[2] These have stressed syllables as follows: Chebyshóv, Kórkin, Zolotaryóv, Márkov, Voronóĭ. *Trans.*

[3] In 1947. *Trans.*

[4] Algebraic integers. *Trans.*

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A List of Works in Number Theory by Chebyshev, Korkin, Zolotarev, Markov, Voronoĭ, and Vinogradov

P. L. Chebyshev

[1] *The theory of congruences*, book; Russian editions: first, St. Petersburg, 1849; second, St. Petersburg, 1879; third, St. Petersburg, 1901; Translations: *Theorie der Congruenzen*, Berlin, 1888; *Theoria delle congruenze*, Roma, 1895. Also reprinted in the Collected Works of P. L. Chebyshev, published by the Academy of Sciences of the USSR in 1944 (Russian).

[2] *On the determination of the number of primes not exceeding a given number*, first published in 1849 as a third appendix to the book *The theory of congruences* (Russian), and also as a separate article under the title *Sur la fonction qui détermine la totalité des nombres premiers inférieures à une limite donnée*, Mémoires des savants étrangers de l'Acad. Imp. Sci. de St.-Pétersbourg, VI, 1848. Published also under the same title in the Journal de math. pures et appl., I série, XVIII, 1852.

[3] *Mémoire sur les nombres premiers*, Mémoires de l'Acad. Imp. Sci. de St.-Pétersbourg, VII, 1850, and under the same title in the Journal de math. pures et appl., I série, XVII, 1852.

[4] *Sur les formes quadratiques*, Journal de math. pures et appliquées, I série, XVI, 1851.

[5] *Note sur différentes séries*, Journal de math. pures et appl., I série, XVI, 1851.

[6] *On an arithmetical question*, An appendix to Notes of the Imp. Akad. of Sciences, X, No. 4, 1868. (Russian)

[7] *Lettre à M. Fuss sur un nouveau théorème relatif aux nombres premiers contenus dans les formes* $4n+1$ *and* $4n+3$, Bull. de la classe phys.-math. de l'Akad. Imp. Sci. de St.-Pétersbourg, XI, 1853.

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