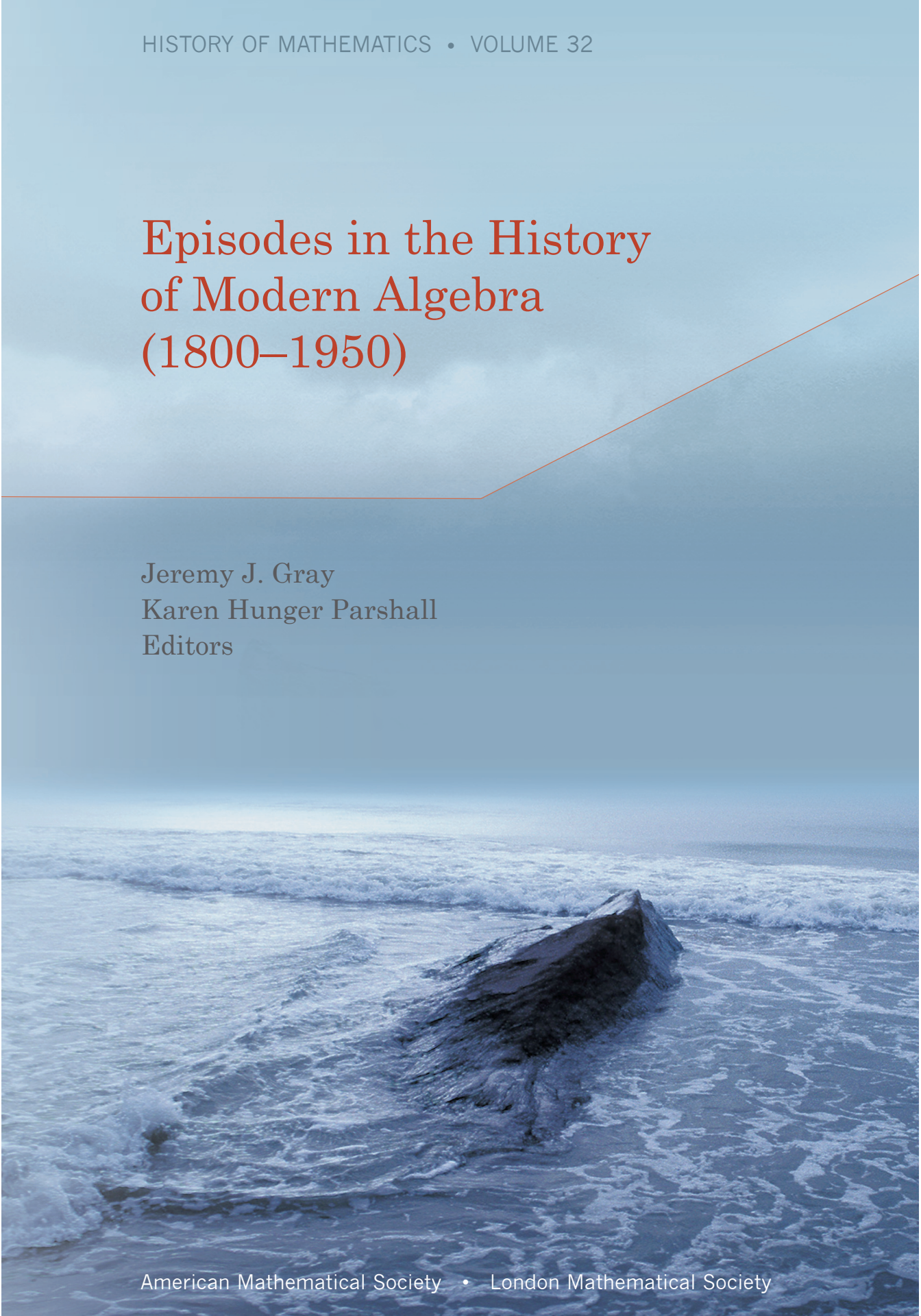
HISTORY OF MATHEMATICS • VOLUME 32

Episodes in the History of Modern Algebra (1800–1950)

Jeremy J. Gray
Karen Hunger Parshall
Editors



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Algebra, as a subdiscipline of mathematics, arguably has a history going back some 4000 years to ancient Mesopotamia. The history, however, of what is recognized today as high school algebra is much shorter, extending back to the sixteenth century, while the history of what practicing mathematicians call "modern algebra" is even shorter still.

The present volume provides a glimpse into the complicated and often convoluted history of this latter conception of algebra by juxtaposing twelve episodes in the evolution of modern algebra from the early nineteenth-century work of Charles Babbage on functional equations to Alexandre Grothendieck's mid-twentieth-century metaphor of a "rising sea" in his categorical approach to algebraic geometry. In addition to considering the technical development of various aspects of algebraic thought, the historians of modern algebra whose work is united in this volume explore such themes as the changing aims and organization of the subject as well as the often complex lines of mathematical communication within and across national boundaries. Among the specific algebraic ideas considered are the concept of divisibility and the introduction of non-commutative algebras into the study of number theory and the emergence of algebraic geometry in the twentieth century.

The resulting volume is essential reading for anyone interested in the history of modern mathematics in general and modern algebra in particular. It will be of particular interest to mathematicians and historians of mathematics.

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