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Vector Fields on Two-Dimensional Surfaces

In a way similar to the case of plane vector fields, one may define vector fields on the sphere. Of course to each point of the sphere we can assign a vector in space. For vector fields in the plane we did not have to worry about the trajectories staying in the plane, this is automatic. But for vector fields on the sphere, this will not happen automatically, we must require the vectors to be tangent to the sphere. Let us say that a *continuous vector field on the sphere* is defined if a vector tangent to the sphere is given at each of its points, and these vectors continuously depend on the point. In the sequel we shall only consider vector fields with a finite number of singular points.

Let us give two examples of vector fields on spheres.

EXAMPLE 1. Consider the rotation of the sphere with some constant angular velocity about an axis passing through the center of the sphere. Then the vector field of the velocities of all the points arises on the sphere (Figure 7.1). This vector field has two singular points, namely the intersection points of the axis of rotation with the sphere. The indices of both singular points are equal to 1.

EXAMPLE 2. It is possible to construct a vector field with only one singular point on the sphere. Such is the vector field whose trajectories are the sections of the sphere by planes passing through some tangent to the sphere (Figure 7.2). The index of the singular point is equal to 2.

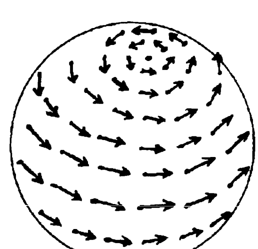


FIGURE 7.1

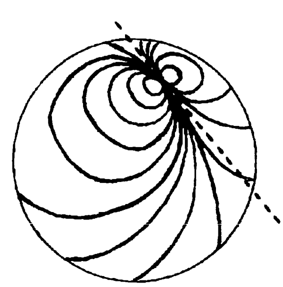
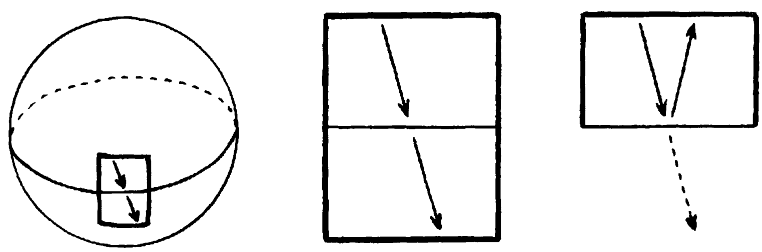


FIGURE 7.2

In both examples the sum of indices of singular points is equal to 2. This is not a coincidence.

THEOREM 7.1. *Suppose that a continuous vector field on the sphere has a finite number of singular points. Then the sum of their indices is equal to* 2.

FIRST PROOF. The sphere may be divided into two parts by a curve not containing any singular points. For our aims, any closed curve not containing singular points will do. But we shall prove a stronger statement, namely that if the number of singular points is finite, then there exists a great circle (i.e., the section of the sphere by a plane containing the sphere's center) that avoids all the singular points. Consider the correspondence $\{A, A^*\} \leftrightarrow a$, where A and A^* are diametrically opposed points of the sphere, and a is the section of the sphere by the plane passing through the center of the sphere perpendicularly to the line AA^*. It is easy to verify that the point A belongs to the plane b (where $b \leftrightarrow \{B, B^*\}$) if and only if the point B belongs to the plane a. Suppose $A_1, \dots, A_n$ are the given (singular) points. Let $a_1, \dots, a_n$ be the corresponding great circles. There exists a point M which does not belong to any of these circles. Then none of the given points lies on the circle m corresponding to M.



(a) (b) (c)

FIGURE 7.3

Now imagine the sphere as a rubber balloon. If we let the air out, we can flatten the balloon so that the two parts into which the circle m divides the sphere coincide with each other, forming a (double) disk. Let us choose some neighborhood of a point of the great circle and see what happens to it. Figure 7.3 shows that on the boundary circle the vectors of the two fields are symmetric with respect to the tangent. It remains to prove the following simple fact, which we state in the form of a problem.

Problem 7.1. Suppose the vector fields v and w have no singular points on the circle S and, at each point X of S, the vectors $v(X)$ and $w(X)$ are symmetric with respect to the tangent. Prove that the sum of indices of the singular points for the fields v and w lying within the circle is equal to 2. □

SECOND PROOF. Take an arbitrary nonsingular point of the given vector field on the sphere and choose a neighborhood of this point so small that all the vectors in it are almost the same (Figure 7.4 (a)). The neighborhood contains no singular points, so we can consider the remaining part of the sphere. This part can be deformed into a disk (Figure 7.4 (b)). We obtain a vector field, which is directed inward on the arcs BC and CD and outward on the arcs DA and AB. The index of the circle with respect to the vector field thus obtained is 2. Indeed, when we move through each of the arcs AB, BC, CD, DA the vector turns by 180°, the direction of rotation coinciding with that chosen to go around the circle. □

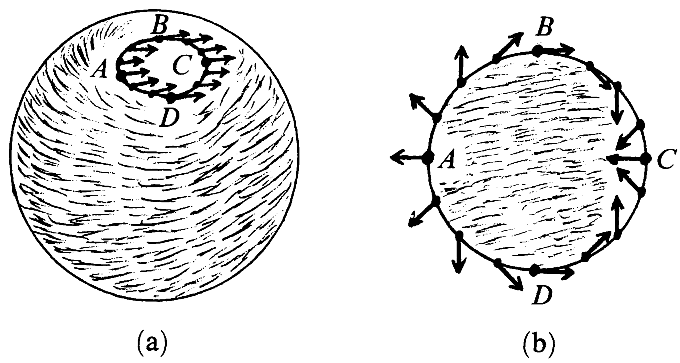


FIGURE 7.4

As an example of the application of Theorem 7.1, let us prove the following *Descartes–Euler Theorem* on convex polyhedra.

THEOREM 7.2. *Let V be the number of vertices of a convex polyhedron, E the number of edges, and F the number of faces. Then $V - E + F = 2$.*

PROOF. Place the polyhedron inside the sphere so that it contains the sphere's center. Project the edges of the polyhedron on the sphere from this center, forming a curvilinear network on its surface. Inside each curvilinear face choose a point and join pairs of such points when they are in neighboring faces by a path passing through the center of their common edge. In Figure 7.5 (a) the new network thus obtained is represented by dotted lines.

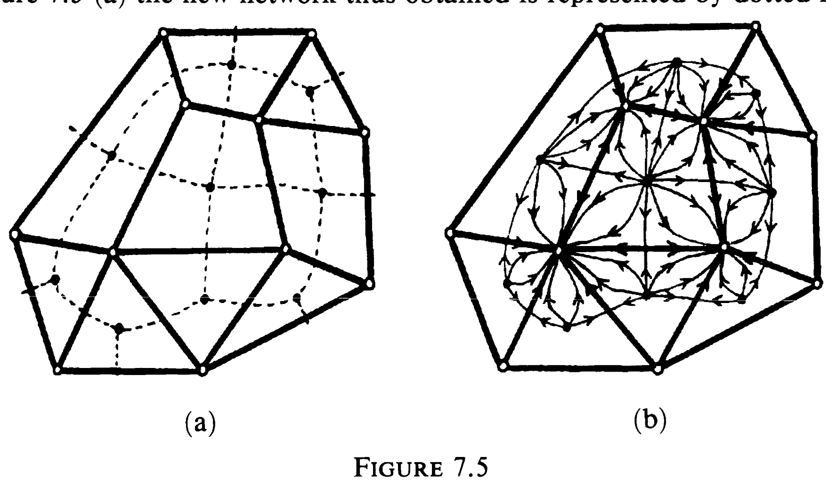
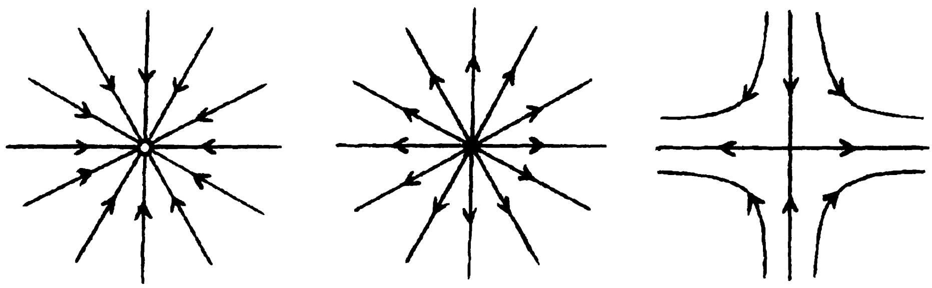


FIGURE 7.5

On the sphere consider the network consisting of both the old and the new networks and draw arrows on its edges, directing them away from the chosen points on the faces and towards the vertices. This network can be included in the system of trajectories of a vector field (Figure 7.5 (b)). The appearance of this vector field at a vertex, at a chosen point of a face, and at the midpoint of an edge is shown in Figure 7.6 (a), (b), and (c) respectively. Our vector field has no other singular points, hence the sum of indices of its singular points equals $V - E + F$. By Theorem 7.1 this number is equal to 2. □

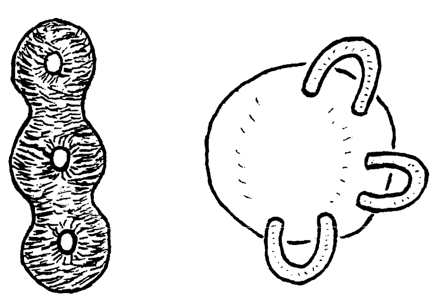


(a) (b) (c)

FIGURE 7.6

Problem 7.2. To each point X of the sphere a nonzero vector $v(X)$ in space is assigned. The vector depends continuously on the point of the sphere, but is not necessarily tangent to it. Prove that at least one of the vectors $v(X)$ is perpendicular to the tangent plane to the sphere at the point X.

Vector fields can also be considered on two-dimensional surfaces other than the sphere. One such surface is represented in Figure 7.7 in different ways. It is called the *sphere with three handles*; the origin of such a name should be clear from Figure 7.7 (b). In a similar way one defines a *sphere with g handles.* On the sphere with g handles, we shall also consider vector fields consisting only of tangent vectors.



(a) (b)

FIGURE 7.7

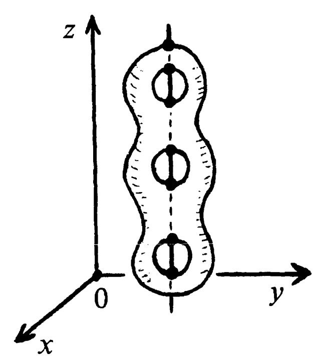


FIGURE 7.8

We can always assume that the sphere with g handles has an axis of symmetry parallel to the z-axis (Figure 7.8). On this surface consider a vector field whose trajectories lie in horizontal planes $z = c$. The singular points of this vector field are the intersection points of the surface with its symmetry axis. The highest and lowest of these points (i.e., those with greatest and least z-coordinate) are of index 1, because the trajectories near these points are like concentric circles. The other singular points all have index -1, because the trajectories near these points look like families of hyperbolas and their asymptotes. The vector field under consideration has 2 singular points of index 1 and $2g$ singular points of index -1, therefore the sum of indices of its singular points equals $2 - 2g$.

Theorem 7.3. *The sum of the indices of all singular points of any continuous vector field on the sphere with g handles equals* $2 - 2g$.

First Proof. The sphere with three handles may be cut into two pieces, each of which may be flattened into the plane region F shown in Figure 7.9.

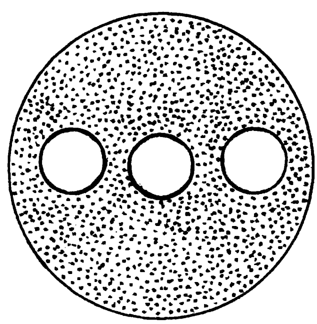


Figure 7.9

The sphere with g handles may be cut in a similar way. If the curve along which we cut happens to pass through a singular point, we can always modify it a little bit so as to avoid it. Hence we can (and will) assume that the curve does not contain any singular points. The sphere with g handles, just like the ordinary sphere, can be flattened so that its two halves coincide with F. From the sphere with g handles we therefore get two vector fields v and w on F such that at each point X of any of the boundary circles the vectors $v(X)$ and $w(X)$ are symmetric with respect to the tangent to the circle at X. We must prove that the sum of indices of these vector fields equals $2 - 2g$.

To this end let us apply the result of Problem 7.1. Suppose a and b are the indices of the outer circle with respect to the vector fields v and w, while $a_1, \dots, a_g$ and $b_1, \dots, b_g$ are the indices of the inner circles with respect to these vector fields. Then we have

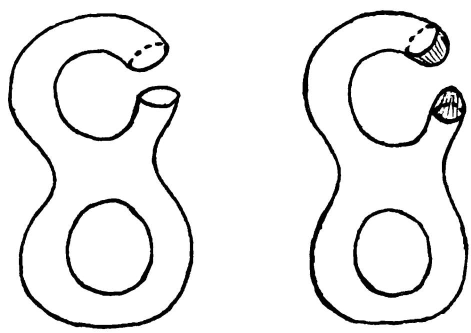
$$a + b = 2, \quad a_1 + b_1 = 2, \quad \dots, \quad a_g + b_g = 2$$

according to Problem 7.1. Let us extend the vector fields v and w to the plane disks bounded by the inner circles. This can be done by means of homotheties with centers at the centers of the inner circles. The sum of indices of all the singular points for the extended vector field v will equal a. On the other hand, it is equal to $a_1+\cdots+a_g+A$, where A is the sum of indices of all the singular points of the given field v, i.e., of those points that belong to F. Therefore $a=a_1+\cdots+a_g+A$. Similarly, $b=b_1+\cdots+b_g+B$. Adding the last two relations, we get

$$a+b=(a_1+b_1)+\cdots+(a_g+b_g)+(A+B),$$

i.e., $A+B=2-2g$, as required. □

SECOND PROOF. Figure 7.10 shows how one can get a sphere with $g-1$ handles from a sphere with g handles. To do that we cut one of the handles (Figure 7.10 (a)) and then attach a little hemisphere to each of the two cuts (Figure 7.10 (b)).



(a) (b)

FIGURE 7.10

For $g=0$, i.e., for the ordinary sphere, Theorem 7.3 has already been proved (see Theorem 7.1). Suppose that the assertion of Theorem 7.3 has been proved for any continuous vector field on the sphere with $g-1$ handles. We must prove it for a sphere with g handles. Let us pass to the sphere with $g-1$ handles as indicated in Figure 7.10. Now the vector field is not defined on the two hemispheres. But we can extend it continuously by flattening the hemispheres to disks and then considering the homothety with center at the disk's center. By slightly modifying the cut, we can assume that it does not contain any singular points. The vector fields on the curves of the cut come from the same vector field on the sphere with g handles. Hence we can glue the two hemispheres together, obtaining an ordinary sphere with a continuous vector field on it. Therefore (by Theorem 7.1 again), the sum of indices of the singular points on the two hemispheres is 2. Thus to find the sum of indices of all the singular points of the remaining part of the sphere with $g-1$ handles we must subtract 2 from the same sum for the entire sphere with $g-1$ handles. So we get

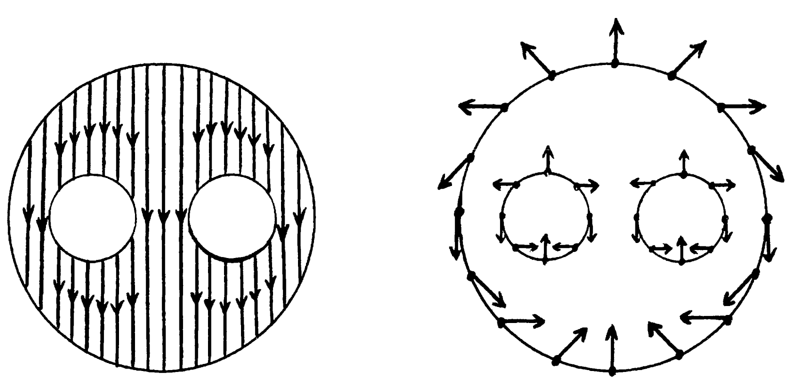
$$2-2(g-1)-2=2-2g,$$

as required. □

COROLLARY. *If $g \neq 1$, then any vector field on the sphere with g handles has at least one singular point.*

The sphere with one handle is the torus. A vector field without singular points can be constructed on it. For example, imagine the torus as a uniformly rotating wheel and take the corresponding velocity field.

On the sphere with g handles it is possible to construct a vector field with exactly one singular point of index $2 - 2g$. To do that take an arbitrary continuous vector field with a finite set of singular points. Let us join two singular points by a path and merge them into one by pulling the path to a point. As a result, the number of singular points will decrease by 1. Repeating this procedure an appropriate number of times, we get a vector field with exactly one singular point. However, it is not easy to picture it on the curved surface of the sphere with g handles. To obtain a clear picture of a vector field with one singular point on the sphere with g handles, we shall use a different approach. Cut the sphere with g handles in half and flatten out the halves so as to obtain the set shown in Figure 7.9. (It will be easier to draw plane pictures.) On one of the halves, let us take a vector field v of parallel vectors. For the sphere with two handles its trajectories are shown in Figure 7.11 (a). On the boundary circles on the other half, we obtain a vector field w for which each vector $w(X)$ is symmetric to $v(X)$ with respect to the tangent to the circle at the point X. The vector field w on the boundary circles (for the sphere with two handles) is represented in Figure 7.11 (b).



(a) (b)

FIGURE 7.11

Problem 7.3. Extend the vector field shown in Figure 7.11 (b) to the entire plane region with two holes.

Solutions.

7.1. Let us use the notation from Figure 7.12. Since the vectors v and w are symmetric with respect to the tangent, we have $\varphi_1 + \varphi_2 = 2(90^\circ + \alpha)$. Therefore $\varphi_2 = 180^\circ + 2\alpha - \varphi_1$. Suppose that the point X performs a complete revolution around the circle, i.e., the angle α varies from 0° to 360°. If at the same time the angle φ_1 changes by $n \cdot 360^\circ$, then the angle φ_2 changes by $2 \cdot 360^\circ - n \cdot 360^\circ$. This means that if the vector v performs n revolutions as we go around the circle, then the vector w performs $2 - n$ revolutions. According to Theorem 6.1, the sum of indices of singular points of the vector fields v and w is equal to n and $2 - n$ respectively. Thus the sum of indices of singular points for the fields v and w is equal to $n + (2 - n) = 2$.

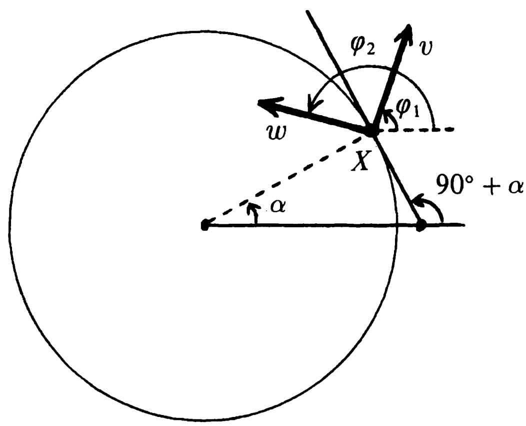


FIGURE 7.12

7.2. To each point X of the sphere assign the projection of the vector $v(X)$ on the tangent plane to the sphere at the point X. The result will be a continuous vector field on the sphere. According to Theorem 7.1, this vector field has a singular point X_0, i.e., the projection of the vector $v(X_0)$ on the tangent plane is zero. But this means that the vector $v(X_0)$ is perpendicular to the tangent plane.

7.3. See Figure 7.13.

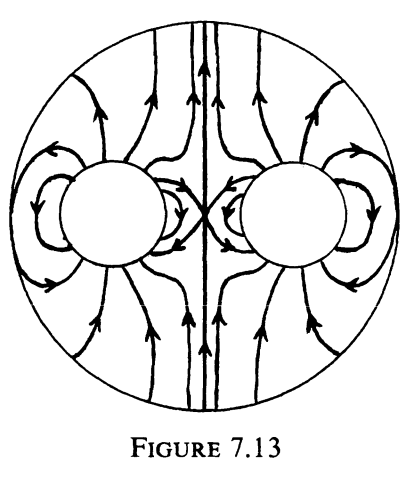


FIGURE 7.13