Groups and Symmetry
A Guide to Discovering Mathematics

David W. Farmer
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Groups and Symmetry
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David W. Farmer

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Preface

This book is a guide to discovering mathematics.

Every mathematics textbook is filled with results and techniques which once were unknown. The results were discovered by mathematicians who experimented, conjectured, discussed their work with others, and then experimented some more. Many promising ideas turned out to be dead-ends, and lots of hard work resulted in little output. Often the first progress was the understanding of some special cases. Continued work led to greater understanding, and sometimes a complex picture began to be seen as simple and familiar. By the time the work reaches a textbook, it bears no resemblance to its early form, and the details of its birth and adolescence have been lost. The precise and methodical exposition of a typical textbook often leads people to mistakenly think that mathematics is a dry, rigid, and unchanging subject.

The most exciting part of mathematics is the process of invention and discovery. The aim of this book is to introduce that process to you. By means of a wide variety of tasks, this book will lead you to discover some real mathematics. There are no formulas to memorize. There are no procedures to follow. By looking at examples, searching for patterns in those examples, and then searching for the reasons behind those patterns, you will develop your own mathematical ideas. The book is only a guide; its job is to start you in the right direction, and to bring you back if you stray too far. The discovery is left to you.

This book is suitable for a one semester course at the beginning undergraduate level. There are no prerequisites. Any college student interested in discovering the beauty of mathematics can enjoy a course taught from this book. An interested high school student will find this book to be a pleasant introduction to some modern areas of mathematics.

I thank Dave Bayer for showing me his method of drawing the Cayley diagrams of wallpattern groups. While preparing this book I was fortunate to have access to excellent notes taken by Hui-Chun Lee and by Elie Levine. It is a pleasure to thank Benji Fisher, Klaus Peters, Sandy Rhoades, Ted Stanford, John Sullivan, and Gretchen Wright for helpful comments on earlier versions of this book.

David W. Farmer
September, 1995
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Bibliography

   Interesting account of the mathematics of the native African peoples.
   Section 5, "Pattern and Shape," is relevant to this book.

   Description of his life and work. 40 color plates of his wallpatterns.

   An interesting book which touches on many of the topics discussed in
   the present book.

[E] *Ethnomathematics: A Multicultural View of Mathematical Ideas*, by Mar-
   Presents an interesting account of the mathematical sophistication of
   'primitive' people. The chapter, "The Logic of Kin Relations" is fasci-
   nating. "Symmetric Strip Decorations" is a nice introduction.

   A collection of 35 papers on mathematical symmetry. Most relate to
   Escher's work, and most have nice pictures. A good place to see how
   others have analyzed real–world symmetry. Some of the papers are
   very mathematical.

   Various papers on pentagonal symmetry. The paper, "800–Year–Old
   Pentagonal Tiling..." suggests that Penrose tilings were invented in
   12th–century Iran.

[FSE] *Fantasy and Symmetry: The Periodic Drawings of M.C. Escher*, by Caroline

   Islam Festival, 1976.
   Mathematics and Islamic Art.

   A comprehensive book on regular patterns. Millions of examples, and
   some reasonable guides on how to construct interesting patterns.
   A book in the same style as *Groups and Symmetry*.

   Nice chapters on a variety of mathematics, written for a general audience. All of it is interesting, and two or three of the chapters are relevant to this book.

   Interesting essays on real–world symmetry.


   Two huge books of papers on all aspects of symmetry.

   Very mathematical and not that easy to just pick up and read. It is comprehensive. Interesting exercises in Chapter 5. Lots of good references.

   A collection 36 papers dealing with mathematical aspects of art. Several papers are relevant to this book. The paper, “Interlace patterns in Islamic and Moorish art” includes Cayley diagrams of wallpaper groups.

   The comprehensive source of Escher plane tilings. Pages and pages of fascinating pictures. Description of Escher’s own classification of patterns.
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Groups and Symmetry
A Guide to Discovering Mathematics

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Mathematics is discovered by looking at examples, noticing patterns, making conjectures, and testing those conjectures. Once discovered, the final results get organized and put in textbooks. The details and the excitement of the discovery are lost.

This book introduces the reader to the excitement of the original discovery. By means of a wide variety of tasks, readers are led to find interesting examples, notice patterns, devise rules to explain the patterns, and discover mathematics for themselves.

The subject studied here is the mathematics behind the idea of symmetry, but the methods and ideas apply to all of mathematics. The only prerequisites are enthusiasm and a knowledge of basic high-school math.

The book is only a guide. It will start you off in the right direction and bring you back if you stray too far. The excitement and the discovery are left to you.