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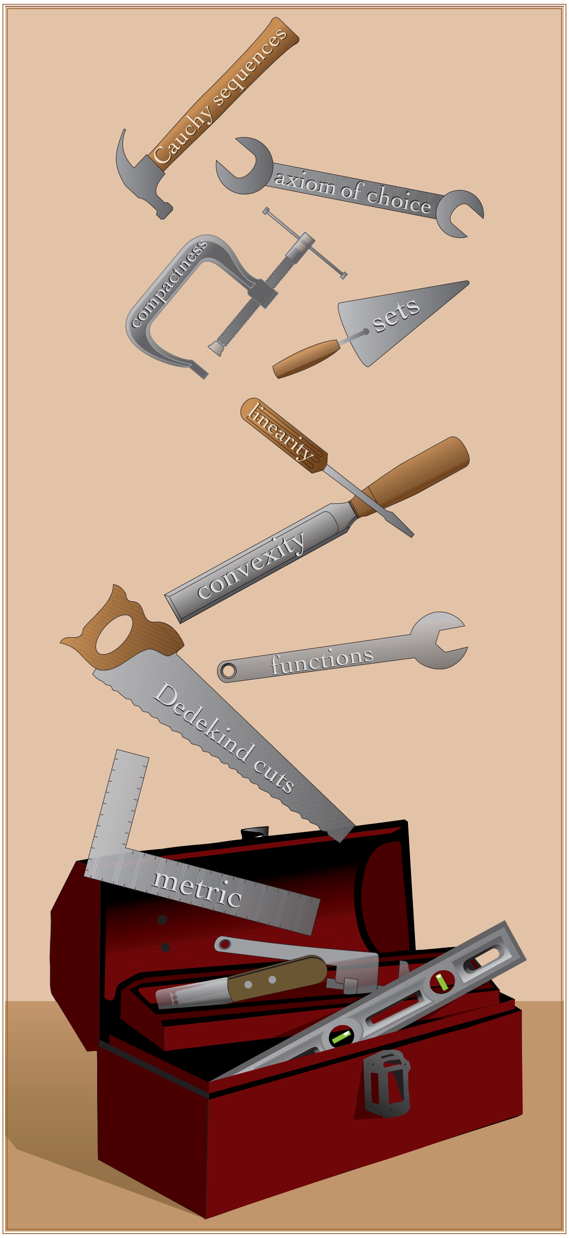

Introduction

to

Advanced

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Paul J. Sally, Jr.

American Mathematical Society

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This book is dedicated to the memory of my father, Paul Joseph Sally. He was born in 1897 into an Irish family then living in Scotland. His family immigrated to the United States in 1907 and lived in Philadelphia.

He left school after the eighth grade and worked at Friends Hospital to help support his family. He joined the Army in 1917 and thereby obtained his citizenship. He fought in World War I and was among the first U.S. troops that landed in Europe. He was a highly intelligent man with unmatched skills as a bricklayer, plasterer, and roofer. He knew the tools of his trade.

Paul J. Sally, Jr.
May 2008

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Introduction

When structuring an undergraduate mathematics program, ordinarily the faculty designs the initial set of courses to provide techniques that permit a student to solve problems of a more or less computational nature. So, for example, students might begin with a one variable calculus course and proceed through multi-variable calculus, ordinary differential equations, and linear algebra without ever encountering the fundamental ideas that underlie this mathematics. If the students are to learn to do mathematics well, they must at some stage come to grips with the idea of proof in a serious way.

In this book, we attempt to provide enough background so that students can gain familiarity and facility with the mathematics required to pursue demanding upper-level courses. The material is designed to provide the depth and rigor necessary for a serious study of advanced topics in mathematics, especially analysis.

There are several unusual features in this book. First, the exercises, of which there are many, are spread throughout the body of the text. They do not occur at the ends of the chapters. Instead Chapters 1–4 close with special projects that allow the teachers and students to extend the material covered in the text to a much wider range of topics. These projects are an integral part of the book, and the results in them are often cited in later chapters. They can be used as a regular part of the class, a source of independent study for the students, or as an Inquiry Based Learning (IBL) experience in which the students study the material and present it to the class. At the end of Chapter 5, there is a collection of Challenge Problems that are intended to test the students' understanding of the material in all five chapters as well as their mathematical creativity. Some of these problems are rather simple while others should challenge even the most able students.

We now give an outline of the content of the individual chapters. Chapter 1 begins with set theory, counting principles, and equivalence relations. This is followed by an axiomatic approach to the integers and the presentation of several basic facts about divisibility and number theory. The notions of a commutative ring with 1 and a field are introduced. Modular systems are given as examples of these structures. The ordered field of rational numbers is constructed as the field of quotients of the integers. Finally, cardinality, especially countability, is discussed. Several equivalent forms of the axiom of choice are stated and the equivalences proved.

Chapter 2 is about linear algebra. The first part of the chapter is devoted to abstract linear algebra up through linear transformations and determinants. In particular, the properties of determinants are attacked with bare knuckles. The final section of the chapter is devoted to geometric linear algebra. This is a study of the algebra and geometry of Euclidean n-space with respect to the usual distance. It is a preparation for the study of metric spaces in Chapter 4 as well as for the geometric ideas that occur in advanced calculus.

Chapter 3 begins with an axiomatic approach to the real numbers as an ordered field in which the least upper bound property holds. Several fundamental topics are addressed including some specific ideas about rational approximation of real numbers. Next, beginning with the rational numbers as an ordered field, the real numbers are constructed via the method of equivalence classes of Cauchy sequences. After this construction, the standard convergence theorems in the real numbers are proved. This includes the one-dimensional versions of the Bolzano-Weierstrass theorem and the Heine-Borel theorem. The last sections involve the construction of the complex numbers and their arithmetic properties. We also study the topic of convergence in the complex numbers.

In Chapter 4, the stakes are raised a bit. There is a complete and thorough treatment of metric spaces and their topology. Such spaces as bounded real valued functions on a set with the sup norm, the infinite-dimensional ℓ^p spaces, and others are given careful treatment. The equivalence between compactness and sequential compactness is proved, and the standard method of completing a metric space is presented. Here it is noted that this process cannot be used to complete the rational numbers to the real numbers since the completeness of the real numbers is fundamental to the proof. At the end of the chapter, several topics such as convexity and connectedness are analyzed.

Chapter 5 is a compendium of results that follow naturally from the theory of complete metric spaces developed in Chapter 4. These results are essential in further developments in advanced mathematics. The Contraction Mapping Theorem has a number of very useful applications, for example, in the proof of the Inverse Function Theorem. We give an application to the solution of ordinary differential equations. The Baire Category theorem is most often used in functional analysis. We give an application to uniformly bounded families of continuous functions on a complete metric space. The Stone–Weierstrass theorem concerns dense families of functions in the algebra of continuous functions on a compact metric space. In particular, this theorem implies the density of the polynomials in the algebra of continuous functions on closed bounded intervals in $\mathbb{R}$. The final section contains the most basic example of completing a metric space, that is, the p-adic completion of the rational numbers relative to a prime p. Along with being an example of the completion process, the p-adic completion yields a family of locally compact fields that currently is prominent in research in number theory, automorphic forms, mathematical physics, and other areas.

As pointed out above, each chapter ends with a set of special projects that are intended to broaden and deepen students' understanding of advanced mathematics. The first project in Chapter 1 is a series of exercises in elementary number theory that serves as an introduction to the subject and provides necessary material for the construction of the p-adic numbers in Chapter 5. Next, we introduce the idea of completely independent axiom systems, so that students working through this project might have some idea of the role of axioms in mathematics. Finally, we discuss ordered integral domains. We ask the students to show that the integers, as an ordered integral domain in which the Well Ordering Principle holds, are contained in every ordered integral domain. This leads naturally to the conclusion that every ordered field contains the rational numbers.

The projects at the end of Chapter 2 provide a set of exercises for the student that form a primer on basic group theory, with special emphasis on the general linear group and its subgroups.

The projects at the end of Chapter 3 present the students with an opportunity to investigate the following topics: an alternate construction of the real numbers using Dedekind cuts; an introduction to the convergence of infinite series; and a careful analysis of the decimal expansions of real numbers. The material about the convergence of infinite series is used extensively throughout the remaining chapters.

The projects in Chapter 4 provide an insight into advanced mathematics. They begin with an exploration for students of general point set topology, building on the theory of metric spaces covered in Chapter 4. Next, the students are asked to study a proof of the Fundamental Theorem of Algebra which establishes one of the basic facts in advanced mathematics.

The first three chapters of this book are used in a one quarter transition course at the University of Chicago. A substantial portion, but not all, of the material in the first three chapters can be covered in ten weeks. The remaining material in the book is used in the first quarter of "Analysis in $\mathbb{R}^n$." This course is intended as an advanced multivariable calculus course for sophomores. It covers geometric linear algebra from Chapter 2, some convergence theorems in $\mathbb{R}$ and $\mathbb{C}$ in Chapter 3, and the theory of metric spaces in Chapter 4, with an introduction to Chapter 5 if time allows. The remaining two quarters of Analysis in $\mathbb{R}^n$ cover differentiation theory and integration theory in $\mathbb{R}^n$ along with the usual theorems in vector calculus. The entire book is more than sufficient for a two quarter or one semester course, and if the projects are covered completely there is more than enough for a three quarter or two semester course.

Acknowledgements

We acknowledge with gratitude the contributions to our book made by our colleagues, students, and friends over the span of twenty years or more. It is not possible to mention by name all those with whom we have had conversations about this material, but we express appreciation to them here.

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We are in great debt to Mitya Boyarchenko who contributed significant ideas to the book and to John Boller and Loren Spice, both of whom also contributed ideas to the book and carefully read the final manuscript.

My ultimate debt is owed to those who worked with me to produce this manuscript. The word colleague describes them appropriately. The word amanuensis could be used as a formal title, but they are much more. We argued, discussed, rewrote, reaffirmed and readjusted parts of the manuscript on many occasions. These friends are Chris Jeris, Nick Ramsey, Kaj Gartz, Nick Ramsey (again), and Nick Longo. Without them, etc.

Paul J. Sally, Jr.
Chicago, Illinois
May 2008

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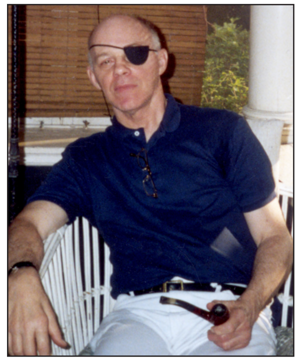

This book provides a transition from the formula-full aspects of the beginning study of college level mathematics to the rich and creative world of more advanced topics. It is designed to assist the student in mastering the techniques of analysis and proof that are required to do mathematics.

Along with the standard material such as linear algebra, construction of the real numbers via Cauchy sequences, metric spaces and complete metric spaces, there are three projects at the end of each chapter that form an integral part of the text. These projects include a detailed discussion of topics such as group theory, convergence of infinite series, decimal expansions of real numbers, point set topology and topological groups. They are carefully designed to guide the student through the subject matter. Together with numerous exercises included in the book, these projects may be used as part of the regular classroom presentation, as self-study projects for students, or for Inquiry Based Learning activities presented by the students.

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