LESSONS IN GEOMETRY

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Translated from the French by
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Translator's Preface

In the late 1890s Gaston Darboux was named as the editor of a set of textbooks, resources for the teaching of mathematics (*Cours Complet Pour la Classe de Mathématiques Élémentaires*). Darboux commissioned several mathematicians to write these materials. Jacques Hadamard, having taught on the high school (*lycée*) level,[1] was asked to prepare the materials for geometry. Two volumes resulted: one on plane geometry in 1898 and a volume on solid geometry in 1901.

Hadamard clearly saw this work as important, as he revised it twelve times during his long life, the last edition appearing in 1947. (Hadamard died in 1963 at the age of 97.)

The present book is a translation of the thirteenth edition of the first volume, first printed by Librarie Armand Colin, Paris, in 1947 and reprinted by Éditions Jacques Gabay, Sceaux, in 1988. It includes all the materials that this reprint contains. The volume on solid geometry has not been included here.

A companion volume to this translation, not based on the work of Hadamard, includes solutions to the problems as well as ideas for classroom use.

Hadamard's vision of geometry is remarkably fresh, even after the passage of 100 years. The classical approach is delicately balanced with modern extensions. The various geometric transformations arise simply and naturally from more static considerations of geometric objects.

The book includes a disk for use with the Texas Instruments TI-NSpire™software*. This disk is not meant to exhaust the possibilities of applying technology to these materials. Rather, it is meant to whet the appetite of the user for exploration of this area.

The same can be said about all the materials in the companion volume: Hadamard's book is a rich source of mathematical and pedagogical ideas, too rich to be exhausted in one supplementary volume. The supplementary materials are intended to invite the reader to consider further the ideas brought up by Hadamard.

A word is in order about the process of translation. Hadamard was a master of mathematics, and of mathematical exposition, but not particularly of the language itself. Some of his sentences are stiffly formal, others clumsy, even ambiguous (although the ambiguity is easily resolved by the logic of the discussion). In some cases (the appendix on Malfatti's problem is a good example) footnotes or dependent clauses seem to have been piled on as afterthoughts, to clarify a phrase or logical point. This circumstance presents an awkward dilemma for the translator.

[1] The best account of Hadamard's life, including those episodes alluded to in this preface, can be found in the excellent book by Vladimir Maz'ya and Tatyana Shaposhnikova, *Jacques Hadamard, A Universal Mathematician*, American Mathematical Society, Providence, Rhode Island, 1998.

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Does he make the English elegant and accessible? Or does he convey to the reader the flavor of the original? I have resolved this problem on a case-by-case basis, hoping that the result reads smoothly without distorting the spirit of the original.

In this work, I have received invaluable help from an initial translation prepared by Hari Bercovici, of Indiana University. While most of his work has been altered and fine-tuned, the core of it remains, and Bercovici made significant contributions to the resolution of a number of difficult problems of translation. In addition, the illustrations—faithful copies of Hadamard's own—are almost entirely the work of Bercovici. I am grateful for this opportunity to thank him for generously allowing me access to his work. In return, I take on myself the responsibility for any errors that may have crept in, and that the patient reader will doubtless find.

Others to whom I am grateful for help in this work include Wing Suet Li, Florence Fasanelli, Al Cuoco, Larry Zimmerman, and Sergei Gelfand. My wife, Carol Saul, a great supporter of everything I do, has been immeasurably tolerant of my preoccupation with this work.

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Hadamard's career as a high school teacher does not seem to have ended successfully. He did not stay long in the profession, and there exist notes by his superiors testifying to his difficulties in getting along with his students.[2] However, he seems to have learned from his experiences how to approach them intellectually, thus allowing other teachers, with other skills, the benefit of his own genius.

It is in this spirit of combining the skills of the teacher and the mathematician that I offer these materials to the field.

Mark Saul May 2008

[2]Hadamard's mentoring of Maurice Fréchet, which started in the latter's high school years, is a notable exception to this circumstance.

Author's Prefaces

Preface to the Second Edition

Since the appearance of this work, the teaching of mathematics, and particularly of geometry, has undergone some profound modifications, not just in its details, but in its whole spirit, changes which have been awaited for a long time and are universally desired. In working with beginners, we now tend to rely on practice and intuition, rather than on the Euclidean method, whose utility they are incapable of understanding.

On the other hand, it is clear that we must return to this method when we revisit these early starts, and complete them. It is to this stage of education that our book corresponds, and thus we have not had to change its character.

But even in the area of rigorous logic, the classical exposition was uselessly complicated and scholastic in its first chapter, the one devoted to angles. The convention—unchanged up to the present—which does not permit us to talk of circles in the first book, renders matters obscure which, in themselves, are perfectly clear and natural. Thus this is a place which we have been able to notably simplify things, by introducing arcs of circles into the discussion of angles. We had already departed from the traditional considerations of continuity on which the existence of perpendiculars is often based; the simple artifice which replaced it has itself now become superfluous.

In the same way, the measure of the central angle is naturally integrated into the theory of angles, its correct logical place.

The second book gains no less than the first by this change in order. The fundamental property of the inscribed angle, indeed, is no longer connected to angle measure, a connection which gives one an idea of this property and its significance which is as false as could be.

With this exception, the plan of the work as a whole has been preserved. In fact, the complementary materials introduced by the program of 1902 had been already covered in our first edition. The program of 1905, which has reduced the importance of these materials, has not until now obliged us to do any essential revision. It requires only a single addition: the inverter of Peaucellier. Having made this addition, the only complementary material remaining in this revision, at least in plane geometry[3] is inversion and its applications, which corresponds to Chapters V–VII of our Complements.

[3] I note in this regard, that I have never attempted—despite the advocacy of such a step by M. Méray, whose initiative has proved so fertile and so fortunate in the teaching of geometry—to mix plane and solid geometry together. As this is preferable from a purely logical point of view, I would like very much to do this. But it seems to me that from a pedagogical point of view, we must think, first and foremost, of dividing up the difficulties. That of "spatial visualization" is so serious in and of itself, that I haven't considered adding it to the other difficulties initially.

Another tendency has appeared in the teaching corps in the last few years, which we have had the bad manners not to applaud. We speak here—and I hope we begin to use it a bit—of the method of heuristics. The Note we have added in 1898 to our first edition (Note A) had exactly the goal of describing how, in our view, this method might be understood: how it might be understood, at least, from a theoretical point of view, since both are needed for the application of the heuristic method. I hope that this Note might now be of some use in indicating, at least, how these principles can be put to work.

I have already explained (Preface to *Solid Geometry*) that the method described in Note C for tangent circles belongs in fact to M. Fouché, or to Poncelet himself, and that a solution to the question of areas of plane figures, different, it is true, from that in Note D, is due to M. Gérard. I seize this occasion to add that an objection concerning the theory of dihedral angles has already been noted and refuted by M. Fontené.

J. Hadamard

Preface to the Eighth Edition

The present addition contains no important changes from those that preceded it. We must note, however, that our ideas about the Postulate of Euclid have been modified considerably by recent progress in physics: I have had to recast the end of Note B to take into account this scientific evolution.

A few modification have been made in the present edition, intended to give a bit more importance to properties of the most common articulated systems.

J. Hadamard

Preface to the Twelfth Edition

This edition differs from the preceding only in the addition of several exercises. The elegant Exercise 421b is due to M. Daynac, a teacher in the French School of Cairo; the simple proof of Morley's theorem which is given by Exercise 422, and which brings into play only the first two Books, is due to M. Sasportès;[4] the supplement added to exercise 107 is borrowed from an article of M. Lapierre (*Enseignement Scientifique*, November 1934), with certain modifications intended to reduce the proof to properties of the inscribed angle; Exercise 314b is from Japanese geometers.

J. Hadamard

Preface to the First Edition

In editing these Lessons in Geometry, I have not lost sight of the very special role played by this science in the area of elementary mathematics.

Placed at the entry point to the teaching of mathematics, it is in fact the simplest and most accessible form of reasoning. The importance of its methods, and their fecundity, are here more immediately tangible then in the relatively abstract theories of arithmetic or algebra. Because of this, geometry reveals itself capable of

[4] An essentially equivalent proof was sent me by M. Gauthier, a student at the École Normale Supérieure.

exercising an undeniable influence on the activity of the mind. I have, first of all, sought to develop this influence in awakening and assisting the student's initiative.

Thus it seemed to me necessary to increase the number of exercises as much as the framework of the work would allow. This requirement has been, so to speak, the only rule guiding me in this part of my work. I believed that I must pose questions of very different and gradually increasing difficulty. While the exercises at the end of each chapter, and especially the first few chapters, are very simple, those which I have inserted after each book have solutions which are less immediate. Finally, I have postponed to the end of the volume the statement of problems which are relatively difficult. Certain questions have been borrowed from some important theories—among these we note problems related to the theory of inversion and to systems of circles, many of which come from the note *On the relations between groups of points, of circles, and of spheres in the plane and in space*, of M. Darboux.[5] Others, on the contrary, have no pretensions other than to train the mind in the rules of reason. I have been no less eclectic in the choice of sources I have drawn on: alongside classic exercises which are immediate applications of the theory, and whose absence in this sort of book would be almost astonishing, can be found exercises which are borrowed from various authors and periodicals, both French and foreign, and also a large number which are original.

I have also included, at the end of the work, a note in which I seek to summarize the basic principles of the mathematical method, methods which students must begin to understand starting from the first year of instruction, and which we find poorly understood even by students in our schools of higher education. The dogmatic form which I have had to adopt is not, it must be admitted, the one that fits this topic best: this sort of subject is best taught through a sort of dialogue in which each rule intervenes at exactly the moment when it applies. I believed, despite all, that I had to attempt this exposition, hoping to find readers who are indulgent of the fact that it is presented imperfectly. Let this essay, imperfect as it is, perform a number of services and contribute to the infusion into the classroom of ideas on whose importance we must not tire of insisting.

The other notes, also placed at the end of the volume, are more special in character. Note B concerns Euclid's postulate. The ideas of modern geometers on this subject have assumed a form which is clear and well enough defined that it is possible to give an account of them even in an elementary work.

Note C concerns the problem of tangent circles. As M. Koenigs has noted,[6] the known solution of Gergonne, even when completed by the synthesis neglected by the author, leaves something to be desired. It is this gap that I seek to fill.

Finally, Note D is devoted to the notion of area. The usual theory of area presents, as we know, a serious logical fault. It supposes a priori that this quantity is well-defined and enjoys certain properties. The theory that I give in the note in question, and in which we do without this *postulatum* , must be preferred, especially if one realizes that it applies to space geometry without any significant change.

In the text itself, various classical arguments might be modified to advantage, sometimes to support more rigor, and sometimes in the interests of simplicity.

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[5] *Annales Scientifiques de l'École Normale Superieure*, 2nd series, Vol. I, 1876. Exercise 401 (the construction of tangent circles) was provided to me by M. Gérard, a teacher at the *Lycée Ampère*

[6] *Leçons de l'agrégation classique de Mathématiques*, p. 92, Paris, Hermann, 1892.

Among these, for example, are the proof at the beginning of the first book that a perpendicular can be erected to a line from a point on that line. The considerations of continuity usually raised at this point can be set aside, as long as one assumes, without proof, that a segment or an angle can be divided into two equal parts. The consideration of the sense of rotation of an angle has permitted me to give the statements of theorems in the second book, as well as several following, all their cleanness and all their generality without rendering them less simple or less elementary.

The theories described in the *Complements to the Third Book* are those that, while not included in the elements of geometry as set forth by Euclid, have not taken a lesser place in education in a definitive way. I have limited myself to the elements of these theories and I have systematically eliminated those without real importance. In any case, this work is edited so that these complements, as well as several passages printed in small characters, can be passed over in a first reading without losing the coherence of the rest.

M. Darboux, who has given me the honor of trusting me with the editing of this work, has rendered the task singularly easy by the valuable advice which he has not ceased to give me for its composition. I would not want to end this preface without offering to him the homage of my recognition.

Jacques Hadamard

This is a book in the tradition of Euclidean synthetic geometry written by one of the twentieth century's great mathematicians. The original audience was pre-college teachers, but it is useful as well to gifted high school students and college students, in particular, to mathematics majors interested in geometry from a more advanced standpoint.

The text starts where Euclid starts, and covers all the basics of plane Euclidean geometry. But this text does much more. It is at once pleasingly classic and surprisingly modern. The problems (more than 450 of them) are well-suited to exploration using the modern tools of dynamic geometry software. For this reason, the present edition includes a CD of dynamic solutions to select problems, created using Texas Instruments' TI-Nspire™ Learning Software. The TI-Nspire™ documents demonstrate connections among problems and—through the free trial software included on the CD—will allow the reader to explore and interact with Hadamard's Geometry in new ways. The material also includes introductions to several advanced topics. The exposition is spare, giving only the minimal background needed for a student to explore these topics. Much of the value of the book lies in the problems, whose solutions open worlds to the engaged reader.

And so this book is in the Socratic tradition, as well as the Euclidean, in that it demands of the reader both engagement and interaction. A forthcoming companion volume that includes solutions, extensions, and classroom activities related to the problems can only begin to open the treasures offered by this work. We are just fortunate that one of the greatest mathematical minds of recent times has made this effort to show to readers some of the opportunities that the intellectual tradition of Euclidean geometry has to offer.

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