Poincaré's Legacies, Part I

pages from year two of a mathematical blog

Terence Tao

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To Garth Gaudry, who set me on the road;
To my family, for their constant support;
And to the readers of my blog, for their feedback and contributions.

Contents

Preface

In February of 2007, I converted my "What's new" web page of research updates into a blog at `terrytao.wordpress.com`. This blog has since grown and evolved to cover a wide variety of mathematical topics, ranging from my own research updates, to lectures and guest posts by other mathematicians, to open problems, to class lecture notes, to expository articles at both basic and advanced levels.

With the encouragement of my blog readers, and also of the AMS, I published many of the mathematical articles from the first year (2007) of the blog as [**Ta2008b**], which will henceforth be referred to as *Structure and Randomness* throughout this book. This gave me the opportunity to improve and update these articles to a publishable (and citeable) standard, and also to record some of the substantive feedback I had received on these articles from the readers of the blog. Given the success of the blog experiment so far, I am now doing the same for the second year (2008) of articles from the blog. This year, the amount of material is large enough that the blog will be published in two volumes.

As with *Structure and Randomness*, each part begins with a collection of expository articles, ranging in level from completely elementary logic puzzles to remarks on recent research, which are only loosely related to each other and to the rest of the book. However, in contrast to the previous book, the bulk of these volumes is dominated by the lecture notes for two graduate courses I gave during the year. The two courses stemmed from two very different but fundamental contributions to mathematics by Henri Poincaré, which explains the title of the book.

This is the first of the two volumes, and it focuses on ergodic theory, combinatorics, and number theory. In particular, Chapter 2 contains the lecture notes for my course on *topological dynamics and ergodic theory*, which originated in part from Poincaré's pioneering work in chaotic dynamical systems. Many situations in mathematics, physics, or other sciences can be modeled by a discrete or continuous *dynamical system*, which at its most abstract level is simply a space X, together with a shift $T : X \to X$ (or family of shifts) acting on that space, and possibly preserving either the topological or measure-theoretic structure of that space. At this level of generality, there are a countless variety of dynamical systems available for study, and it may seem hopeless to say much of interest without specialising to much more concrete systems. Nevertheless, there is a remarkable phenomenon that dynamical systems can largely be classified into "structured" (or "periodic") components, and "random" (or "mixing") components,[1] which then can be used to prove various *recurrence theorems* that apply to very large classes of dynamical systems, not the least of which is the *Furstenberg multiple recurrence theorem* (Theorem 2.10.3). By means of various *correspondence principles*, these recurrence theorems can then be used to prove some deep theorems in combinatorics and other areas of mathematics, in particular yielding one of the shortest known proofs of *Szemerédi's theorem* (Theorem 2.10.1) that all sets of integers of positive upper density contain arbitrarily long arithmetic progressions. The road to these recurrence theorems, and several related topics (e.g. ergodicity, and Ratner's theorem on the equidistribution of unipotent orbits in homogeneous spaces) will occupy the bulk of this course. I was able to cover all but the last two sections in a 10-week course at UCLA, using the exercises provided within the notes to assess the students (who were generally second or third-year graduate students, having already taken a course or two in graduate real analysis).

Finally, I close this volume with a third (and largely unrelated) topic (Chapter 3), namely a series of lectures on recent developments in additive prime number theory, both by myself and my coauthors, and by others. These lectures are derived from a lecture I gave at the annual meeting of the AMS at San Diego in January of 2007, as well as a lecture series I gave at Penn State University in November 2007.

A remark on notation

For reasons of space, we will not be able to define every single mathematical term that we use in this book. If a term is italicised for reasons other than emphasis or definition, then it denotes a standard mathematical object, result, or concept, which can be easily looked up in any number of references.

[1] One also has to consider *extensions* of systems of one type by another, e.g. mixing extensions of periodic systems; see Section 2.15 for a precise statement.

(In the blog version of the book, many of these terms were linked to their Wikipedia pages, or other on-line reference pages.)

I will however mention a few notational conventions that I will use throughout. The cardinality of a finite set E will be denoted $|E|$. We will use the asymptotic notation $X = O(Y)$, $X \ll Y$, or $Y \gg X$ to denote the estimate $|X| \leq CY$ for some absolute constant $C > 0$. In some cases we will need this constant C to depend on a parameter (e.g. d), in which case we shall indicate this dependence by subscripts, e.g. $X = O_d(Y)$ or $X \ll_d Y$. We also sometimes use $X \sim Y$ as a synonym for $X \ll Y \ll X$.

In many situations there will be a large parameter n that goes off to infinity. When that occurs, we also use the notation $o_{n\to\infty}(X)$ or simply $o(X)$ to denote any quantity bounded in magnitude by $c(n)X$, where $c(n)$ is a function depending only on n that goes to zero as n goes to infinity. If we need $c(n)$ to depend on another parameter, e.g. d, we indicate this by further subscripts, e.g. $o_{n\to\infty;d}(X)$.

We will occasionally use the averaging notation

$$\mathbf{E}_{x\in X} f(x) := \frac{1}{|X|}\sum_{x\in X} f(x)$$

to denote the average value of a function $f : X \to \mathbf{C}$ on a non-empty finite set X.

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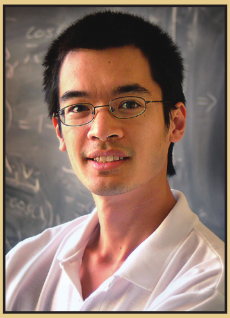

Reed Hutchinson/UCLA

There are many bits and pieces of folklore in mathematics that are passed down from advisor to student, or from collaborator to collaborator, but which are too fuzzy and non-rigorous to be discussed in the formal literature. Traditionally, it was a matter of luck and location as to who learned such folklore mathematics. But today, such bits and pieces can be communicated effectively and efficiently via the semiformal medium of research blogging. This book grew from such a blog.

In 2007, Terry Tao began a mathematical blog to cover a variety of topics, ranging from his own research and other recent developments in mathematics, to lecture notes for his classes, to non-technical puzzles and expository articles. The articles from the first year of that blog have already been published by the AMS. The posts from 2008 are being published in two volumes.

This book is Part I of the second-year posts, focusing on ergodic theory, combinatorics, and number theory. Chapter 2 consists of lecture notes from Tao's course on topological dynamics and ergodic theory. By means of various correspondence principles, recurrence theorems about dynamical systems are used to prove some deep theorems in combinatorics and other areas of mathematics. The lectures are as self-contained as possible, focusing more on the "big picture" than on technical details.

In addition to these lectures, a variety of other topics are discussed, ranging from recent developments in additive prime number theory to expository articles on individual mathematical topics such as the law of large numbers and the Lucas–Lehmer test for Mersenne primes. Some selected comments and feedback from blog readers have also been incorporated into the articles.

The book is suitable for graduate students and research mathematicians interested in broad exposure to mathematical topics.

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