Poincaré’s Legacies, Part II

pages from year two of a mathematical blog

Terence Tao
To Garth Gaudry, who set me on the road;
To my family, for their constant support;
And to the readers of my blog, for their feedback and contributions.
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Preface

In February of 2007, I converted my “What’s new” web page of research updates into a blog at terrytao.wordpress.com. This blog has since grown and evolved to cover a wide variety of mathematical topics, ranging from my own research updates, to lectures and guest posts by other mathematicians, to open problems, to class lecture notes, to expository articles at both basic and advanced levels.

With the encouragement of my blog readers, and also of the AMS, I published many of the mathematical articles from the first year (2007) of the blog as [Ta2008b], which will henceforth be referred to as Structure and Randomness throughout this book. This gave me the opportunity to improve and update these articles to a publishable (and citeable) standard, and also to record some of the substantive feedback I had received on these articles by the readers of the blog. Given the success of the blog experiment so far, I am now doing the same for the second year (2008) of articles from the blog. This year, the amount of material is large enough that the blog will be published in two volumes.

As with Structure and Randomness, each part begins with a collection of expository articles, ranging in level from completely elementary logic puzzles to remarks on recent research, which are only loosely related to each other and to the rest of the book. However, in contrast to the previous book, the bulk of these volumes is dominated by the lecture notes for two graduate courses I gave during the year. The two courses stemmed from two very different but fundamental contributions to mathematics by Henri Poincaré, which explains the title of the book.

This is the second of the two volumes, and it focuses on geometry, topology, and partial differential equations. In particular, Chapter 2 contains
the lecture notes for my course on the famous Poincaré conjecture that
every simply connected compact three-dimensional manifold is homeomor-
phic to a sphere, and its recent spectacular solution [Pe2002], [Pe2003],
[Pe2003b] by Perelman. This conjecture is purely topological in nature,
and yet Perelman’s proof uses remarkably little topology, instead working
almost entirely in the realm of Riemannian geometry and partial differential
equations, and specifically in a detailed analysis of solutions to Ricci flows on
three-dimensional manifolds, and the singularities formed by these flows. As
such, the course will incorporate, along the way, a review of many of the ba-
cic concepts and results from Riemannian geometry (and to a lesser extent,
from parabolic PDE), while being focused primarily on the single objective
of proving the Poincaré conjecture. Due to the complexity and technical
intricacy of the argument, we will not be providing a fully complete proof of
this conjecture here (see [MoTi2007] for a careful and detailed treatment);
but we will be able to cover the high-level features of the argument, as well
as many of the specific components of that argument, in full detail, and the
remaining components are sketched and motivated, with references to more
complete arguments given. In principle, the course material is sufficiently
self-contained that prior exposure to Riemannian geometry, PDE, or topol-
ogy at the graduate level is not strictly necessary, but in practice, one would
probably need some comfort with at least one of these three areas in order
to not be totally overwhelmed by the material. (I ran this course as a topics
course; in particular, I did not assign homework.)

A remark on notation

For reasons of space, we will not be able to define every single mathematical
term that we use in this book. If a term is italicised for reasons other
than emphasis or definition, then it denotes a standard mathematical object,
result, or concept, which can be easily looked up in any number of references.
(In the blog version of the book, many of these terms were linked to their
Wikipedia pages, or other on-line reference pages.)

I will however mention a few notational conventions that I will use
throughout. The cardinality of a finite set $E$ will be denoted $|E|$. We
will use the asymptotic notation $X = O(Y)$, $X \ll Y$, or $Y \gg X$ to denote
the estimate $|X| \leq CY$ for some absolute constant $C > 0$. In some cases
we will need this constant $C$ to depend on a parameter (e.g. $d$), in which
case we shall indicate this dependence by subscripts, e.g. $X = O_d(Y)$ or
$X \ll_ d Y$. We also sometimes use $X \sim Y$ as a synonym for $X \ll Y \ll X$.

In many situations there will be a large parameter $n$ that goes off to
infinity. When that occurs, we also use the notation $o_{n \to \infty}(X)$ or simply
$o(X)$ to denote any quantity bounded in magnitude by $c(n)X$, where $c(n)$
is a function depending only on \( n \) that goes to zero as \( n \) goes to infinity. If we need \( c(n) \) to depend on another parameter, e.g. \( d \), we indicate this by further subscripts, e.g. \( o_{n \to \infty d}(X) \).

We will occasionally use the averaging notation

\[
E_{x \in X} f(x) := \frac{1}{|X|} \sum_{x \in X} f(x)
\]

to denote the average value of a function \( f : X \to \mathbb{C} \) on a non-empty finite set \( X \).

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There are many bits and pieces of folklore in mathematics that are passed down from advisor to student, or from collaborator to collaborator, but which are too fuzzy and non-rigorous to be discussed in the formal literature. Traditionally, it was a matter of luck and location as to who learned such folklore mathematics. But today, such bits and pieces can be communicated effectively and efficiently via the semiformal medium of research blogging. This book grew from such a blog.

In 2007, Terry Tao began a mathematical blog to cover a variety of topics, ranging from his own research and other recent developments in mathematics, to lecture notes for his classes, to non-technical puzzles and expository articles. The articles from the first year of that blog have already been published by the AMS. The posts from 2008 are being published in two volumes.

This book is Part II of the second-year posts, focusing on geometry, topology, and partial differential equations. The major part of the book consists of lecture notes from Tao’s course on the Poincaré conjecture and its recent spectacular solution by Perelman. The course incorporates a review of many of the basic concepts and results needed from Riemannian geometry and, to a lesser extent, from parabolic PDE. The aim is to cover in detail the high-level features of the argument, along with selected specific components of that argument, while sketching the remaining elements, with ample references to more complete treatments. The lectures are as self-contained as possible, focusing more on the “big picture” than on technical details.

In addition to these lectures, a variety of other topics are discussed, including expository articles on topics such as gauge theory, the Kakeya needle problem, and the Black–Scholes equation. Some selected comments and feedback from blog readers have also been incorporated into the articles.

The book is suitable for graduate students and research mathematicians interested in broad exposure to mathematical topics.