

# HADAMARD'S PLANE GEOMETRY

A Reader's Companion

Mark Saul

 **AMS**  
American  
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## Foreword

A well-written book invites the reader into the mind of the author. Novels, stories, books of poetry do this on an unconscious level, sweeping the reader along with the words. A book of mathematics does it differently: well-written mathematics impels the reader to take a pencil in hand and have paper at the ready. The present volume is offered as an enhancement to just such a reading of Jacques Hadamard: *Lessons in Geometry, I. Plane Geometry*, by the American Mathematical Society, Providence, RI, and Education Development Center, Inc., Newton, MA, 2008.

Jacques Hadamard was among the greatest mathematicians of the twentieth century. He made signal contributions to a number of fields, including number theory, differential geometry, and differential equations. But his mind could not be confined to the upper reaches of mathematical thought. His legacy includes a book<sup>1</sup> in which he reflects on the process of creating mathematics, both his own and others. He was active in the Dreyfuss Affair (his wife was related to Alfred Dreyfuss), and held and expressed strong political and philosophical views all his life.<sup>2</sup>

And he was a teacher. For several years, as a graduate student, he worked in a *lycée*, teaching elementary mathematics. Later, the mathematician Gaston Darboux, involved in rewriting the French school mathematics program, thought of Hadamard's experience, and asked him to write a book for teachers on elementary synthetic geometry. The result was a massive two-volume work on plane and solid geometry. The present book is a reader's companion to the translation of the first volume of this work referred to above.

As might be expected of a great mathematician, Hadamard was a master poser of problems. Although termed "exercises", the problems in his *Geometry* are an integral part of the plan of the book. Indeed, the text can be read as a minimal exposition, providing the mathematics that will support the solution of the ensuing problems.

That is, the problems interpret the text, in the way that the harmony interprets the melody in a well-composed piece of music.

The problems are rich and complex. Some of them embroider the text, digging deep into the intuitions behind Hadamard's theorems and lemmas. Others, such as the problems about the Simson line or the nine-point circle, extend the text in various directions. Often these give results which are important in their own right. Very few of them are in any way routine: rarely will the solver read the problem and know immediately how to approach it.

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<sup>1</sup>*The Psychology of Invention in the Mathematical Field*, Princeton University Press, 1945.

<sup>2</sup>For many more interesting details of Hadamard's life, see Vladimir Mazya, Tatyana Shaposhnikova, *Jacques Hadamard, A Universal Mathematician*, American Mathematical Society, Providence, Rhode Island, 1998.



For all these reasons, this companion volume to Hadamard's *Geometry* can add significantly to the reader's experience. It requires of the reader only the background of high school geometry, which the text itself provides. The solutions strive to connect the general methods given in the text with intuitions that are natural to the subject, giving as much motivation as possible as well as rigorous and formal solutions. Ideas for further exploration are often suggested.

Another aspect of this companion volume is pedagogical. The work provides indications of possible motivation for difficult turns in the argument, all drawn from classroom experience. Software explorations (using dynamic geometry software) are suggested throughout. (The text of *Geometry* comes with a sample of such explorations, implemented on the TI-Nspire™ Learning Software.) While directly addressed to high school teachers, this style of exposition will speak to any sophisticated lover of classical synthetic geometry, whatever her or his professional interest.

A few notes about the form of the solutions are in order. Hadamard made a judicious choice of material for his text, but omitted certain results that come up often (and are usually easy to prove) in working the problems. These are stated here as lemmas. This is not the usual use of this term: a "lemma" is more often a special case of a more general result to come, or a result about a very specific situation which will not arise later. The lemmas here are separated out from the problems, usually, because they are of interest in their own right. For the same reason, a lemma in one problem is sometimes referenced in another problem.

Other references are to the text of Hadamard's *Geometry* itself. These appear in the form "see **27**", where **27** refers to article (*not* page) 27 in *Geometry*. For reasons of space, this volume omits solutions to Exercises 343 to 422 in Hadamard's original text. These "miscellaneous problems" (Hadamard's own term) are some of the most interesting and complicated in the book. Solutions to these problems can be found on the internet at [www.ams.org/bookpages/mbk-70](http://www.ams.org/bookpages/mbk-70). In some cases, reference is made in the present volume to solutions to these exercises.

It takes a village to write a book like this, and I welcome this occasion to express my gratitude to many people who contributed to it, either by suggesting solutions, or reviewing the content, or assisting with the logistics of getting the manuscript out the door. Vincent Matsko, Don Barry, Yvonne Lai, Alon Amit, Sergei Markelov, and Jordan Tabov all read through the manuscript and corrected some egregious errors in the first versions. Al Cuoco, Jim Sotiros, Tatyana Shubin, Valeriy Ryzhek, Alexander Shen, Borislav Lazarov, and Jenny Sendova all pitched in at those awkward moments when something needed to be done that I couldn't do myself. Sergei Gelfand at the American Mathematical Society was at once patient and prodding in his support of my efforts, urging me on just when my energy flagged. Jennifer Wright Sharp and Gil Poulin, also at AMS, spent hours on the tedious but vital tasks of copy editing, catching errors, omissions, and inconsistencies that the author's jaded eye could not find.

A special mention must be made of Alexei Kopylov, whose detailed and thoughtful reading of most of the manuscript turned up errors large and small, whose suggestions smoothed over rough patches in the exposition, and whose ideas for alternatives enriched the presentation. The work bears the stamp of his careful, detailed, and inspired work.

But perhaps the most important credit goes to the late Dmitri Ivanovitch Perepelkin, a member of the mathematics faculty at Moscow State University from the mid-1930s to his death in 1954. A first-rate geometer, he prepared, in several editions, a translation of Hadamard's *Geometry* into Russian, and supplied solutions to the problems. His solutions were written for an audience of considerable mathematical sophistication. I have borrowed heavily and shamelessly from his work. Many of the solutions—and most of the really clever solutions—are based on his. However, they are reworked significantly, in the hope of making them accessible to a wider audience.

Of course, I hold myself responsible for any incompleteness or errors that remain despite the help of Perepelkin and all the others I have mentioned. A list of errata for Hadamard's *Geometry* appears at <http://www.ams.org/bookpages/mbk-57>. A list of errata for this volume appears at <http://www.ams.org/bookpages/mbk-70>.

This work was supported by grant number NSF ESI 0242476-03 from the National Science Foundation. Additional support was provided by the John Templeton Foundation through a grant to the Mathematical Sciences Research Institute. I am grateful to all these donors for making this work possible.

On a personal level, I would like to thank my high school French teachers, Ann Bosch, Louis Fuhrman, Josephine Burstein, and Dorothy Roth, for giving me access to that language. I thank my grandfather, Froim Camenir, for transmitting to me a love of language and learning, and helping me in my faltering first steps in learning both mathematics and Russian. And I must thank my wife, Carol Saul, whose support means more to me than is appropriate to express in this context.

I drew on the support of all these people in completing this project. One of the greatest difficulties was the richness of the problems themselves. I found myself thinking for days on some of them, examining and re-examining solutions, extensions, and generalizations, as the book did its work of drawing me into the mind of the author himself. At some point—actually at many points—I had to force myself to relinquish the problem, and remind myself that my role is not to keep them for my own amusement or edification, but to offer them to the mathematical community.

I do so in the hope that readers will enjoy going further than I have in exploring these problems.

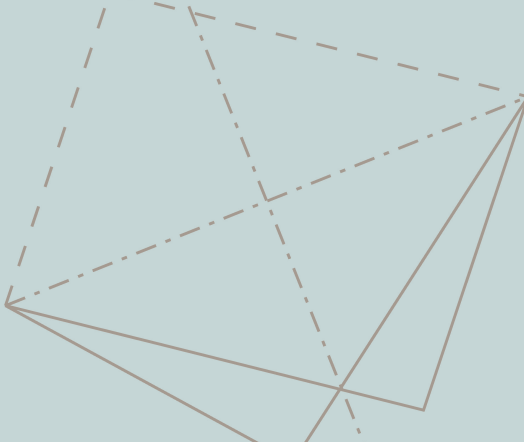
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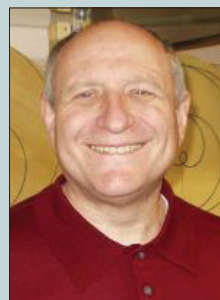






Jacques Hadamard, among the greatest mathematicians of the twentieth century, made signal contributions to a number of fields. But his mind could not be confined to the upper reaches of mathematical thought. He also produced a massive two-volume work, on plane and solid geometry, for pre-college teachers in the French school system.

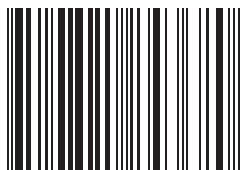
In those books, Hadamard's style invites participation. His exposition is minimal, providing only the results necessary to support the solution of the many elegant problems he poses afterwards. That is, the problems interpret the text in the way that harmony interprets melody in a well-composed piece of music.



The present volume offers solutions to the problems in the first part of Hadamard's work (*Lessons in Geometry. I. Plane Geometry*, Jacques Hadamard, Amer. Math. Soc. (2008)), and can be viewed as a reader's companion to that book. It requires of the reader only the background of high school plane geometry, which *Lessons in Geometry* provides. The solutions strive to connect the general methods given in the text with intuitions that are natural to the subject, giving as much motivation as possible as well as rigorous and formal solutions. Ideas for further exploration are often suggested, as well as hints for classroom use.

This book will be of interest to high school teachers, gifted high school students, college students, and those mathematics majors interested in geometry.

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