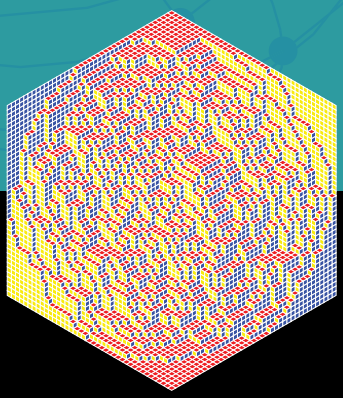
Markov Chains and Mixing Times

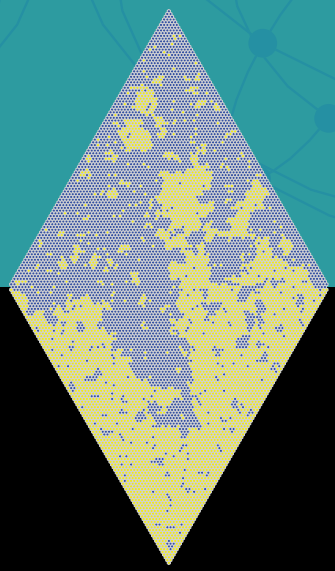
Second Edition

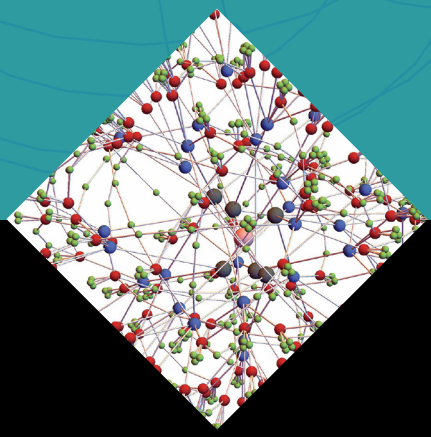
David A. Levin

Yuval Peres

With contributions by
Elizabeth L. Wilmer







AMS
American Mathematical Society

10.1090/mbk/107

Markov Chains and Mixing Times

Second Edition

David A. Levin
University of Oregon

Yuval Peres
Microsoft Research

With contributions by
Elizabeth L. Wilmer

With a chapter on "Coupling from the Past" by
James G. Propp and David B. Wilson

AMS
American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification.* Primary 60J10, 60J27, 60B15, 60C05, 65C05, 60K35, 68W20, 68U20, 82C22.

FRONT COVER: The figure on the bottom left of the front cover, courtesy of David B. Wilson, is a uniformly random lozenge tiling of a hexagon (see Section 25.2). The figure on the bottom center, also from David B. Wilson, is a random sample of an Ising model at its critical temperature (see Sections 3.3.5 and 25.2) with mixed boundary conditions. The figure on the bottom right, courtesy of Eyal Lubetzky, is a portion of an expander graph (see Section 13.6).

For additional information and updates on this book, visit
www.ams.org/bookpages/mbk-107

Library of Congress Cataloging-in-Publication Data

Names: Levin, David Asher, 1971- | Peres, Y. (Yuval) | Wilmer, Elizabeth L. (Elizabeth Lee), 1970- | Propp, James, 1960- | Wilson, David B. (David Bruce)

Title: Markov chains and mixing times / David A. Levin, Yuval Peres ; with contributions by Elizabeth L. Wilmer.

Description: Second edition. | Providence, Rhode Island : American Mathematical Society, [2017] | "With a chapter on Coupling from the past, by James G. Propp and David B. Wilson." | Includes bibliographical references and indexes.

Identifiers: LCCN 2017017451 | ISBN 9781470429621 (alk. paper)

Subjects: LCSH: Markov processes–Textbooks. | Distribution (Probability theory)–Textbooks. | AMS: Probability theory and stochastic processes – Markov processes – Markov chains (discrete-time Markov processes on discrete state spaces). msc | Probability theory and stochastic processes – Markov processes – Continuous-time Markov processes on discrete state spaces. msc | Probability theory and stochastic processes – Probability theory on algebraic and topological structures – Probability measures on groups or semigroups, Fourier transforms, factorization. msc | Probability theory and stochastic processes – Combinatorial probability – Combinatorial probability. msc | Numerical analysis – Probabilistic methods, simulation and stochastic differential equations – Monte Carlo methods. msc | Probability theory and stochastic processes – Special processes – Interacting random processes; statistical mechanics type models; percolation theory. msc | Computer science – Algorithms – Randomized algorithms. msc | Computer science – Computing methodologies and applications – Simulation. msc | Statistical mechanics, structure of matter – Time-dependent statistical mechanics (dynamic and nonequilibrium) – Interacting particle systems. msc

Classification: LCC QA274.7 .L48 2017 | DDC 519.2/33–dc23 LC record available at https://lccn.loc.gov/2017017451

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 17 16 15 14 13 12

Contents

Preface

Preface to the Second Edition

Since the publication of the first edition, the field of mixing times has continued to enjoy rapid expansion. In particular, many of the open problems posed in the first edition have been solved. The book has been used in courses at numerous universities, motivating us to update it.

In the eight years since the first edition appeared, we have made corrections and improvements throughout the book. We added three new chapters: Chapter 22 on monotone chains, Chapter 23 on the exclusion process, and Chapter 24, which relates mixing times and hitting time parameters to stationary stopping times. Chapter 4 now includes an introduction to mixing times in ℓ^p, which reappear in Chapter 10. The latter chapter has several new topics, including estimates for hitting times on trees and Eulerian digraphs. A bound for cover times using spanning trees has been added to Chapter 11, which also now includes a general bound on cover times for regular graphs. The exposition in Chapter 6 and Chapter 17 now employs filtrations rather than relying on the random mapping representation. To reflect the key developments since the first edition, especially breakthroughs on the Ising model and the cutoff phenomenon, the Notes at the end of chapters and the open problems have been updated.

We thank the many careful readers who sent us comments and corrections: Anselm Adelmann, Amitabha Bagchi, Nathanael Berestycki, Olena Bormashenko, Krzysztof Burdzy, Gerandy Brito, Darcy Camargo, Varsha Dani, Sukhada Fadnavis, Tertuliano Franco, Alan Frieze, Reza Gheissari, Jonathan Hermon, Ander Holroyd, Kenneth Hu, John Jiang, Svante Janson, Melvin Kianmanesh Rad, Yin Tat Lee, Zhongyang Li, Eyal Lubetzky, Abbas Mehrabian, R. Misturini, L. Morgado, Asaf Nachmias, Fedja Nazarov, Joe Neeman, Ross Pinsky, Anthony Quas, Miklos Racz, Dinah Shender, N. J. A. Sloane, Jeff Steif, Izabella Stuhl, Jan Swart, Ryokichi Tanaka, Daniel Wu, and Zhen Zhu. We are particularly grateful to Daniel Jerison, Pawel Pralat, and Perla Sousi, who sent us long lists of insightful comments.

Preface to the First Edition

Markov first studied the stochastic processes that came to be named after him in 1906. Approximately a century later, there is an active and diverse interdisciplinary community of researchers using Markov chains in computer science, physics, statistics, bioinformatics, engineering, and many other areas.

The classical theory of Markov chains studied *fixed* chains, and the goal was to estimate the rate of convergence to stationarity of the distribution at time t, as $t \to \infty$. In the past two decades, as interest in chains with large state spaces has increased, a different asymptotic analysis has emerged. Some target distance to

the stationary distribution is prescribed; the number of steps required to reach this target is called the *mixing time* of the chain. Now, the goal is to understand how the mixing time grows as the size of the state space increases.

The modern theory of Markov chain mixing is the result of the convergence, in the 1980s and 1990s, of several threads. (We mention only a few names here; see the chapter Notes for references.)

For statistical physicists Markov chains become useful in Monte Carlo simulation, especially for models on finite grids. The mixing time can determine the running time for simulation. However, Markov chains are used not only for simulation and sampling purposes, but also as models of dynamical processes. Deep connections were found between rapid mixing and spatial properties of spin systems, e.g., by Dobrushin, Shlosman, Stroock, Zegarlinski, Martinelli, and Olivieri.

In theoretical computer science, Markov chains play a key role in sampling and approximate counting algorithms. Often the goal was to prove that the mixing time is polynomial in the logarithm of the state space size. (In this book, we are generally interested in more precise asymptotics.)

At the same time, mathematicians including Aldous and Diaconis were intensively studying card shuffling and other random walks on groups. Both spectral methods and probabilistic techniques, such as coupling, played important roles. Alon and Milman, Jerrum and Sinclair, and Lawler and Sokal elucidated the connection between eigenvalues and expansion properties. Ingenious constructions of "expander" graphs (on which random walks mix especially fast) were found using probability, representation theory, and number theory.

In the 1990s there was substantial interaction between these communities, as computer scientists studied spin systems and as ideas from physics were used for sampling combinatorial structures. Using the geometry of the underlying graph to find (or exclude) bottlenecks played a key role in many results.

There are many methods for determining the asymptotics of convergence to stationarity as a function of the state space size and geometry. We hope to present these exciting developments in an accessible way.

We will only give a taste of the applications to computer science and statistical physics; our focus will be on the common underlying mathematics. The prerequisites are all at the undergraduate level. We will draw primarily on probability and linear algebra, but we will also use the theory of groups and tools from analysis when appropriate.

Why should mathematicians study Markov chain convergence? First of all, it is a lively and central part of modern probability theory. But there are ties to several other mathematical areas as well. The behavior of the random walk on a graph reveals features of the graph's geometry. Many phenomena that can be observed in the setting of finite graphs also occur in differential geometry. Indeed, the two fields enjoy active cross-fertilization, with ideas in each playing useful roles in the other. Reversible finite Markov chains can be viewed as resistor networks; the resulting discrete potential theory has strong connections with classical potential theory. It is amusing to interpret random walks on the symmetric group as card shuffles—and real shuffles have inspired some extremely serious mathematics—but these chains are closely tied to core areas in algebraic combinatorics and representation theory.

In the spring of 2005, mixing times of finite Markov chains were a major theme of the multidisciplinary research program *Probability, Algorithms, and Statistical Physics,* held at the Mathematical Sciences Research Institute. We began work on this book there.

Overview

We have divided the book into two parts.

In **Part I**, the focus is on techniques, and the examples are illustrative and accessible. Chapter 1 defines Markov chains and develops the conditions necessary for the existence of a unique stationary distribution. Chapters 2 and 3 both cover examples. In Chapter 2, they are either classical or useful—and generally both; we include accounts of several chains, such as the gambler's ruin and the coupon collector, that come up throughout probability. In Chapter 3, we discuss Glauber dynamics and the Metropolis algorithm in the context of "spin systems." These chains are important in statistical mechanics and theoretical computer science.

Chapter 4 proves that, under mild conditions, Markov chains do, in fact, converge to their stationary distributions and defines *total variation distance* and *mixing time*, the key tools for quantifying that convergence. The techniques of Chapters 5, 6, and 7, on coupling, strong stationary times, and methods for lower bounding distance from stationarity, respectively, are central to the area.

In Chapter 8, we pause to examine card shuffling chains. Random walks on the symmetric group are an important mathematical area in their own right, but we hope that readers will appreciate a rich class of examples appearing at this stage in the exposition.

Chapter 9 describes the relationship between random walks on graphs and electrical networks, while Chapters 10 and 11 discuss hitting times and cover times.

Chapter 12 introduces eigenvalue techniques and discusses the role of the relaxation time (the reciprocal of the spectral gap) in the mixing of the chain.

In **Part II**, we cover more sophisticated techniques and present several detailed case studies of particular families of chains. Much of this material appears here for the first time in textbook form.

Chapter 13 covers advanced spectral techniques, including comparison of Dirichlet forms and Wilson's method for lower bounding mixing.

Chapters 14 and 15 cover some of the most important families of "large" chains studied in computer science and statistical mechanics and some of the most important methods used in their analysis. Chapter 14 introduces the path coupling method, which is useful in both sampling and approximate counting. Chapter 15 looks at the Ising model on several different graphs, both above and below the critical temperature.

Chapter 16 revisits shuffling, looking at two examples—one with an application to genomics—whose analysis requires the spectral techniques of Chapter 13.

Chapter 17 begins with a brief introduction to martingales and then presents some applications of the evolving sets process.

Chapter 18 considers the cutoff phenomenon. For many families of chains where we can prove sharp upper and lower bounds on mixing time, the distance from stationarity drops from near 1 to near 0 over an interval asymptotically smaller than the mixing time. Understanding why cutoff is so common for families of interest is a central question.

Chapter 19, on lamplighter chains, brings together methods presented throughout the book. There are many bounds relating parameters of lamplighter chains to parameters of the original chain: for example, the mixing time of a lamplighter chain is of the same order as the cover time of the base chain.

Chapters 20 and 21 introduce two well-studied variants on finite discrete time Markov chains: continuous time chains and chains with countable state spaces. In both cases we draw connections with aspects of the mixing behavior of finite discrete-time Markov chains.

Chapter 25, written by Propp and Wilson, describes the remarkable construction of coupling from the past, which can provide exact samples from the stationary distribution.

Chapter 26 closes the book with a list of open problems connected to material covered in the book.

For the Reader

Starred sections, results, and chapters contain material that either digresses from the main subject matter of the book or is more sophisticated than what precedes them and may be omitted.

Exercises are found at the ends of chapters. Some (especially those whose results are applied in the text) have solutions at the back of the book. We of course encourage you to try them yourself first!

The Notes at the ends of chapters include references to original papers, suggestions for further reading, and occasionally "complements." These generally contain related material not required elsewhere in the book—sharper versions of lemmas or results that require somewhat greater prerequisites.

The Notation Index at the end of the book lists many recurring symbols.

Much of the book is organized by method, rather than by example. The reader may notice that, in the course of illustrating techniques, we return again and again to certain families of chains—random walks on tori and hypercubes, simple card shuffles, proper colorings of graphs. In our defense we offer an anecdote.

In 1991 one of us (Y. Peres) arrived as a postdoc at Yale and visited Shizuo Kakutani, whose rather large office was full of books and papers, with bookcases and boxes from floor to ceiling. A narrow path led from the door to Kakutani's desk, which was also overflowing with papers. Kakutani admitted that he sometimes had difficulty locating particular papers, but he proudly explained that he had found a way to solve the problem. He would make four or five copies of any really interesting paper and put them in different corners of the office. When searching, he would be sure to find at least one of the copies. . . .

Cross-references in the text and the Index should help you track earlier occurrences of an example. You may also find the chapter dependency diagrams below useful.

We have included brief accounts of some background material in Appendix A. These are intended primarily to set terminology and notation, and we hope you will consult suitable textbooks for unfamiliar material.

Be aware that we occasionally write symbols representing a real number when an integer is required (see, e.g., the $\binom{n}{\delta k}$'s in the proof of Proposition 13.37). We

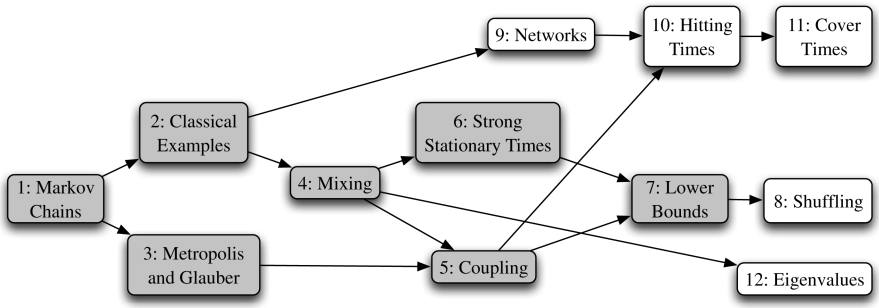
hope the reader will realize that this omission of floor or ceiling brackets (and the details of analyzing the resulting perturbations) is in her or his best interest as much as it is in ours.

For the Instructor

The prerequisites this book demands are a first course in probability, linear algebra, and, inevitably, a certain degree of mathematical maturity. When introducing material which is standard in other undergraduate courses—e.g., groups—we provide definitions but often hope the reader has some prior experience with the concepts.

In Part I, we have worked hard to keep the material accessible and engaging for students. (Starred material is more sophisticated and is not required for what follows immediately; they can be omitted.)

Here are the dependencies among the chapters of Part I:

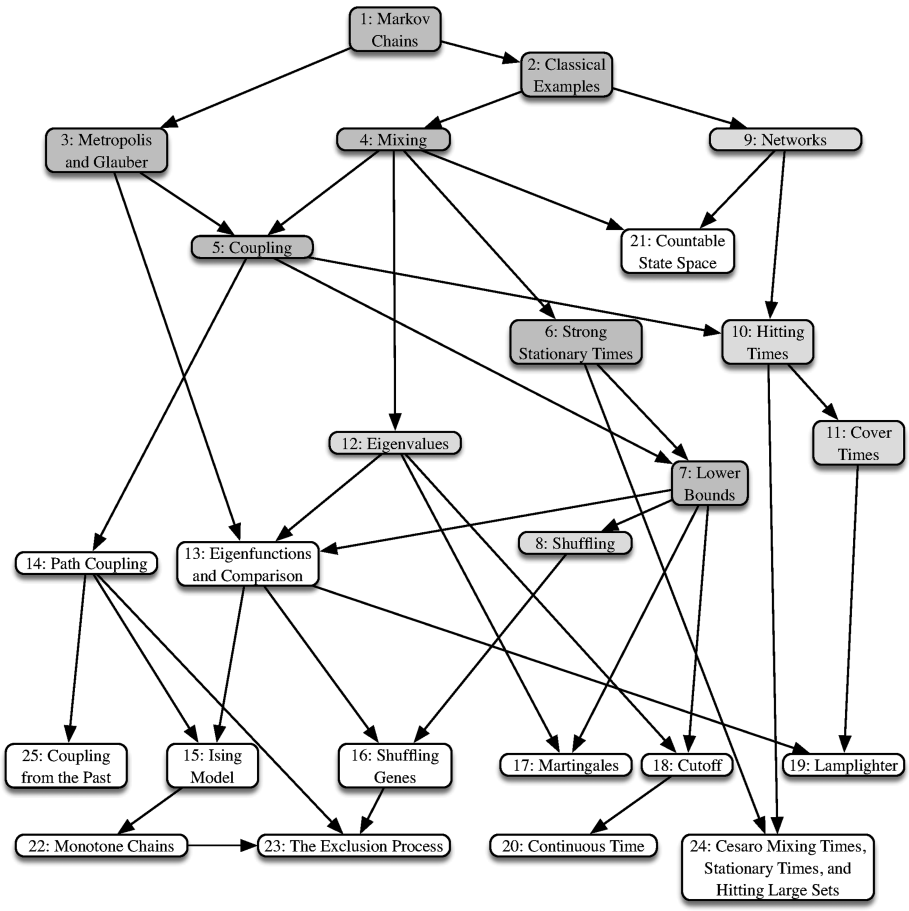


Chapters 1 through 7, shown in gray, form the core material, but there are several ways to proceed afterwards. Chapter 8 on shuffling gives an early rich application but is not required for the rest of Part I. A course with a probabilistic focus might cover Chapters 9, 10, and 11. To emphasize spectral methods and combinatorics, cover Chapters 8 and 12 and perhaps continue on to Chapters 13 and 16.

While our primary focus is on chains with finite state spaces run in discrete time, continuous-time and countable-state-space chains are both discussed—in Chapters 20 and 21, respectively.

We have also included Appendix B, an introduction to simulation methods, to help motivate the study of Markov chains for students with more applied interests. A course leaning towards theoretical computer science and/or statistical mechanics might start with Appendix B, cover the core material, and then move on to Chapters 14, 15, and 25.

Of course, depending on the interests of the instructor and the ambitions and abilities of the students, any of the material can be taught! Below we include a full diagram of dependencies of chapters. Its tangled nature results from the interconnectedness of the area: a given technique can be applied in many situations, while a particular problem may require several techniques for full analysis.



The logical dependencies of chapters. The core Chapters 1 through 7 are in dark gray, the rest of Part I is in light gray, and Part II is in white.

For the Expert

Several other recent books treat Markov chain mixing. Our account is more comprehensive than those of **Häggström (2002)**, **Jerrum (2003)**, or **Montenegro and Tetali (2006)**, yet not as exhaustive as **Aldous and Fill (1999)**. **Norris (1998)** gives an introduction to Markov chains and their applications but does not focus on mixing. Since this is a textbook, we have aimed for accessibility and comprehensibility, particularly in Part I.

What is different or novel in our approach to this material?

- Our approach is probabilistic whenever possible. We also integrate "classical" material on networks, hitting times, and cover times and demonstrate its usefulness for bounding mixing times.
- We provide an introduction to several major statistical mechanics models, most notably the Ising model, and collect results on them in one place.

- We give expository accounts of several modern techniques and examples, including evolving sets, the cutoff phenomenon, lamplighter chains, and the L-reversal chain.
- We systematically treat lower bounding techniques, including several applications of Wilson's method.
- We use the transportation metric to unify our account of path coupling and draw connections with earlier history.
- We present an exposition of coupling from the past by Propp and Wilson, the originators of the method.

Acknowledgements

The authors thank the Mathematical Sciences Research Institute, the National Science Foundation VIGRE grant to the Department of Statistics at the University of California, Berkeley, and National Science Foundation grants DMS-0244479 and DMS-0104073 for support. We also thank Hugo Rossi for suggesting we embark on this project. Thanks to Blair Ahlquist, Tonci Antunovic, Elisa Celis, Paul Cuff, Jian Ding, Ori Gurel-Gurevich, Tom Hayes, Itamar Landau, Yun Long, Karola Mészáros, Shobhana Murali, Weiyang Ning, Tomoyuki Shirai, Walter Sun, Sithparran Vanniasegaram, and Ariel Yadin for corrections to an earlier version and making valuable suggestions. Yelena Shvets made the illustration in Section 6.5.4. The simulations of the Ising model in Chapter 15 are due to Raissa D'Souza. We thank László Lovász for useful discussions. We are indebted to Alistair Sinclair for his work co-organizing the M.S.R.I. program *Probability, Algorithms, and Statistical Physics* in 2005, where work on this book began. We thank Robert Calhoun for technical assistance.

Finally, we are greatly indebted to David Aldous and Persi Diaconis, who initiated the modern point of view on finite Markov chains and taught us much of what we know about the subject.

Bibliography

The pages on which a reference appears follow the symbol ↑.

Ahlfors, L. V. 1978. *Complex analysis*, 3rd ed., McGraw-Hill Book Co., New York. An introduction to the theory of analytic functions of one complex variable; International Series in Pure and Applied Mathematics. ↑126

Aldous, D. J. 1982. *Some inequalities for reversible Markov chains*, J. London Math. Soc. (2) **25**, no. 3, 564–576. ↑337, 348

Aldous, D. J. 1983a. *On the time taken by random walks on finite groups to visit every state*, Z. Wahrsch. Verw. Gebiete **62**, no. 3, 361–374. ↑158

Aldous, D. J. 1983b. *Random walks on finite groups and rapidly mixing Markov chains*, Seminar on probability, XVII, Lecture Notes in Math., vol. 986, Springer, Berlin, pp. 243–297. ↑58, 242

Aldous, D. J. 1989a. *An introduction to covering problems for random walks on graphs*, J. Theoret. Probab. **2**, no. 1, 87–89. ↑158

Aldous, D. J. 1989b. *Lower bounds for covering times for reversible Markov chains and random walks on graphs*, J. Theoret. Probab. **2**, no. 1, 91–100. ↑158

Aldous, D. J. 1990. *A random walk construction of uniform spanning trees and uniform labelled trees*, SIAM Journal on Discrete Mathematics **3**, 450–465. ↑359

Aldous, D. J. 1991a. *Random walk covering of some special trees*, J. Math. Anal. Appl. **157**, no. 1, 271–283. ↑157

Aldous, D. J. 1991b. *Threshold limits for cover times*, J. Theoret. Probab. **4**, no. 1, 197–211. ↑158, 279

Aldous, D. J. 1995. *On simulating a Markov chain stationary distribution when transition probabilities are unknown* (D. Aldous, P. Diaconis, J. Spencer, and J. M. Steele, eds.), IMA Volumes in Mathematics and Its Applications, vol. 72, Springer-Verlag. ↑358

Aldous, D. J. 1999. unpublished note. ↑199

Aldous, D. J. 2004, American Institute of Mathematics (AIM) research workshop "Sharp Thresholds for Mixing Times" (Palo Alto, December 2004), available at `http://www.aimath.org/WWN/mixingtimes`. ↑271

Aldous, D. and P. Diaconis. 1986. *Shuffling cards and stopping times*, Amer. Math. Monthly **93**, no. 5, 333–348. ↑58, 86, 95, 113

Aldous, D. and P. Diaconis. 1987. *Strong uniform times and finite random walks*, Adv. in Appl. Math. **8**, no. 1, 69–97. ↑86

Aldous, D. and P. Diaconis. 2002. *The asymmetric one-dimensional constrained Ising model: Rigorous results*, J. Statist. Phys. **107**, no. 5-6, 945–975. ↑98

Aldous, D. and J. Fill. 1999. *Reversible Markov chains and random walks on graphs*, available at `http://www.stat.berkeley.edu/~aldous/RWG/book.html`. ↑xiv, 19, 58, 141, 147, 149, 158, 360

Aleliunas, R., R. M. Karp, R. J. Lipton, L. Lovász, and C. Rackoff. 1979. *Random walks, universal traversal sequences, and the complexity of maze problems*, 20th Annual Symposium on Foundations of Computer Science (San Juan, Puerto Rico, 1979), IEEE, New York, pp. 218–223. ↑158

Alon, N. 1986. *Eigenvalues and expanders*, Combinatorica **6**, no. 2, 83–96. ↑199

Alon, N. and V. D. Milman. 1985. λ_1, *isoperimetric inequalities for graphs, and superconcentrators*, J. Combin. Theory Ser. B **38**, no. 1, 73–88. ↑199

Alon, N. and J. H. Spencer. 2008. *The probabilistic method*, 3rd ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., Hoboken, NJ. With an appendix on the life and work of Paul Erdős. ↑319

Anantharam, V. and P. Tsoucas. 1989. *A proof of the Markov chain tree theorem*, Statistics and Probability Letters **8**, 189–192. ↑359

Angel, O., Y. Peres, and D. B. Wilson. 2008. *Card shuffling and Diophantine approximation*, Ann. Appl. Probab. **18**, no. 3, 1215–1231. ↑

Archer, A. F. 1999. *A modern treatment of the 15 puzzle*, Amer. Math. Monthly **106**, no. 9, 793–799. ↑404

Artin, M. 1991. *Algebra*, Prentice Hall Inc., Englewood Cliffs, NJ. ↑35, 112

Asmussen, S., P. Glynn, and H. Thorisson. 1992. *Stationary detection in the initial transient problem*, ACM Transactions on Modeling and Computer Simulation **2**, 130–157. ↑358

Balázs, M. and Á. Folly. 2016. *An electric network for nonreversible Markov chains*, Amer. Math. Monthly **123**, no. 7, 657–682. ↑148

Barlow, M. T. 2017. *Random walks and heat kernels on graphs*, Cambridge University Press. ↑305

Barlow, M. T., T. Coulhon, and T. Kumagai. 2005. *Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs*, Comm. Pure Appl. Math. **58**, no. 12, 1642–1677. ↑305

Barnes, G. and U. Feige. 1996. *Short random walks on graphs*, SIAM J. Discrete Math. **9**, no. 1, 19–28. ↑158

Barrera, J., B. Lachaud, and B. Ycart. 2006. *Cut-off for n-tuples of exponentially converging processes*, Stochastic Process. Appl. **116**, no. 10, 1433–1446. ↑291

Basharin, G. P., A. N. Langville, and V. A. Naumov. 2004. *The life and work of A. A. Markov*, Linear Algebra Appl. **386**, 3–26. ↑19

Basu, R., J. Hermon, and Y. Peres. 2017. *Characterization of cutoff for reversible Markov chains*, Ann. Probab. **45**, no. 3, 1448–1487. ↑272

Baxter, J. R. and R. V. Chacon. 1976. *Stopping times for recurrent Markov processes*, Illinois J. Math. **20**, no. 3, 467–475. ↑

Baxter, R. J. 1982. *Exactly solved models in statistical mechanics*, Academic Press. ↑350

Bayer, D. and P. Diaconis. 1992. *Trailing the dovetail shuffle to its lair*, Ann. Appl. Probab. **2**, no. 2, 294–313. ↑111, 113, 114, 178

Ben-Hamou, A. and J. Salez. 2017. *Cutoff for non-backtracking random walks on sparse random graphs*, Ann. Probab. **45**, no. 3, 1752–1770. ↑272

Benjamin, A. T. and J. J. Quinn. 2003. *Proofs that really count: The art of combinatorial proof*, Dolciani Mathematical Expositions, vol. 27, Math. Assoc. Amer., Washington, D. C. ↑210

Benjamini, I., N. Berger, C. Hoffman, and E. Mossel. 2005. *Mixing times of the biased card shuffling and the asymmetric exclusion process*, Trans. Amer. Math. Soc. **357**, no. 8, 3013–3029 (electronic). ↑335

Berestycki, N., E. Lubetzky, Y. Peres, and A. Sly. 2018. *Random walks on the random graph*, Annals of Probability, to appear. ↑272

Berger, N., C. Kenyon, E. Mossel, and Y. Peres. 2005. *Glauber dynamics on trees and hyperbolic graphs*, Probab. Theory Related Fields **131**, no. 3, 311–340. ↑230

Bernstein, M. and E. Nestoridi. 2017. *Cutoff for random to random card shuffle*, available at `https://arxiv.org/abs/1703.06210`. ↑363

Billingsley, P. 1995. *Probability and measure*, 3rd ed., Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons Inc., New York. ↑365, 371

Boczkowski, L., Y. Peres, and P. Sousi. 2018. *Sensitivity of mixing times in Eulerian digraphs*, SIAM Journal Discrete Math, to appear. ↑158

Borgs, C., J. T. Chayes, A. Frieze, J. H. Kim, P. Tetali, E. Vigoda, and V. H. Vu. 1999. *Torpid mixing of some Monte Carlo Markov chain algorithms in statistical physics*, 40th Annual Symposium on Foundations of Computer Science (New York, 1999), IEEE Computer Soc., Los Alamitos, CA, pp. 218–229, DOI 10.1109/SFFCS.1999.814594. MR1917562 ↑232

Borgs, C., J. T. Chayes, and P. Tetali. 2012. *Tight bounds for mixing of the Swendsen-Wang algorithm at the Potts transition point*, Probab. Theory Related Fields **152**, no. 3-4, 509–557, DOI 10.1007/s00440-010-0329-0. MR2892955 ↑232

Borovkov, A. A. and S. G. Foss. 1992. *Stochastically recursive sequences and their generalizations*, Siberian Advances in Mathematics **2**, 16–81. ↑350

Bodineau, T. 2005. *Slab percolation for the Ising model*, Probab. Theory Related Fields **132**, no. 1, 83–118. ↑231

Bollobás, B. 1988. *The isoperimetric number of random regular graphs*, European J. Combin. **9**, no. 3, 241–244, DOI 10.1016/S0195-6698(88)80014-3. MR947025 ↑200

Bollobás, B. 1998. *Modern graph theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York. ↑126

Boucheron, S., G. Lugosi, and P. Massart. 2013. *Concentration inequalities*, Oxford University Press, Oxford. A nonasymptotic theory of independence. With a foreword by Michel Ledoux. ↑200

Brémaud, P. 1999. *Markov chains*, Texts in Applied Mathematics, vol. 31, Springer-Verlag, New York. Gibbs fields, Monte Carlo simulation, and queues. ↑45

Broder, A. 1989. *Generating random spanning trees*, 30th Annual Symposium on Foundations of Computer Science, 442–447. ↑359

Broder, A. Z. and A. R. Karlin. 1989. *Bounds on the cover time*, J. Theoret. Probab. **2**, no. 1, 101–120. ↑158

Bubley, R. and M. Dyer. 1997. *Path coupling: A technique for proving rapid mixing in Markov chains*, Proceedings of the 38th Annual Symposium on Foundations of Computer Science, pp. 223–231. ↑203, 204

Cancrini, N., P. Caputo, and F. Martinelli. 2006. *Relaxation time of L-reversal chains and other chromosome shuffles*, Ann. Appl. Probab. **16**, no. 3, 1506–1527. ↑242

Cancrini, N., F. Martinelli, C. Roberto, and C. Toninelli. 2008. *Kinetically constrained spin models*, Probab. Theory Related Fields **140**, no. 3-4, 459–504. ↑98

Caputo, P., T. M. Liggett, and T. Richthammer. 2010. *Proof of Aldous' spectral gap conjecture*, J. Amer. Math. Soc. **23**, no. 3, 831–851. ↑335, 363

Caputo, P., E. Lubetzky, F. Martinelli, A. Sly, and F. L. Toninelli. 2014. *Dynamics of* $(2+1)$*-dimensional SOS surfaces above a wall: Slow mixing induced by entropic repulsion*, Ann. Probab. **42**, no. 4, 1516–1589. ↑323

Carne, T. K. 1985. *A transmutation formula for Markov chains* (English, with French summary), Bull. Sci. Math. (2) **109**, no. 4, 399–405. ↑98

Cerf, R. and A. Pisztora. 2000. *On the Wulff crystal in the Ising model*, Ann. Probab. **28**, no. 3, 947–1017. ↑

Cesi, F., G. Guadagni, F. Martinelli, and R. H. Schonmann. 1996. *On the two-dimensional stochastic Ising model in the phase coexistence region near the critical point*, J. Statist. Phys. **85**, no. 1-2, 55–102. ↑231

Chandra, A. K., P. Raghavan, W. L. Ruzzo, R. Smolensky, and P. Tiwari. 1996. *The electrical resistance of a graph captures its commute and cover times*, Comput. Complexity **6**, no. 4, 312–340. Extended abstract originally published in *Proc. 21st ACM Symp. Theory of Computing* (1989) 574–586. ↑147

Chayes, J. T., L. Chayes, and R. H. Schonmann. 1987. *Exponential decay of connectivities in the two-dimensional Ising model*, J. Statist. Phys. **49**, no. 3-4, 433–445. ↑230

Cheeger, J. 1970. *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in analysis (Papers dedicated to Salomon Bochner, 1969), Princeton Univ. Press, Princeton, pp. 195–199. ↑199

Chen, F., L. Lovász, and I. Pak. 1999. *Lifting Markov chains to speed up mixing*, Annual ACM Symposium on Theory of Computing (Atlanta, GA, 1999), ACM, New York, pp. 275–281 (electronic). ↑98

Chen, G.-Y. and L. Saloff-Coste. 2008. *The cutoff phenomenon for ergodic Markov processes*, Electron. J. Probab. **13**, no. 3, 26–78. ↑270

Chen, G.-Y. and L. Saloff-Coste. 2013. *Comparison of cutoffs between lazy walks and Markovian semigroups*, J. Appl. Probab. **50**, no. 4, 943–959. ↑291

Chen, M.-F. 1998. *Trilogy of couplings and general formulas for lower bound of spectral gap*, Probability towards 2000 (New York, 1995), Lecture Notes in Statist., vol. 128, Springer, New York, pp. 123–136. ↑180, 274

Chung, F., P. Diaconis, and R. Graham. 2001. *Combinatorics for the East model*, Adv. in Appl. Math. **27**, no. 1, 192–206. ↑98

Chyakanavichyus, V. and P. Vaĭtkus. 2001. *Centered Poisson approximation by the Stein method* (Russian, with Russian and Lithuanian summaries), Liet. Mat. Rink. **41**, no. 4, 409–423; English transl.,. 2001, Lithuanian Math. J. **41**, no. 4, 319–329. ↑291

Coppersmith, D., U. Feige, and J. Shearer. 1996. *Random walks on regular and irregular graphs*, SIAM J. Discrete Math. **9**, no. 2, 301–308. ↑158

Coppersmith, D., P. Tetali, and P. Winkler. 1993. *Collisions among random walks on a graph*, SIAM J. Discrete Math. **6**, no. 3, 363–374. ↑148

Cox, J. T., Y. Peres, and J. E. Steif. 2016. *Cutoff for the noisy voter model*, Ann. Appl. Probab. **26**, no. 2, 917–932. ↑230

Cuff, P., J. Ding, O. Louidor, E. Lubetzky, Y. Peres, and A. Sly. 2012. *Glauber dynamics for the mean-field Potts model*, J. Stat. Phys. **149**, no. 3, 432–477. ↑232

Dembo, A., J. Ding, J. Miller, and Y. Peres. 2013. *Cut-off for lamplighter chains on tori: Dimension interpolation and phase transition*, arXiv preprint arXiv:1312.4522. ↑279

Dembo, A., Y. Peres, J. Rosen, and O. Zeitouni. 2004. *Cover times for Brownian motion and random walk in two dimensions*, Ann. Math. **160**, 433–464. ↑158

Denardo, E. V. 1977. *Periods of connected networks and powers of nonnegative matrices*, Mathematics of Operations Research **2**, no. 1, 20–24. ↑20

Devroye, L. 1986. *Nonuniform random variate generation*, Springer-Verlag, New York. ↑391

Diaconis, P. 1988a. *Group representations in probability and statistics*, Lecture Notes—Monograph Series, vol. 11, Inst. Math. Stat., Hayward, CA. ↑35, 105, 112, 113, 177

Diaconis, P. 1988b. *Applications of noncommutative Fourier analysis to probability problems*, École d'Été de Probabilités de Saint-Flour XV–XVII, 1985–87, Lecture Notes in Math., vol. 1362, Springer, Berlin, pp. 51–100. ↑272

Diaconis, P. 1996. *The cutoff phenomenon in finite Markov chains*, Proc. Nat. Acad. Sci. U.S.A. **93**, no. 4, 1659–1664. ↑

Diaconis, P. 2003. *Mathematical developments from the analysis of riffle shuffling*, Groups, combinatorics & geometry (Durham, 2001), World Sci. Publ., River Edge, NJ, pp. 73–97. ↑114

Diaconis, P. 2013. *Some things we've learned (about Markov chain Monte Carlo)*, Bernoulli **19**, no. 4, 1294–1305. ↑323

Diaconis, P. and J. A. Fill. 1990. *Strong stationary times via a new form of duality*, Ann. Probab. **18**, no. 4, 1483–1522. ↑86, 177, 258, 270

Diaconis, P. and D. Freedman. 1999. *Iterated random functions*, SIAM Review **41**, 45–76. ↑350

Diaconis, P., M. McGrath, and J. Pitman. 1995. *Riffle shuffles, cycles, and descents*, Combinatorica **15**, no. 1, 11–29. ↑114

Diaconis, P. and L. Saloff-Coste. 1993a. *Comparison theorems for reversible Markov chains*, Ann. Appl. Probab. **3**, no. 3, 696–730. ↑200

Diaconis, P. and L. Saloff-Coste. 1993b. *Comparison techniques for random walk on finite groups*, Ann. Probab. **21**, no. 4, 2131–2156. ↑200, 242

Diaconis, P. and L. Saloff-Coste. 1996a. *Nash inequalities for finite Markov chains*, J. Theoret. Probab. **9**, no. 2, 459–510. ↑148

Diaconis, P. and L. Saloff-Coste. 1996b. *Logarithmic Sobolev inequalities for finite Markov chains*, Ann. Appl. Probab. **6**, no. 3, 695–750. ↑291

Diaconis, P. and L. Saloff-Coste. 1996c. *Walks on generating sets of abelian groups*, Probab. Theory Related Fields **105**, no. 3, 393–421. ↑362

Diaconis, P. and L. Saloff-Coste. 1998. *What do we know about the Metropolis algorithm?*, J. Comput. System Sci. **57**, no. 1, 20–36. 27th Annual ACM Symposium on the Theory of Computing (STOC'95) (Las Vegas, NV). ↑45

Diaconis, P. and L. Saloff-Coste. 2006. *Separation cut-offs for birth and death chains*, Ann. Appl. Probab. **16**, no. 4, 2098–2122. ↑270, 272, 363

Diaconis, P. and M. Shahshahani. 1981. *Generating a random permutation with random transpositions*, Z. Wahrsch. Verw. Gebiete **57**, no. 2, 159–179. ↑101, 164, 177, 242

Diaconis, P. and M. Shahshahani. 1987. *Time to reach stationarity in the Bernoulli-Laplace diffusion model*, SIAM J. Math. Anal. **18**, no. 1, 208–218. ↑272

Diaconis, P. and D. Stroock. 1991. *Geometric bounds for eigenvalues of Markov chains*, Ann. Appl. Probab. **1**, no. 1, 36–61. ↑199

Ding, J., J. R. Lee, and Y. Peres. 2012. *Cover times, blanket times, and majorizing measures*, Ann. of Math. (2) **175**, no. 3, 1409–1471. ↑158

Ding, J., E. Lubetzky, and Y. Peres. 2009. *The mixing time evolution of Glauber dynamics for the mean-field Ising model*, Comm. Math. Phys. **289**, no. 2, 725–764. ↑230

Ding, J., E. Lubetzky, and Y. Peres. 2010a. *Total variation cutoff in birth-and-death chains*, Probab. Theory Related Fields **146**, no. 1-2, 61–85. ↑271, 363

Ding, J., E. Lubetzky, and Y. Peres. 2010b. *Mixing time of critical Ising model on trees is polynomial in the height*, Comm. Math. Phys. **295**, no. 1, 161–207. ↑323

Ding, J. and Y. Peres. 2011. *Mixing time for the Ising model: A uniform lower bound for all graphs*, Ann. Inst. Henri Poincaré Probab. Stat. **47**, no. 4, 1020–1028, available at `arXiv:0909.5162v2`. ↑323, 362

Dobrushin, R., R. Kotecký, and S. Shlosman. 1992. *Wulff construction. A global shape from local interaction*, Translations of Mathematical Monographs, vol. 104, American Mathematical Society, Providence, RI. Translated from the Russian by the authors. ↑231

Dobrushin, R. L. and S. B. Shlosman. 1987. *Completely analytical interactions: Constructive description*, J. Statist. Phys. **46**, no. 5-6, 983–1014. ↑231

Doeblin, W. 1938. *Esposé de la théorie des chaînes simple constantes de Markov à un nombre fini d'états*, Rev. Math. Union Interbalkan. **2**, 77–105. ↑73

Doob, J. L. 1953. *Stochastic processes*, John Wiley & Sons Inc., New York. ↑260

Doyle, P. G. and E. J. Snell. 1984. *Random walks and electrical networks*, Carus Math. Monographs, vol. 22, Math. Assoc. Amer., Washington, D. C. ↑126, 303, 305

Doyle, P. G. and J. Steiner. 2011. *Commuting time geometry of ergodic Markov chains*, available at `arxiv:1107.2612`. ↑148

Dubins, L. E. 1968. *On a theorem of Skorohod*, Ann. Math. Statist. **39**, 2094–2097. ↑86

Dudley, R. M. 2002. *Real analysis and probability*, Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, Cambridge. Revised reprint of the 1989 original. ↑215

Durrett, R. 2003. *Shuffling chromosomes*, J. Theoret. Probab. **16**, 725–750. ↑237, 242

Durrett, R. 2005. *Probability: Theory and examples*, 3rd ed., Brooks/Cole, Belmont, CA. ↑284, 365

Dyer, M., L. A. Goldberg, and M. Jerrum. 2006a. *Dobrushin conditions and systematic scan*, Approximation, randomization and combinatorial optimization, Lecture Notes in Comput. Sci., vol. 4110, Springer, Berlin, pp. 327–338. ↑361

Dyer, M., L. A. Goldberg, and M. Jerrum. 2006b. *Systematic scan for sampling colorings*, Ann. Appl. Probab. **16**, no. 1, 185–230. ↑361

Dyer, M., L. A. Goldberg, M. Jerrum, and R. Martin. 2006. *Markov chain comparison*, Probab. Surv. **3**, 89–111 (electronic). ↑200

Dyer, M. and C. Greenhill. 2000. *On Markov chains for independent sets*, J. Algorithms **35**, no. 1, 17–49. ↑358

Dyer, M., C. Greenhill, and M. Molloy. 2002. *Very rapid mixing of the Glauber dynamics for proper colorings on bounded-degree graphs*, Random Structures Algorithms **20**, no. 1, 98–114. ↑214

Dyer, M., C. Greenhill, and M. Ullrich. 2014. *Structure and eigenvalues of heat-bath Markov chains*, Linear Algebra Appl. **454**, 57–71, DOI 10.1016/j.laa.2014.04.018. ↑178

Eggenberger, F. and G Pólya. 1923. *Über die Statistik vorketter vorgänge*, Zeit. Angew. Math. Mech. **3**, 279–289. ↑35

Einstein, A. 1934. *On the method of theoretical physics*, Philosophy of Science **1**, no. 2, 163–169. ↑1

Elias, P. 1972. *The efficient construction of an unbiased random sequence*, Ann. Math. Statist. **43**, 865–870. ↑391

Erdős, P. and A. Rényi. 1961. *On a classical problem of probability theory* (English, with Russian summary), Magyar Tud. Akad. Mat. Kutató Int. Közl. **6**, 215–220. ↑35

Feige, U. 1995a. *A tight lower bound on the cover time for random walks on graphs*, Random Structures Algorithms **6**, no. 4, 433–438. ↑158

Feige, U. 1995b. *A tight upper bound on the cover time for random walks on graphs*, Random Structures Algorithms **6**, no. 1, 51–54. ↑

Feige, U. 1997. *Collecting coupons on trees, and the cover time of random walks*, Comput. Complexity **6**, no. 4, 341–356. ↑158

Feller, W. 1968. *An introduction to probability theory and its applications*, 3rd ed., Vol. 1, Wiley, New York. ↑19, 35, 177, 305, 365

Fill, J. A. 1991. *Eigenvalue bounds on convergence to stationarity for nonreversible Markov chains, with an application to the exclusion process*, Ann. Appl. Probab. **1**, no. 1, 62–87. ↑98

Fill, J. A. 1998. *An interruptible algorithm for perfect sampling via Markov chains*, Annals of Applied Probability **8**, 131–162. ↑350, 354

Fill, J. A. 2009. *On hitting times and fastest strong stationary times for skip-free and more general chains*, J. Theoret. Probab. **22**, no. 3, 587–600. ↑177

Fill, J. A. and M. Huber. 2000. *The randomness recycler: A new technique for perfect sampling*, 41st Annual Symposium on Foundations of Computer Science, 503–511. ↑350

Fill, J. A., M. Machida, D. J. Murdoch, and J. S. Rosenthal. 2000. *Extension of Fill's perfect rejection sampling algorithm to general chains*, Random Structure and Algorithms **17**, 290–316. ↑350

Folland, G. B. 1999. *Real analysis*, 2nd ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York. Modern techniques and their applications; A Wiley-Interscience Publication. ↑59

Frieze, A. and E. Vigoda. 2007. *A survey on the use of Markov chains to randomly sample colourings*, Combinatorics, complexity, and chance, Oxford Lecture Ser. Math. Appl., vol. 34, Oxford Univ. Press, Oxford, pp. 53–71. ↑215

Ganapathy, M. K. 2007. *Robust mixing*, Electronic Journal of Probability **12**, 262–299. ↑113

Ganguly, S., E. Lubetzky, and F. Martinelli. 2015. *Cutoff for the east process*, Comm. Math. Phys. **335**, no. 3, 1287–1322. ↑272

Ganguly, S. and F. Martinelli. 2016. *Upper triangular matrix walk: Cutoff for finitely many columns*, available at `arXiv:1612.08741`. ↑364

Gaudillière, A. and C. Landim. 2014. *A Dirichlet principle for non reversible Markov chains and some recurrence theorems*, Probab. Theory Related Fields **158**, no. 1-2, 55–89, DOI 10.1007/s00440-012-0477-5. MR3152780 ↑148

Gheissari, R. and E. Lubetzky. 2016. *Mixing times of critical 2D Potts models*, available at `arxiv:1607.02182`. ↑232

Godbole, A. P. and S. G. Papastavridis (eds.) 1994. *Runs and patterns in probability: Selected papers*, Mathematics and Its Applications, vol. 283, Kluwer Academic Publishers Group, Dordrecht. ↑158

Graham, R. L., D. E. Knuth, and O. Patashnik. 1994. *Concrete mathematics: A foundation for computer science*, 2nd ed., Addison Wesley, Reading, Massachusetts. ↑111, 210

Greenberg, S., A. Pascoe, and D. Randall. 2009. *Sampling biased lattice configurations using exponential metrics*, ACM-SIAM Symposium on Discrete Algorithms (New York, New York, 2009), pp. 76-85. ↑335

Griffeath, D. 1974/75. *A maximal coupling for Markov chains*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **31**, 95–106. ↑73

Griffiths, S., R. J. Kang, R. Imbuzeiro Oliveira, and V. Patel. 2012. *Tight inequalities among set hitting times in Markov chains*, Proc. Amer. Math. Soc. ↑348

Grinstead, C. and L. Snell. 1997. *Introduction to probability*, 2nd revised ed., American Mathematical Society, Providence. ↑35, 58

Guo, H. and M. Jerrum. 2017. *Random cluster dynamics for the Ising model is rapidly mixing*, Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 1818–1827. ↑231

Häggström, O. 2002. *Finite Markov chains and algorithmic applications*, London Mathematical Society Student Texts, vol. 52, Cambridge University Press, Cambridge. ↑xiv, 58

Häggström, O. 2007. *Problem solving is often a matter of cooking up an appropriate Markov chain*, Scand. J. Statist. **34**, no. 4, 768–780. ↑46

Häggström, O. and J. Jonasson. 1997. *Rates of convergence for lamplighter processes*, Stochastic Process. Appl. **67**, no. 2, 227–249. ↑279

Häggström, O. and K. Nelander. 1998. *Exact sampling from anti-monotone systems*, Statist. Neerlandica **52**, no. 3, 360–380. ↑359

Häggström, O. and K. Nelander. 1999. *On exact simulation of Markov random fields using coupling from the past*, Scand. J. Statist. **26**, no. 3, 395–411. ↑356, 357

Hajek, B. 1988. *Cooling schedules for optimal annealing*, Math. Oper. Res. **13**, no. 2, 311–329. ↑45

Handjani, S. and D. Jungreis. 1996. *Rate of convergence for shuffling cards by transpositions*, J. Theoret. Probab. **9**, no. 4, 983–993. ↑363

Hastings, W. K. 1970. *Monte Carlo sampling methods using Markov chains and their applications*, Biometrika **57**, no. 1, 97 –109. ↑45

Hayes, T. P. and A. Sinclair. 2007. *A general lower bound for mixing of single-site dynamics on graphs*, Ann. Appl. Probab. **17**, no. 3, 931–952. ↑98, 323, 362

Hayes, T. P and E. Vigoda. 2003. *A non-Markovian coupling for randomly sampling colorings*, Proceedings of the 44th Annual IEEE Symposium on Foundations of Computer Science, 618–627. ↑74

Hermon, J., H. Lacoin, and Y. Peres. 2016. *Total variation and separation cutoffs are not equivalent and neither one implies the other*, Electron. J. Probab. **21**, Paper No. 44. ↑272, 363

Herstein, I. N. 1975. *Topics in algebra*, 2nd ed., John Wiley and Sons, New York. ↑35, 112

Hoeffding, W. 1963. *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. **58**, 13–30. MR0144363 (26 #1908) ↑369

Holley, R. 1974. *Remarks on the* FKG *inequalities*, Comm. Math. Phys. **36**, 227–231. ↑323

Holley, R. and D. Stroock. 1988. *Simulated annealing via Sobolev inequalities*, Comm. Math. Phys. **115**, no. 4, 553–569. ↑45

Holroyd, A. E. 2011. *Some circumstances where extra updates can delay mixing*, J. Stat. Phys. **145**, no. 6, 1649–1652. ↑323

Hoory, S., N. Linial, and A. Wigderson. 2006. *Expander graphs and their applications*, Bull. Amer. Math. Soc. (N.S.) **43**, no. 4, 439–561 (electronic). ↑200

Horn, R. A. and C. R. Johnson. 1990. *Matrix analysis*, Cambridge University Press, Cambridge. ↑177, 376

Huber, M. 1998. *Exact sampling and approximate counting techniques*, Proceedings of the 30th Annual ACM Symposium on the Theory of Computing, 31–40. ↑356

Ioffe, D. 1995. *Exact large deviation bounds up to T_c for the Ising model in two dimensions*, Probab. Theory Related Fields **102**, no. 3, 313–330. ↑231

Ising, E. 1925. *Beitrag zur theorie der ferromagnetismus*, Zeitschrift Fur Physik **31**, 253–258. ↑232

Jerison, D. 2013. *General mixing time bounds for finite Markov chains via the absolute spectral gap*, available at `arxiv:1310.8021`. ↑177

Jerrum, M. R. 1995. *A very simple algorithm for estimating the number of k-colorings of a low-degree graph*, Random Structures Algorithms **7**, no. 2, 157–165. ↑214

Jerrum, M. R. 2003. *Counting, sampling and integrating: Algorithms and complexity*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel. ↑xiv, 58

Jerrum, M. R. and A. J. Sinclair. 1989. *Approximating the permanent*, SIAM Journal on Computing **18**, 1149–1178. ↑199, 231

Jerrum, M. and A. Sinclair. 1996. *The Markov chain Monte Carlo method: An approach to approximate counting and integration*, Approximation Algorithms for NP-hard Problems. ↑210, 211

Jerrum, M., A. Sinclair, and E. Vigoda. 2004. *A polynomial-time approximation algorithm for the permanent of a matrix with nonnegative entries*, J. ACM **51**, no. 4, 671–697 (electronic). ↑215

Jerrum, M. R., L. G. Valiant, and V. V. Vazirani. 1986. *Random generation of combinatorial structures from a uniform distribution*, Theoret. Comput. Sci. **43**, no. 2-3, 169–188. ↑215

Johnson, N. L. and S. Kotz. 1977. *Urn models and their application*, John Wiley & Sons, New York-London-Sydney. An approach to modern discrete probability theory; Wiley Series in Probability and Mathematical Statistics. ↑35

Jonasson, J. 2012. *Mixing times for the interchange process*, ALEA Lat. Am. J. Probab. Math. Stat. **9**, no. 2, 667–683. ↑335, 363

Kahn, J., J. H. Kim, L. Lovász, and V. H. Vu. 2000. *The cover time, the blanket time, and the Matthews bound*, 41st Annual Symposium on Foundations of Computer Science (Redondo Beach, CA, 2000), IEEE Comput. Soc. Press, Los Alamitos, CA, pp. 467–475. ↑158

Kahn, J. D., N. Linial, N. Nisan, and M. E. Saks. 1989. *On the cover time of random walks on graphs*, J. Theoret. Probab. **2**, no. 1, 121–128. ↑158

Kaĭmanovich, V. A. and A. M. Vershik. 1983. *Random walks on discrete groups: Boundary and entropy*, Ann. Probab. **11**, no. 3, 457–490. ↑279

Kakutani, S. 1948. *On equivalence of infinite product measures*, Ann. of Math. (2) **49**, 214–224. ↑291

Kandel, D., Y. Matias, R. Unger, and P. Winkler. 1996. *Shuffling biological sequences*, Discrete Applied Mathematics **71**, 171–185. ↑243

Kantorovich, L. V. 1942. *On the translocation of masses*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 199–201. ↑203, 214

Kantorovich, L. V. and G. S. Rubinstein. 1958. *On a space of completely additive functions* (Russian, with English summary), Vestnik Leningrad. Univ. **13**, no. 7, 52–59. ↑215

Karlin, S. and J. McGregor. 1959. *Coincidence properties of birth and death processes*, Pacific J. Math. **9**, 1109–1140. ↑177

Karlin, S. and H. M. Taylor. 1975. *A first course in stochastic processes*, 2nd ed., Academic Press, New York. ↑19

Karlin, S. and H. M. Taylor. 1981. *A second course in stochastic processes*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York. ↑177

Kassabov, M. 2005. *Kazhdan constants for* $\mathrm{SL}_n(\mathbb{Z})$, Internat. J. Algebra Comput. **15**, no. 5-6, 971–995. ↑362

Kasteleyn, P. W. 1961. *The statistics of dimers on a lattice I. The number of dimer arrangements on a quadratic lattice*, Physica **27**, no. 12, 1209–1225. ↑386

Keilson, J. 1979. *Markov chain models—rarity and exponentiality*, Applied Mathematical Sciences, vol. 28, Springer-Verlag, New York. ↑177

Kemeny, J. G., J. L. Snell, and A. W. Knapp. 1976. *Denumerable Markov chains*, 2nd ed., Springer-Verlag, New York. With a chapter on Markov random fields, by David Griffeath; Graduate Texts in Mathematics, No. 40. ↑305

Kendall, W. S. and J. Møller. 2000. *Perfect simulation using dominating processes on ordered spaces, with application to locally stable point processes*, Adv. in Appl. Probab. **32**, no. 3, 844–865. ↑350

Kenyon, C., E. Mossel, and Y. Peres. 2001. *Glauber dynamics on trees and hyperbolic graphs*, 42nd IEEE Symposium on Foundations of Computer Science (Las Vegas, NV, 2001), IEEE Computer Soc., Los Alamitos, CA, pp. 568–578. ↑230, 231

Knuth, D. 1997. *The art of computer programming*, 3rd ed., Vol. 2: Seminumerical Algorithms, Addison-Wesley, Reading, Massachusetts. ↑385

Kobe, S. 1997. *Ernst Ising—physicist and teacher*, J. Statist. Phys. **88**, no. 3-4, 991–995. ↑232

Kolata, G. January 9, 1990. *In shuffling cards, 7 is winning number*, New York Times, C1. ↑110, 113

Komjáthy, J., J. Miller, and Y. Peres. 2014. *Uniform mixing time for random walk on lamplighter graphs* (English, with English and French summaries), Ann. Inst. Henri Poincaré Probab. Stat. **50**, no. 4, 1140–1160. ↑279

Komjáthy, J. and Y. Peres. 2013. *Mixing and relaxation time for random walk on wreath product graphs*, Electron. J. Probab. **18**, no. 71, 23, DOI 10.1214/EJP.v18-2321. MR3091717 ↑279

Labbé, C. and H. Lacoin. 2016. *Cutoff phenomenon for the asymmetric simple exclusion process and the biased card shuffling*, available at `arxiv:1610.07383v1`. ↑335

Lacoin, H. 2015. *A product chain without cutoff*, Electron. Commun. Probab. **20**, no. 19, 9. ↑272, 291

Lacoin, H. 2016a. *Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion*, Ann. Probab. **44**, no. 2, 1426–1487. ↑242, 334, 363

Lacoin, H. 2016b. *The cutoff profile for the simple exclusion process on the circle*, Ann. Probab. **44**, no. 5, 3399–3430. ↑334

Laslier, B. and F. L. Toninelli. 2015. *How quickly can we sample a uniform domino tiling of the* $2L \times 2L$ *square via Glauber dynamics?*, Probab. Theory Related Fields **161**, no. 3-4, 509–559. ↑323

Lawler, G. F. 1991. *Intersections of random walks*, Probability and Its Applications, Birkhäuser Boston, Inc., Boston, MA. ↑126

Lawler, G. and A. Sokal. 1988. *Bounds on the* L^2 *spectrum for Markov chains and Markov processes: A generalization of Cheeger's inequality*, Trans. Amer. Math. Soc. **309**, 557–580. ↑183, 199

Lee, J. R. and T. Qin. 2012. *A note on mixing times of planar random walks*, available at `arxiv:1205.3980`. ↑

León, C. A. and F. Perron. 2004. *Optimal Hoeffding bounds for discrete reversible Markov chains*, Ann. Appl. Probab. **14**, no. 2, 958–970. ↑178

Letac, G. 1986. *A contraction principle for certain Markov chains and its applications*, Contemporary Mathematics **50**, 263–273. ↑350

Levin, D. A., M. J. Luczak, and Y. Peres. 2010. *Glauber dynamics for the mean-field Ising model: Cut-off, critical power law, and metastability*, Probab. Theory Related Fields **146**, no. 1-2, 223–265. ↑230, 271, 362

Levin, D. A. and Y. Peres. 2010. *Pólya's theorem on random walks via Pólya's urn*, Amer. Math. Monthly **117**, no. 3, 220–231. ↑305

Levin, D. A. and Y. Peres. 2016. *Mixing of the exclusion process with small bias*, J. Stat. Phys. **165**, no. 6, 1036–1050. ↑335

Li, S.-Y. R. 1980. *A martingale approach to the study of occurrence of sequence patterns in repeated experiments*, Ann. Probab. **8**, no. 6, 1171–1176. ↑260

Liggett, T. M. 1985. *Interacting particle systems*, Springer-Verlag, New York. ↑126, 335

Liggett, T. M. 1999. *Stochastic interacting systems: Contact, voter and exclusion processes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 324, Springer-Verlag, Berlin. ↑335

Lindvall, T. 2002. *Lectures on the coupling method*, Dover, Mineola, New York. ↑74

Littlewood, J. E. 1948. *Large numbers*, Mathematical Gazette **32**, no. 300. ↑99

Logan, B. F. and L. A. Shepp. 1977. *A variational problem for random Young tableaux*, Advances in Math. **26**, no. 2, 206–222. ↑

Long, Y., A. Nachmias, W. Ning, and Y. Peres. 2014. *A power law of order* 1/4 *for critical mean field Swendsen-Wang dynamics*, Mem. Amer. Math. Soc. **232**, no. 1092, vi+84. ↑231

Lovász, L. 1993. *Random walks on graphs: A survey*, Combinatorics, Paul Erdős is Eighty, pp. 1–46. ↑58, 149

Lovász, L. and R. Kannan. 1999. *Faster mixing via average conductance*, Annual ACM Symposium on Theory of Computing (Atlanta, GA, 1999), ACM, New York, pp. 282–287 (electronic). ↑261

Lovász, L. and P. Winkler. 1993. *On the last new vertex visited by a random walk*, J. Graph Theory **17**, 593–596. ↑86

Lovász, L. and P. Winkler. 1995a. *Exact mixing in an unknown Markov chain*, Electronic Journal of Combinatorics **2**. Paper #R15. ↑358

Lovász, L. and P. Winkler. 1995b. *Efficient stopping rules for Markov chains*, Proc. 27th ACM Symp. on the Theory of Computing, 76–82. ↑86, 337, 348

Lovász, L. and P. Winkler. 1998. *Mixing times*, Microsurveys in discrete probability (Princeton, NJ, 1997), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 41, Amer. Math. Soc., Providence, RI, pp. 85–133. ↑58, 84, 86, 336, 337, 348

Loynes, R. M. 1962. *The stability of a queue with non-independent inter-arrival and service times*, Proceedings of the Cambridge Philosophical Society **58**, 497–520. ↑350

Lubetzky, E., F. Martinelli, A. Sly, and F. L. Toninelli. 2013. *Quasi-polynomial mixing of the 2D stochastic Ising model with "plus" boundary up to criticality*, J. Eur. Math. Soc. (JEMS) **15**, no. 2, 339–386. ↑360

Lubetzky, E. and Y. Peres. 2016. *Cutoff on all Ramanujan graphs*, Geom. Funct. Anal. **26**, no. 4, 1190–1216, DOI 10.1007/s00039-016-0382-7. MR3558308 ↑272

Lubetzky, E. and A. Sly. 2010. *Cutoff phenomena for random walks on random regular graphs*, Duke Math. J. **153**, no. 3, 475–510. ↑272

Lubetzky, E. and A. Sly. 2012. *Critical Ising on the square lattice mixes in polynomial time*, Comm. Math. Phys. **313**, no. 3, 815–836. ↑231

Lubetzky, E. and A. Sly. 2013. *Cutoff for the Ising model on the lattice*, Invent. Math. **191**, no. 3, 719–755. ↑230, 231, 362

Lubetzky, E. and A. Sly. 2014a. *Cutoff for general spin systems with arbitrary boundary conditions*, Comm. Pure Appl. Math. **67**, no. 6, 982–1027. ↑231, 291, 362

Lubetzky, E. and A. Sly. 2014b. *Universality of cutoff for the Ising model*. Preprint. ↑362

Lubetzky, E. and A. Sly. 2016. *Information percolation and cutoff for the stochastic Ising model*, J. Amer. Math. Soc. **29**, no. 3, 729–774. ↑231, 362

Lubotzky, A. 1994. *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics, vol. 125, Birkhäuser Verlag, Basel. With an appendix by Jonathan D. Rogawski. ↑200

Luby, M., D. Randall, and A. Sinclair. 1995. *Markov chain algorithms for planar lattice structures*, Proceedings of the 36th IEEE Symposium on Foundations of Computing, 150–159. ↑73, 386

Luby, M., D. Randall, and A. Sinclair. 2001. *Markov chain algorithms for planar lattice structures*, SIAM J. Comput. **31**, 167–192. ↑73

Luby, M. and E. Vigoda. 1995. *Approximately counting up to four*, Proceedings of the Twenty-Ninth Annual ACM Symposium on Theory of Computing, 150–159. Extended abstract. ↑74

Luby, M. and E. Vigoda. 1999. *Fast convergence of the Glauber dynamics for sampling independent sets*, Random Structures and Algorithms **15**, no. 3-4, 229–241. ↑74

Lyons, R. 2005. *Asymptotic enumeration of spanning trees*, Combin. Probab. Comput. **14**, no. 4, 491–522. ↑255

Lyons, R and S. Oveis Gharan. 2017. *Sharp bounds on random walk eigenvalues via spectral embedding*, International Mathematics Research Notices. ↑148

Lyons, R. and Y. Peres. 2016. *Probability on trees and networks*, Cambridge University Press, Cambridge. ↑98, 126, 148, 305

Lyons, T. 1983. *A simple criterion for transience of a reversible Markov chain*, Ann. Probab. **11**, no. 2, 393–402. ↑296, 305

Madras, N. and D. Randall. 1996. *Factoring graphs to bound mixing rates*, Proceedings of the 37th IEEE Symposium on Foundations of Computing, 194–203. ↑200

Madras, N. and G. Slade. 1993. *The self-avoiding walk*, Birkhäuser, Boston. ↑386, 387

Mann, B. 1994. *How many times should you shuffle a deck of cards?*, UMAP J. **15**, no. 4, 303–332. ↑113

Markov, A. A. 1906. *Rasprostranenie zakona bol'shih chisel na velichiny, zavisyaschie drug ot druga*, Izvestiya Fiziko-matematicheskogo obschestva pri Kazanskom universitete, 2-ya seriya **15**, 135–156. ↑19

Martinelli, F. 1994. *On the two-dimensional dynamical Ising model in the phase coexistence region*, J. Statist. Phys. **76**, no. 5-6, 1179–1246. ↑360

Martinelli, F. 1999. *Lectures on Glauber dynamics for discrete spin models*, Lectures on probability theory and statistics (Saint-Flour, 1997), Lecture Notes in Math., vol. 1717, Springer, Berlin, pp. 93–191. ↑227, 232, 323, 361

Martinelli, F. and E. Olivieri. 1994. *Approach to equilibrium of Glauber dynamics in the one phase region. I. The attractive case*, Comm. Math. Phys. **161**, no. 3, 447–486. ↑231

Martinelli, F., E. Olivieri, and R. H. Schonmann. 1994. *For 2-D lattice spin systems weak mixing implies strong mixing*, Comm. Math. Phys. **165**, no. 1, 33–47. ↑231

Martinelli, F., A. Sinclair, and D. Weitz. 2004. *Glauber dynamics on trees: Boundary conditions and mixing time*, Comm. Math. Phys. **250**, no. 2, 301–334. ↑230

Martinelli, F. and M. Wouts. 2012. *Glauber dynamics for the quantum Ising model in a transverse field on a regular tree*, J. Stat. Phys. **146**, no. 5, 1059–1088. ↑323

Matthews, P. 1988a. *Covering problems for Markov chains*, Ann. Probab. **16**, 1215–1228. ↑149, 150, 157

Matthews, P. 1988b. *A strong uniform time for random transpositions*, J. Theoret. Probab. **1**, no. 4, 411–423. ↑112, 164

Matthews, P. 1989. *Some sample path properties of a random walk on the cube*, J. Theoret. Probab. **2**, no. 1, 129–146. ↑157

McKay, B. D. and C. E. Praeger. 1996. *Vertex-transitive graphs that are not Cayley graphs. II*, J. Graph Theory **22**, no. 4, 321–334. ↑29

Menshikov, M., S. Popov, and A. Wade. 2017. *Non-homogeneous random walks: Lyapunov function methods for near-critical stochastic systems*, Vol. 209, Cambridge University Press. ↑305

Metropolis, N., A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller. 1953. *Equation of state calculations by fast computing machines*, J. Chem. Phys. **21**, 1087–1092. ↑45

Mihail, M. 1989. *Conductance and convergence of Markov chains—A combinatorial treatment of expanders*, Proceedings of the 30th Annual Conference on Foundations of Computer Science, 1989, pp. 526–531. ↑98

Miller, J. and Y. Peres. 2012. *Uniformity of the uncovered set of random walk and cutoff for lamplighter chains*, Ann. Probab. **40**, no. 2, 535–577. ↑279, 363

Mironov, I. 2002. *(Not so) random shuffles of RC4*, Advances in cryptology—CRYPTO 2002, Lecture Notes in Comput. Sci., vol. 2442, Springer, Berlin, pp. 304–319. ↑113

Montenegro, R. and P. Tetali. 2006. *Mathematical aspects of mixing times in Markov chains*, Vol. 1. ↑xiv, 58

Móri, T. F. 1987. *On the expectation of the maximum waiting time*, Ann. Univ. Sci. Budapest. Sect. Comput. **7**, 111–115 (1988). ↑158

Morris, B. 2006. *The mixing time for simple exclusion*, Ann. Appl. Probab. **16**, no. 2, 615–635. ↑334

Morris, B. 2009. *Improved mixing time bounds for the Thorp shuffle and L-reversal chain*, Ann. Probab. **37**, no. 2, 453–477. ↑243

Morris, B., W. Ning, and Y. Peres. 2014. *Mixing time of the card-cyclic-to-random shuffle*, Ann. Appl. Probab. **24**, no. 5, 1835–1849. ↑361

Morris, B. and Y. Peres. 2005. *Evolving sets, mixing and heat kernel bounds*, Probab. Theory Related Fields **133**, no. 2, 245–266. ↑250, 261

Morris, B. and C. Qin. 2017. *Improved bounds for the mixing time of the random-to-random insertion shuffle*, Electron. Commun. Probab. **22**. ↑363

Mossel, E., Y. Peres, and A. Sinclair. 2004. *Shuffling by semi-random transpositions*, Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science (FOCS'04) October 17–19, 2004, Rome, Italy, pp. 572–581. ↑113, 200, 361

Nacu, Ş. 2003. *Glauber dynamics on the cycle is monotone*, Probab. Theory Related Fields **127**, no. 2, 177–185. ↑323, 360

Nash-Williams, C. St. J. A. 1959. *Random walks and electric currents in networks*, Proc. Cambridge Philos. Soc. **55**, 181–194. ↑126

von Neumann, J. 1951. *Various techniques used in connection with random digits*, National Bureau of Standards Applied Mathematics Series **12**, 36–38. ↑378, 391

Norris, J. R. 1998. *Markov chains*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 2, Cambridge University Press, Cambridge. Reprint of 1997 original. ↑xiv, 305

Oliveira, R. I. 2012. *Mixing and hitting times for finite Markov chains*, Electron. J. Probab. **17**, no. 70, 12. ↑348

Oliveira, R. I. 2013. *Mixing of the symmetric exclusion processes in terms of the corresponding single-particle random walk*, Ann. Probab. **41**, no. 2, 871–913. ↑335

Pak, I. 1997. *Random walks on groups : Strong uniform time approach*, Ph.D. thesis, Harvard University. ↑86

Pemantle, R. 2007. *A survey of random processes with reinforcement*, Probab. Surv. **4**, 1–79 (electronic). ↑35

Peres, Y. 1992. *Iterating von Neumann's procedure for extracting random bits*, Ann. Stat. **20**, no. 1, 590–597. ↑379, 391

Peres, Y. 1999. *Probability on trees: An introductory climb*, Lectures on Probability Theory and Statistics, École d'Été de Probabilités de Saint-Flour XXVII — 1997, pp. 193–280. ↑126

Peres, Y. 2002. *Brownian intersections, cover times and thick points via trees*, (Beijing, 2002), Higher Ed. Press, Beijing, pp. 73–78. ↑157

Peres, Y., S. Popov, and P. Sousi. 2013. *On recurrence and transience of self-interacting random walks*, Bull. Braz. Math. Soc. (N.S.) **44**, no. 4, 841–867. ↑305

Peres, Y. and D. Revelle. 2004. *Mixing times for random walks on finite lamplighter groups*, Electron. J. Probab. **9**, no. 26, 825–845 (electronic). ↑279

Peres, Y. and A. Sly. 2013. *Mixing of the upper triangular matrix walk*, Probab. Theory Related Fields **156**, no. 3-4, 581–591. ↑364

Peres, Y. and P. Sousi. 2015a. *Mixing times are hitting times of large sets*, J. Theoret. Probab. **28**, no. 2, 488–519. ↑336, 348

Peres, Y. and P. Sousi. 2015b. *Total variation cutoff in a tree* (English, with English and French summaries), Ann. Fac. Sci. Toulouse Math. (6) **24**, no. 4, 763–779, DOI 10.5802/afst.1463. MR3434255 ↑272

Peres, Y. and P. Winkler. 2013. *Can extra updates delay mixing?*, Comm. Math. Phys. **323**, no. 3, 1007–1016. ↑323, 362

Pinsker, M. S. 1973. *On the complexity of a concentrator*, Proc. 7th Int. Teletraffic Conf., Stockholm, Sweden, pp. 318/1–318/4. ↑196, 200

Pisztora, A. 1996. *Surface order large deviations for Ising, Potts and percolation models*, Probab. Theory and Related Fields **104**, no. 4, 427–466. ↑231

Pitman, J. W. 1974. *Uniform rates of convergence for Markov chain transition probabilities*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **29**, 193–227. ↑73

Pitman, J. W. 1976. *On coupling of Markov chains*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **35**, no. 4, 315–322. ↑73

Pólya, G. 1931. *Sur quelques points de la théorie des probabilités*, Ann. Inst. H. Poincaré **1**, 117–161. ↑35

Pourmiri, A. and T. Sauerwald. 2014. *Cutoff phenomenon for random walks on Kneser graphs*, Discrete Appl. Math. **176**, 100–106. ↑272

Preston, C. J. 1974. *Gibbs states on countable sets*, Cambridge University Press, London-New York. Cambridge Tracts in Mathematics, No. 68. ↑

Propp, J. and D. Wilson. 1996. *Exact sampling with coupled Markov chains and applications to statistical mechanics*, Random Structure and Algorithms **9**, 223–252. ↑320, 349, 350, 355

Propp, J. and D. Wilson. 1998. *How to get a perfectly random sample from a generic Markov chain and generate a random spanning tree of a directed graph*, Journal of Algorithms (SODA '96 special issue) **27**, no. 2, 170–217. ↑358, 359

Quastel, J. 1992. *Diffusion of color in the simple exclusion process*, Comm. Pure Appl. Math. **45**, no. 6, 623–679. ↑200

Randall, D. 2006. *Slow mixing of Glauber dynamics via topological obstructions*, SODA (2006). Available at `http://www.math.gatech.edu/~randall/reprints.html`. ↑227, 229

Randall, D. and A. Sinclair. 2000. *Self-testing algorithms for self-avoiding walks*, Journal of Mathematical Physics **41**, no. 3, 1570–1584. ↑386, 388

Randall, D. and P. Tetali. 2000. *Analyzing Glauber dynamics by comparison of Markov chains*, J. Math. Phys. **41**, no. 3, 1598–1615. Probabilistic techniques in equilibrium and nonequilibrium statistical physics. ↑200

Restrepo, R., D. Štefankovič, J. C. Vera, E. Vigoda, and L. Yang. 2014. *Phase transition for Glauber dynamics for independent sets on regular trees*, SIAM J. Discrete Math. **28**, no. 2, 835–861. ↑323

Riordan, J. 1944. *Three-line Latin rectangles*, Amer. Math. Monthly **51**, 450–452. ↑198

Röllin, A. 2007. *Translated Poisson approximation using exchangeable pair couplings*, Ann. Appl. Probab. **17**, no. 5-6, 1596–1614. ↑291

Salas, J. and A. D. Sokal. 1997. *Absence of phase transition for antiferromagnetic Potts models via the Dobrushin uniqueness theorem*, J. Statist. Phys. **86**, no. 3-4, 551–579. ↑214

Saloff-Coste, L. 1997. *Lectures on finite Markov chains*, Lectures on Probability Theory and Statistics, École d'Été de Probabilités de Saint-Flour XXVI - 1996, pp. 301–413. ↑58, 148

Saloff-Coste, L. and J. Zúñiga. 2007. *Convergence of some time inhomogeneous Markov chains via spectral techniques*, Stochastic Process. Appl. **117**, no. 8, 961–979. ↑113

Saloff-Coste, L. and J. Zúñiga. 2008. *Refined estimates for some basic random walks on the symmetric and alternating groups*, ALEA Lat. Am. J. Probab. Math. Stat. **4**, 359–392. ↑363

Sarnak, P. 2004. *What is... an expander?*, Notices Amer. Math. Soc. **51**, no. 7, 762–763. ↑200

Scarabotti, F. and F. Tolli. 2008. *Harmonic analysis of finite lamplighter random walks*, J. Dyn. Control Syst. **14**, no. 2, 251–282, available at `arXiv:math.PR/0701603`. ↑279

Schonmann, R. H. 1987. *Second order large deviation estimates for ferromagnetic systems in the phase coexistence region*, Comm. Math. Phys. **112**, no. 3, 409–422. ↑227, 230

Seneta, E. 2006. *Non-negative matrices and Markov chains*, Springer Series in Statistics, Springer, New York. Revised reprint of the second (1981) edition, Springer-Verlag, New York. ↑19, 58

Simon, B. 1993. *The statistical mechanics of lattice gases. Vol. I*, Princeton Series in Physics, Princeton University Press, Princeton, NJ. ↑231, 232

Sinclair, A. 1992. *Improved bounds for mixing rates of Markov chains and multicommodity flow*, Combin. Probab. Comput. **1**, no. 4, 351–370. ↑

Sinclair, A. 1993. *Algorithms for random generation and counting*, Progress in Theoretical Computer Science, Birkhäuser Boston Inc., Boston, MA. A Markov chain approach. ↑58, 215

Sinclair, A. and M. Jerrum. 1989. *Approximate counting, uniform generation and rapidly mixing Markov chains*, Inform. and Comput. **82**, no. 1, 93–133. ↑183, 199, 227

Snell, J. L. 1997. *A conversation with Joe Doob*, Statist. Sci. **12**, no. 4, 301–311. ↑365

Soardi, P. M. 1994. *Potential theory on infinite networks*, Lecture Notes in Mathematics, vol. 1590, Springer-Verlag, Berlin. ↑126, 305

Spielman, D. A. and S.-H. Teng. 1996. *Spectral partitioning works: Planar graphs and finite element meshes*, 37th Annual Symposium on Foundations of Computer Science (Burlington, VT, 1996), IEEE Comput. Soc. Press, Los Alamitos, CA. ↑177

Spitzer, F. 1976. *Principles of random walks*, 2nd ed., Springer-Verlag, New York. Graduate Texts in Mathematics, Vol. 34. ↑291

Stanley, R. P. 1986. *Enumerative combinatorics*, Vol. 1, Wadsworth & Brooks/Cole, Belmont, California. ↑210

Stanley, R. P. 1999. *Enumerative combinatorics*, Vol. 2, Cambridge University Press. ↑35

Stanley, R. P. 2008. *Catalan addendum*. Available at `http://www-math.mit.edu/~rstan/ec/catadd.pdf`. ↑35

Steele, J. M. 1997. *Probability theory and combinatorial optimization*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 69, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. ↑369

Strassen, V. 1965. *The existence of probability measures with given marginals*, Ann. Math. Statist. **36**, 423–439. MR0177430 (31 #1693) ↑323

Stroock, D. W. and B. Zegarliński. 1992. *The equivalence of the logarithmic Sobolev inequality and the Dobrushin-Shlosman mixing condition*, Comm. Math. Phys. **144**, no. 2, 303–323. ↑231

Subag, E. 2013. *A lower bound for the mixing time of the random-to-random insertions shuffle*, Electron. J. Probab. **18**, no. 20, 20. ↑363

Sugimine, N. 2002. *A lower bound on the spectral gap of the 3-dimensional stochastic Ising models*, J. Math. Kyoto Univ. **42**, no. 4, 751–788. ↑360

Tetali, P. 1999. *Design of on-line algorithms using hitting times*, SIAM J. Comput. **28**, no. 4, 1232–1246 (electronic). ↑148

Thomas, L. E. 1989. *Bound on the mass gap for finite volume stochastic Ising models at low temperature*, Comm. Math. Phys. **126**, no. 1, 1–11. ↑227, 231

Thorisson, H. 1988. *Backward limits*, Annals of Probability **16**, 914–924. ↑350

Thorisson, H. 2000. *Coupling, stationarity, and regeneration*, Probability and Its Applications (New York), Springer-Verlag, New York. ↑74, 305

Thorp, E. O. 1965. *Elementary problem E1763*, Amer. Math. Monthly **72**, no. 2, 183. ↑112, 113

Thurston, W. P. 1990. *Conway's tiling groups*, Amer. Math. Monthly **97**, no. 8, 757–773. ↑8

Uyemura-Reyes, J. C. 2002. *Random walk, semidirect products, and card shuffling*, Ph.D. thesis, Stanford University. ↑363

Varopoulos, N. Th. 1985. *Long range estimates for Markov chains* (English, with French summary), Bull. Sci. Math. (2) **109**, no. 3, 225–252. ↑98

Vasershtein, L. N. 1969. *Markov processes over denumerable products of spaces describing large system of automata* (Russian), Problemy Peredači Informacii **5**, no. 3, 64–72; English transl.,. 1969, Problems of Information Transmission **5**, no. 3, 47–52. ↑214

Vershik, A. M. 2004. *The Kantorovich metric: The initial history and little-known applications* (Russian, with English and Russian summaries), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **312**, no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 11, 69–85, 311; English transl.,. 2004, J. Math. Sci. (N. Y.) **133**, no. 4, 1410–1417, available at `arxiv:math.FA/0503035`. ↑214

Veršik, A. M. and S. V. Kerov. 1977. *Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux* (Russian), Dokl. Akad. Nauk SSSR **233**, no. 6, 1024–1027. ↑

Vigoda, E. 2000. *Improved bounds for sampling colorings*, J. Math. Phys. **41**, no. 3, 1555–1569. ↑214

Vigoda, E. 2001. *A note on the Glauber dynamics for sampling independent sets*, Electron. J. Combin. **8**, no. 1, Research Paper 8, 8 pp. (electronic). ↑74, 358

Villani, C. 2003. *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI. ↑214

Wilf, H. S. 1989. *The editor's corner: The white screen problem*, Amer. Math. Monthly **96**, 704–707. ↑157

Williams, D. 1991. *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge. ↑260

Wilson, D. B. 2000a. *Layered multishift coupling for use in perfect sampling algorithms (with a primer on CFTP)* (N. Madras, ed.), Fields Institute Communications, vol. 26, American Mathematical Society. ↑359

Wilson, D. B. 2000b. *How to couple from the past using a read-once source of randomness*, Random Structures and Algorithms **16**, 85–113. ↑350, 354

Wilson, D. B. 2003. *Mixing time of the Rudvalis shuffle*, Electron. Comm. Probab. **8**, 77–85 (electronic). ↑200

Wilson, D. B. 2004a. *Mixing times of lozenge tiling and card shuffling Markov chains*, Ann. Appl. Probab. **14**, no. 1, 274–325. ↑193, 200, 242, 323, 386

Wilson, D. B. 2004b. *Perfectly random sampling with Markov chains*. Available at `http://dbwilson/exact`. ↑359

Woess, W. 2000. *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, Cambridge. ↑305

Zeitouni, O. 2004. *Random walks in random environment*, Lectures on probability theory and statistics, Lecture Notes in Math., vol. 1837, Springer, Berlin, pp. 189–312. ↑305

Zuckerman, D. 1989. *Covering times of random walks on bounded degree trees and other graphs*, J. Theoret. Probab. **2**, no. 1, 147–157. ↑158

Zuckerman, D. 1992. *A technique for lower bounding the cover time*, SIAM J. Discrete Math. **5**, 81–87. ↑157, 158

van Zuylen, A. and F. Schalekamp. 2004. *The Achilles heel of the GSR shuffle. A note on new age solitaire*, Probab. Engrg. Inform. Sci. **18**, no. 3, 315–328. ↑114

Notation Index

The symbol := means *defined as*.

The set $\{\dots, -1, 0, 1, \dots\}$ of integers is denoted $\mathbb{Z}$ and the set of real numbers is denoted $\mathbb{R}$.

For sequences (a_n) and (b_n), the notation $a_n = O(b_n)$ means that for some $c > 0$ we have $a_n/b_n \le c$ for all n, while $a_n = o(b_n)$ means that $\lim_{n\to\infty} a_n/b_n = 0$, and $a_n \asymp b_n$ means both $a_n = O(b_n)$ and $b_n = O(a_n)$ are true.

Index

Italics indicate that the reference is to an exercise.

Photo courtesy of David A. Levin

Photo courtesy of Yuval Peres

This book is an introduction to the modern theory of Markov chains, whose goal is to determine the rate of convergence to the stationary distribution, as a function of state space size and geometry. This topic has important connections to combinatorics, statistical physics, and theoretical computer science. Many of the techniques presented originate in these disciplines.

The central tools for estimating convergence times, including coupling, strong stationary times, and spectral methods, are developed. The authors discuss many examples, including card shuffling and the Ising model, from statistical mechanics, and present the connection of random walks to electrical networks and apply it to estimate hitting and cover times.

The first edition has been used in courses in mathematics and computer science departments of numerous universities. The second edition features three new chapters (on monotone chains, the exclusion process, and stationary times) and also includes smaller additions and corrections throughout. Updated notes at the end of each chapter inform the reader of recent research developments.

Markov Chains and Mixing Times *is a magical book, managing to be both friendly and deep. It gently introduces probabilistic techniques so that an outsider can follow. At the same time, it is the first book covering the geometric theory of Markov chains and has much that will be new to experts. It is certainly THE book that I will use to teach from. I recommend it to all comers, an amazing achievement.*

—Persi Diaconis, Mary V. Sunseri Professor of Statistics and Mathematics, Stanford University

Mixing times are an active research topic within many fields from statistical physics to the theory of algorithms, as well as having intrinsic interest within mathematical probability and exploiting discrete analogs of important geometry concepts. The first edition became an instant classic, being accessible to advanced undergraduates and yet bringing readers close to current research frontiers. This second edition adds chapters on monotone chains, the exclusion process and hitting time parameters. Having both exercises and citations to important research papers it makes an outstanding basis for either a lecture course or self-study.

—David Aldous, University of California, Berkeley

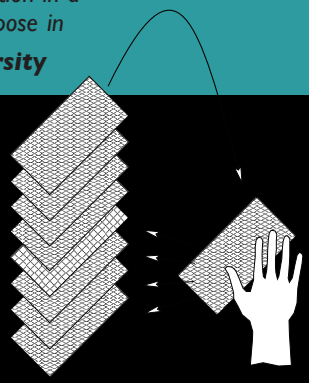
Mixing time is the key to Markov chain Monte Carlo, the queen of approximation techniques. With new chapters on monotone chains, exclusion processes, and set-hitting, Markov Chains and Mixing Times *is more comprehensive and thus more indispensable than ever. Prepare for an eye-opening mathematical tour!*

—Peter Winkler, Dartmouth College

The study of finite Markov chains has recently attracted increasing interest from a variety of researchers. This is the second edition of a very valuable book on the subject. The main focus is on the mixing time of Markov chains, but there is a lot of additional material.

In this edition, the authors have taken the opportunity to add new material and bring the reader up to date on the latest research. I have used the first edition in a graduate course and I look forward to using this edition for the same purpose in the near future.

—Alan Frieze, Carnegie Mellon University



ISBN 978-1-4704-2962-1

9 781470 429621

MBK/107

AMS on the Web
www.ams.org

For additional information and updates on this book, visit
www.ams.org/bookpages/mbk-107