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**MATHEMATICAL  
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Volume 2

**Some Questions  
in the Theory  
of Moments**

N. I. Ahiezer  
M. G. Kreĭn



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N. I. Ahiezer  
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**American Mathematical Society**  
Providence, Rhode Island

Н. АХИЕЗЕР и М. КРЕЙН  
О НЕКОТОРЫХ ВОПРОСАХ ТЕОРИИ МОМЕНТОВ

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## FOREWORD

The present book deals with several specialized questions pertaining to the so-called moment problem which have for one reason or another been of interest to the authors.

The book is divided into separate articles which are basically independent of one another; however, it should not be thought that these articles are not connected with one another. On the contrary, there is a close connection and the articles are arranged in a discernible logical order.

The basic article (the  $L$ -moment problem) has as its starting point several (little-known) ideas and problems advanced by the late academician A. A. Markov [13b, c, d].<sup>1)</sup> In a series of papers [1a-i] the authors have completed and generalized A. A. Markov's results, connecting them with present-day theories apparently unknown to A. A. Markov.

The three articles which follow treat in the light of functional analysis some of the questions solved or discussed in the first article.

The subsequent two articles are devoted to various applications of one class of functions (named  $N$ -functions by the authors) which play a large role in the moment problem.

As is usually the case, the authors propose to write at least one more book which will deal with all the important questions of the theory which have not found a place in this book.

They hope, therefore, that no one will reproach them because questions of determinateness of the moment problem remain unanswered.

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1) The numbers in square brackets refer to the bibliography located at the end of the book.



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zeros in the left half-plane.

From Theorem 11, as we shall see, it does not follow at all that the roots of an entire real function  $f(z)$  which satisfies Hurwitz's conditions lie within the left half-plane. Still, we may assert that together with every root  $\alpha$  lying in the right half-plane  $\Re z \geq 0$  the function  $f(z)$  also has roots  $-\alpha$ ,  $\bar{\alpha}$  and  $-\bar{\alpha}$  (since  $\alpha^2$  and  $\bar{\alpha}^2$  are in this case roots of the function  $E(z)$ ).

Therefore, if it is known in advance that a function  $f(z)$  does not have roots which differ in sign only, then from Hurwitz's conditions it does indeed follow that all the roots of the function  $f(z)$  lie in the left half-plane.

In the somewhat more special case in which among the roots of  $f(z)$  there are none equal in modulus and not conjugate, interesting results to the question of determining how many of the first  $m$  in modulus of the roots of  $f(z)$  lie in the left half-plane were obtained by N. G. Čebotarev in his paper cited repeatedly.

We also comment that our methods make it possible to give an exhaustive characterisation of meromorphic functions  $f(z)$  which satisfy Hurwitz's conditions. However, this is hardly of interest and we shall not present it.

#### BIBLIOGRAPHY <sup>1)</sup>

[1] N. Abiezer and M. Kreiñ

- a) *Über Fouriersche Reihen beschränkter summierbarer Funktionen und ein neues Extremumproblem*, I, Zap. Har'kov. Mat. Obšč. (4) 9 (1934), 9–23.
- b) *Über Fouriersche Reihen beschränkter summierbarer Funktionen und ein neues Extremumproblem*, II, *ibid.* 10 (1934), 3–32.
- c) *Über eine Transformation der reellen Toeplitzischen Formen und das Momentenproblem in einem endlichen Intervalle*, *ibid.* 11 (1935), 21–26.
- d) *Das Momentenproblem bei der zusätzlichen Bedingung von A. Markoff*, *ibid.* 12 (1936), 13–36.
- e) *Bemerkung zur Arbeit "Über Fouriersche Reihen beschränkter summierbarer Funktionen und ein neues Extremumproblem"*, *ibid.*, 37–40.
- f) *The moment problem on two intervals with the additional conditions of A. A. Markov*, *ibid.* 14 (1937), 47–60.
- g) *Some remarks about three papers of M. S. Verblunsky*, *ibid.* 16 (1940), 129–134.

---

1) The article by G. Pick cited on page 150 and the article by L. G. Šnirel'man cited on page 191 have inadvertently been omitted from the bibliography.

- h) *Concerning quadrature formulas of P. Čebyšev and A. Markov*, Memorial volume dedicated to D. A. Grave, Moscow, 1940, pp. 15–28. (Russian)
- i) *Concerning two minimum problems connected with the problem of moments* Dokl. Akad. Nauk SSSR 1 (1936), 331–334. (Russian)
- [2] N. Ahiezer  
 a) *On a theorem of S. Bochner*, Zap. Har'kov. Mat. Obšč. (4) 14 (1937), 75–80. (Russian)  
 b) *Selected writings in the theory of approximation of functions*, DNTVU, 1938. (Ukrainian)
- [3] S. Bernštejn  
 a) *Sur une propriété des polynomes de Tchebycheff*, Dokl. Akad. Nauk SSSR Ser. A 1927, 405–407.  
 b) *Sur les fonctions absolument monotones*, Acta. Math. 52 (1928), 1–66.
- [4] Ya. Geronimus, *On several extremal problems*, Izv. Akad. Nauk SSSR Ser. Mat. 1937, 185–202. (Russian)
- [5] V. Gončarov, *The theory of interpolation and approximation of functions*, ONTI, Moscow, 1934. (Russian)
- [6] L. Kantorovič  
 a) *On a moment problem for finite intervals*, Dokl. Akad. Nauk SSSR 14 (1937), 531–536. (Russian)  
 b) *Lineare halbgeordnete Räume*, Mat. Sb. (N.S.) 2(44) (1937), 121–168.
- [7] A. Korĭin and E. Zolotarev, *Sur un certain minimum*, Collected works of E. I. Zolotarev, Vol. 1, Izdat. Akad. Nauk SSSR, Leningrad, 1931.
- [8] D. Kotelyanskiĭ, *About certain applications of quadratic forms to the Nevanlinna-Pick problem*, Ž. Inst. Mat. Akad. Nauk URSR 1937, no. 1, 73–88. (Ukrainian)
- [9] M. Kreĭn and M. Naĭmark, *Concerning the methods of symmetric and Hermite forms in the theory of separation of roots of algebraic equations*, Kharkov, 1936, pp. 1–41. (Russian)
- [10] M. Kreĭn and P. Rehtman, *Concerning the Nevanlinna-Pick problem*, Trudy Odess. Gos. Univ. 2 (1938), 63–77. (Russian)
- [11] M. Kreĭn  
 a) *Concerning several questions about the geometry of convex sets belonging to a linear, normed and complete space*, Dokl. Akad. Nauk SSSR 14 (1937),

- 5–8. (Russian)
- b) *Positive additive functionals in linear normed spaces*, Zap. Har'kov. Mat. Obšč. (4) 14 (1937), 227–237. (Russian)
- c) *A generalization of the research of Academician A. A. Markov on the limiting values of integrals*, Transactions of the Second All-Union Congress of Mathematicians, Vol. 2, pp. 152–154, Leningrad, 1934. (Russian)
- [12] L. Lyusternik, *Basic concepts of functional analysis*, Uspehi Mat. Nauk 1 (1936), 77–140. (Russian)
- [13] A. A. Markov
- a) *Concerning some applications of algebraic continued fractions*, St. Petersburg, 1884. (Russian)
- b) *New applications of continued fractions*, Zap. Akad. Nauk, 1896. (Russian)
- c) *Concerning limit values of integrals in connection with interpolation*, *ibid.*, 1898. (Russian)
- d) *Research on limit values of integrals*, *ibid.*, 1900. (Russian)
- [14] N. Meïman, *Concerning poles of meromorphic functions*, Zap. Har'kov. Mat. Obšč. (4) 14 (1937), 97–104. (Russian)
- [15] K. Posse, *Sur quelques applications des fractions continues algébriques*, St. Petersburg, 1886.
- [16] F. Remez, *On methods of finding the best approximations of functions by Čebyšev's method*, Izdat. Akad. Nauk USSR, Kiev, 1935. (Ukrainian)
- [17] N. Čebotarev, *Über die Realität von Nullstellen ganzer transzendenter Funktionen*, Math. Ann. 99 (1928), 660–686.
- [18] P. Čebyšev
- a) *Concerning interpolation for a large number of given observations*, Collected works, Vol. I, Izdat. Akad. Nauk SSSR, Moscow, 1946.
- b) *Concerning quadratures*, *ibid.*, Vol. II, 1947. (Russian)
- [19] Ascoli, *Sugli spazi lineari metrici e le loro varietà lineari*, Ann. Mat. Pura Appl. 10 (1932), 33–81.
- [20] S. Banach, *Théorie des opérations linéaires*, Monogr. Mat., Vol. 1, Warsaw, 1932.
- [21] S. Bochner
- a) *Vorlesungen über Fouriersche Integrale*, Akad. Verlagsges., Leipzig, 1932.

- b) *Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse*, Math. Ann. 108 (1933), 378–410.
- [22] C. Carathéodory, *Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen*, Rend. Palermo 32 (1911).
- [23] C. Carathéodory and L. Fejér, *Über den Zusammenhang der extremen von harmonischen Funktionen*, Rend. Palermo 32 (1911).
- [24] T. Carleman, *Sur les équations intégrales singulières à noyau réel et symétrique*, Uppsala, 1923.
- [25] Fischer, *Über das Carathéodorysche Problem, Potenzreihen mit positivem reellen Teil betreffend*, Rend. Palermo 32 (1911).
- [26] Fujiwara  
 a) *Über die Polynome von der kleinsten totalen Schwankung*, Tôhoku Math. J. 13 (1913).  
 b) *Über die Nullstellen der ganzen Funktionen vom Geschlecht Null und Eins*, *ibid.* 25 (1925).
- [27] Grommer, *Ganze transzendente Funktionen mit reellen Nullstellen*, J. Reine Angew. Math. 144 (1914).
- [28] A. Haar, *Über die Minkowskische Geometrie und die Annäherung an stetige Funktionen*, Math. Ann. 78.
- [29] Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, I, II, III, Math. Ann. 81 (1920); 82 (1920).
- [30] Herglotz, *Über potenzreihen mit positiven reellen Teil im Einheitskreise*, Leipziger Berichte 63 (1911).
- [31] D. Jackson, *Note on a class of polynomials of approximation*, Trans. Amer. Math. Soc. 22 (1921), 320–326.
- [32] Mathias, *Über positive Fourier-Integrale*, Math. Z. 16 (1923).
- [33] Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Studia Math. 4 (1933), 70–84.
- [34] R. Nevanlinna  
 a) *Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentenproblem*, Acta Fenn. Ser. A 18 (1922).  
 b) *Über beschränkte analytische Funktionen*, *ibid.* 32 (1929).
- [35] G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, Vol. 2, Berlin, 1925.

- [36] F. Riesz  
 a) *Sur certains systèmes singuliers d'équations intégrales*, Ann. Ecole Norm. 28 (1911).  
 b) *Über ein Problem des Herrn Carathéodory*, J. Reine Angew. Math., 1916.  
 c) *Über Sätze von Stone und Rochner*, Acta Litt. Sci. Szeged 6 (1934), 184–198.
- [37] M. Riesz, *Sur le problème des moments*, I, II, III, Ark. Mat. Astr. Fys. 16 (1922); 17 (1923).
- [38] I. Schoenberg, *Convex domains and linear combinations of continuous functions*, Bull. Amer. Math. Soc. 39 (1933), 273–280.
- [39] Th. Stieltjes  
 a) *Recherches sur les fractions continues*, Ann. Toulouse 8 (1894); 9 (1895).  
 b) *Iets over de benaderde voorstelling van eene functie door eene andere*, Delft, 1876.
- [40] Stridsberg, *Nagra aritmetiska undersökningar rörande fakulteter och vissa allmänna Koefficientsviter*, Not. 3, Ark. Mat. Astr. Fys. 15 (1921).
- [41] O. Toeplitz  
 a) *Über die Fouriersche Entwicklung positiver Funktionen*, Rend. Palermo 32 (1911).  
 b) *Zur Theorie der quadratischen Formen von unendlichvielen Veränderlichen*, Gött. Nachr., 1910.
- [42] D. Widder, *Necessary and sufficient conditions for the representation of a function by a doubly infinite Laplace integral*, Bull. Amer. Math. Soc. 40 (1934), 321–326.

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