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Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains

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HARMONIC ANALYSIS
of Functions of Several Complex Variables in the
CLASSICAL DOMAINS

by
L. K. Hua

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ГАРМОНИЧЕСКИЙ АНАЛИЗ
ФУНКЦИЙ МНОГИХ КОМПЛЕКСНЫХ ПЕРЕМЕННЫХ
В КЛАССИЧЕСКИХ ОБЛАСТЯХ

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Appendix 1
SOME EQUALITIES

(This appendix enlists several elementary means for application in this book. Its purpose is for: (1) the possible application of these means in other fields, and (2) the extracurricular exercises for undergraduate students.)

1. Let $D(x_1, \ldots, x_n) = \Pi_{1 \leqslant i<j \leqslant n}(x_i - x_j)$; then we have the identity

$$\sum_{i_1, i_2, \ldots, i_n} \delta^{1,\ldots,n}_{i_1,\ldots,i_n} \frac{x_{i_1}^{n-1} x_{i_2}^{n-2} \cdots x_{i_{n-1}}^1}{(1 - x_{i_1}^2)(1 - x_{i_1}^2 x_{i_2}^2) \cdots (1 - x_{i_1}^2 x_{i_2}^2 \cdots x_{i_n}^2)}$$

$$= \frac{D(x_1, \ldots, x_n)}{\Pi_{1 \leqslant i \leqslant j \leqslant n} (1 - x_i x_j)},$$

where $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$; $\delta^{1,2,\ldots,n}_{i_1,i_2,\ldots,i_n} = 1$ when $i_1, i_2, \ldots, i_n$ is an even permutation of $1, 2, \ldots, n$; $\delta^{1,2,\ldots,n}_{i_1,i_2,\ldots,i_n} = -1$ when $i_1, i_2, \ldots, i_n$ is an odd permutation of $1, 2, \ldots, n$.

2. We have the identity

$$\sum_{i_1, i_2, \ldots, i_n} \delta^{1,\ldots,n}_{i_1,\ldots,i_n} \frac{x_{i_1}^{n-1} x_{i_2}^{n-2} \cdots x_{i_{n-1}}^1}{(1 - x_{i_1} x_{i_2})(1 - x_{i_1} x_{i_2} x_{i_3} x_{i_4}) \cdots (1 - x_{i_1} \cdots x_{i_{2\nu}})}$$

$$= \frac{D(x_1, \ldots, x_n)}{\Pi_{1 \leqslant i<j \leqslant n}(1 - x_i x_j)},$$

where ν is the integer part of $\tfrac{1}{2}n$.

3.

$$\sum_{\substack{l_1 + \cdots + l_n = m \\ l_\nu \geqslant 0}} \frac{[D(l_1, \ldots, l_n)]^2}{l_1! \cdots l_n!} = n^{m - \frac{1}{2}n(n-1)} \frac{n!(n-1)! \cdots 1!}{(m - \frac{1}{2}n(n-1))!}.$$

4. Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be the power series which converges in $|z| < \rho$, when $|x_j y_k| < \rho$. Then

$$\begin{vmatrix} f(x_1 y_1) \cdots f(x_1 y_n) \\ \cdots\cdots\cdots\cdots \\ f(x_n y_1) \cdots f(x_n y_n) \end{vmatrix}$$

$$= \sum_{l_1 > l_2 > \cdots > l_n \geqslant 0} a_{l_1} \cdots a_{l_n} \begin{vmatrix} x_1^{l_1} \cdots x_1^{l_n} \\ \cdots\cdots \\ x_n^{l_1} \cdots x_n^{l_n} \end{vmatrix} \begin{vmatrix} y_1^{l_1} \cdots y_1^{l_n} \\ \cdots\cdots \\ y_n^{l_1} \cdots y_n^{l_n} \end{vmatrix}.$$

5. Let $f_1(x), \ldots, f_n(x)$ be n differentiable functions with necessary order. Then we have

$$\lim_{\substack{x_1 \to x \\ \cdots \\ x_n \to x}} \frac{\begin{vmatrix} f_1(x_1) \cdots f_n(x_1) \\ \cdots\cdots\cdots \\ f_1(x_n) \cdots f_n(x_n) \end{vmatrix}}{D(x_1, \ldots, x_n)}$$

$$= \frac{(-1)^{\frac{1}{2}n(n-1)}}{1!2! \cdots (n-1)!} \begin{vmatrix} f_1(x) & \cdots & f_n(x) \\ f_1^1(x) & \cdots & f_n^1(x) \\ \cdots & \cdots & \cdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

6. When $\alpha > \frac{1}{2}n$, we have

$$\int \cdots \int_T \frac{\dot{T}}{(\det(I + T^2))^\alpha}$$

$$= 2^{\frac{1}{4}n(n-1)} \pi^{\frac{1}{4}n(n+1)} \frac{\Gamma(\alpha - \frac{1}{2}n)}{\Gamma(\alpha)} \prod_{\nu=1}^{n-1} \frac{\Gamma(2\alpha - \frac{1}{2}(n+\nu))}{\Gamma(2\alpha - \nu)},$$

where T runs over all $n \times n$ real symmetric matrices, $T = (t_{jk})$, $\dot{T} = 2^{\frac{1}{4}n(n-1)} \prod_{j \leqslant k} dt_{jk}$.

7. When $n \geqslant 2$, and $\alpha > \frac{1}{4}(2n-3)$, we have

$$\int \cdots \int_K \frac{\dot{K}}{(\det(I+KK'))^{\alpha}}$$

$$= 2^{\frac{1}{4}n(n-1)}\pi^{\frac{1}{4}n(n-1)} \prod_{\nu=2}^{n} \frac{\Gamma(2\alpha - n + \frac{1}{2}(\nu+1))}{\Gamma(2\alpha - n + \nu)},$$

where K runs over all $n \times n$ real skew-symmetric matrices, $K = (k_{ij})$, $\dot{K} = 2^{n(n-1)/4}\Pi_{i<j} dk_{ij}$.

8. When $\alpha > n - \frac{1}{2}$, we have

$$\int \cdots \int_H \frac{\dot{H}}{(\det(I+H^2))^{\alpha}}$$

$$= 2^{\frac{1}{2}n(n-1)}\pi^{\frac{1}{2}n^2} \prod_{j=0}^{n-1} \frac{\Gamma(\alpha - j - \frac{1}{2})}{\Gamma(\alpha - j)} \prod_{k=0}^{n-2} \frac{\Gamma(2\alpha - n - k)}{\Gamma(2\alpha - 2k - 1)},$$

where H runs over all Hermitian matrices (h_{jk}), $h_{jj} = h_j$, $h_{jk} = h'_{jk} + ih''_{jk}$ $(j \neq k)$, and

$$\dot{H} = 2^{\frac{1}{2}n(n-1)} \prod_{j=1}^{n} dh_j \prod_{j<k} dh'_{jk} dh''_{jk}.$$

9. Let Z be an $m \times n$ matrix with complex number elements. When $\lambda > -1$, we have

$$\int \cdots \int_{I-Z\bar{Z}'>0} (\det(I - Z\bar{Z}'))^{\lambda}\dot{Z} = \frac{\Pi_{j=1}^{n}\Gamma(\lambda+j)\Pi_{k=1}^{m}\Gamma(\lambda+k)}{\Pi_{l=1}^{n+m}\Gamma(\lambda+l)}\pi^{mn},$$

where

$$Z = (z_{jk}), \qquad z_{jk} = x_{jk} + iy_{jk}, \qquad \dot{Z} = \prod_{j=1}^{m}\prod_{k=1}^{n} dx_{jk}dy_{jk}.$$

10. Let Z be a symmetric matrix with complex number elements. When $\lambda > -1$, we have

$$\int \cdots \int_{I-Z\bar{Z}'>0} (\det(I - Z\bar{Z}'))^{\lambda}\dot{Z}$$

$$= \frac{\pi^{\frac{1}{2}n(n+1)}}{(\lambda+1)\cdots(\lambda+n)} \cdot \frac{\Gamma(2\lambda+3)\Gamma(2\lambda+5)\cdots\Gamma(2\lambda+2n-1)}{\Gamma(2\lambda+n+2)\Gamma(2\lambda+n+3)\cdots\Gamma(2\lambda+2n)}$$

where $\dot{Z} = \Pi_{j\leqslant k} dx_{jk}dy_{jk}$.

11. Let Z be an $n \times n$ skew-symmetric matrix with complex number elements. When $\lambda > -\frac{1}{2}$, we have

$$\int_{I+Z\bar{Z}>0} \cdots \int (\det(I + Z\bar{Z}))^{\lambda} \dot{Z}$$

$$= \pi^{\frac{1}{2}n(n-1)} \frac{\Gamma(2\lambda + 1)\Gamma(2\lambda + 3) \cdots \Gamma(2\lambda + 2n - 3)}{\Gamma(2\lambda + n)\Gamma(2\lambda + n + 1) \cdots \Gamma(2\lambda + 2n - 2)},$$

where $\dot{Z} = \Pi_{j<k} dx_{jk}\, dy_{jk}$.

12. When $\alpha > -1$, $\beta > -(n + \alpha)$, we have

$$\int_{\substack{|zz'|^2 - 2z\bar{z}' > 0 \\ |zz'| < 1}} \cdots \int \left(1 - \bar{z}z' - \sqrt{(\bar{z}z')^2 - |zz'|^2}\right)^{\alpha} \left(1 - \bar{z}z' + \sqrt{(\bar{z}z')^2 - |zz'|^2}\right)^{\beta} \dot{z}$$

$$= \frac{\pi^n}{2^{n-1}} \cdot \frac{1}{\alpha + \beta + n} \cdot \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + n)},$$

where $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$, $\dot{z} = \Pi_{j=1}^{n} dx_j dy_j$.

13.

$$\int_{U\bar{U}=I} \cdots \int \frac{\dot{U}}{|\det(I - Z\bar{U}')|^{2n}} = \frac{(2\pi)^{mn - \frac{1}{2}m(m-1)}}{(n-m)! \cdots (n-1)!} \det(I - Z\bar{Z}')^{-n},$$

where Z and U are $m \times n$ complex number matrices, $\dot{U}$ denotes the volume element of the manifold $U\bar{U}' = I$.

14.

$$\int_{U\bar{U}=I} \cdots \int \frac{\dot{U}}{|\det(I - Z\bar{U})|^{n+1}} = 2^{\frac{1}{4}n(3n+1)} \pi^{\frac{1}{4}n(n+1)} \frac{\Gamma(1/2)}{\Gamma((n+1)/2)}$$

$$\times \prod_{\nu=1}^{n-1} \frac{\Gamma(n/2 - \nu/2 + 1)}{\Gamma(n - \nu + 1)} \det(I - Z\bar{Z})^{-\frac{1}{2}(n-1)},$$

where Z and U are $n \times n$ complex number symmetric matrices, $\dot{U}$ denotes the volume element of the manifold $U\bar{U} = I$.

15. Let Z and K be complex number skew-symmetric matrices. When n is even,

$$\int \cdots \int_{K\bar{K}'=I} \frac{\dot{K}}{|\det(I+Z\bar{K})|^{n-1}} = \frac{1}{2}\cdot\frac{((\nu-1)!)^2}{2^{(\nu-1)^2}}(8\pi)^{\frac{1}{4}n(n-1)} \times \prod_{\alpha=1}^{n-1}\frac{\Gamma(\alpha/2)}{\Gamma(\alpha)}\det(I+Z\bar{Z})^{-\frac{1}{2}(n-1)}, \qquad \nu=[n/2],$$

where $\dot{K}$ denotes the volume element of the manifold $K\bar{K}'=I$. When n is odd,

$$\int \cdots \int_{K} \frac{\dot{K}}{|\det(I-Z\bar{K})|^{n-1}} = 2^{-\nu}\frac{((\nu-1)!)^2}{2^{(\nu-1)^2}}(8\pi)^{\frac{1}{4}n(n-1)} \times \prod_{\alpha=1}^{n-1}\frac{\Gamma(\alpha/2)}{\Gamma(\alpha)}\det(I+Z\bar{Z})^{-\frac{1}{2}(n-1)},$$

where K runs over all matrices in the form

$$K=\Gamma\left[\begin{pmatrix}0 & e^{i\theta_1}\\ -e^{i\theta_1} & 0\end{pmatrix}\dotplus\cdots\dotplus\begin{pmatrix}0 & e^{i\theta_\nu}\\ -e^{i\theta_\nu} & 0\end{pmatrix}\dotplus 0\right]\Gamma'$$

and Γ is any real orthogonal matrix.

16.

$$\int \cdots \int_{\mathfrak{C}_{\mathrm{IV}}} \frac{\dot{\xi}}{|(z-\xi)(z-\xi)'|^n} = \frac{2\pi^{n/2+1}}{\Gamma(n/2)}(1+|zz'|^2-2\bar{z}z')^{-n/2},$$

where $z=(z_1,\ldots,z_n)$, $\xi=e^{i\theta}(x_1,\ldots,x_n)$, and $\dot{\xi}$ is the volume element of the manifold $0\leqslant\theta\leqslant 2\pi$, $x_1^2+\cdots+x_n^2=1$.

17.

$$\int \cdots \int_{\pi\geqslant\theta_1\geqslant\cdots\geqslant\theta_n\geqslant-\pi} \frac{1}{|\prod_{j=1}^n(1-re^{-i\theta_j})|^{2n}}\prod_{j<k}|e^{i\theta_j}-e^{i\theta_k}|^2\,d\theta_1\cdots d\theta_n = \frac{1!2!\cdots(n-1)!}{(2\pi)^{\frac{1}{2}n(n-1)}}(1-r^2)^{-n^2} \qquad (r<1).$$

18.

$$\int_1^\infty (t+\sqrt{t^2-1})^{-(2l+m+n)}\left(\frac{d}{dt}\right)^m(t^2-1)^{m+\frac{1}{2}(n-3)}\,dt = \frac{\Gamma(n/2)(2l+m+n)l!\,\Gamma(m+n/2+l)\Gamma(2m+n-2)}{2^{m+n-1}\Gamma(l+n/2+1)\Gamma(m+n+l)\Gamma(m+n/2-1)}.$$

APPENDIX 2

COORDINATES TRANSFORMATION FORMULAS FOR MATRICES

1. Let U be a unitary matrix, H a Hermitian matrix, $\dot{U}$ the volume element of the space of unitary matrices, $\dot{H}$ the volume element of the space of Hermitian matrices. Then these two spaces are in one-one correspondence under the transformation

$$U = (I + iH)(I - iH)^{-1}$$

except for lower dimensional manifolds. Moreover,

$$\dot{U} = 2^{n^2} \det(I + H^2)^{-n} \dot{H}.$$

2. Any unitary matrix can be represented in the form

$$U = V\Lambda V^{-1}, \qquad \Lambda = [e^{i\theta_1}, \dots, e^{i\theta_n}], \qquad 2\pi \geqslant \theta_1 \geqslant \cdots \geqslant \theta_n \geqslant 0,$$

where V is a unitary matrix. The diagonal unitary matrices $\Lambda = [e^{i\theta_1}, \dots, e^{i\theta_n}]$ form a subgroup of the unitary group U_n. We denote by $[U_n]$ the manifold of left cosets of U_n with respect to this subgroup. Let $[\dot{U}]$ be the volume element of $[U_n]$; then

$$\dot{U} = [\dot{U}] \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2 \, d\theta_1 \cdots d\theta_n.$$

3. Any Hermitian matrix can be represented in the form

$$H = U\Lambda U^{-1}, \qquad \Lambda = [\lambda_1, \dots, \lambda_n], \quad \lambda_1 \geqslant \cdots \geqslant \lambda_n,$$

where U is a unitary matrix. We have

$$\dot{H} = [\dot{U}] \prod_{j<k} (\lambda_j - \lambda_k)^2 \, d\lambda_1 \cdots d\lambda_k.$$

4. Let Z be any complex number matrix, $\dot{Z}$ the volume element of space of these matrices. Z can be represented in the form

$$Z = U\Lambda V, \quad \Lambda = [\lambda_1, \dots, \lambda_n], \qquad \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0,$$

where U and V are unitary matrices. We have

$$\dot{Z} = [\dot{U}]\dot{V}\prod_{j<k}(\lambda_j - \lambda_k)^2 d\lambda_1 \cdots d\lambda_n.$$

5. Every complex number symmetric matrix Z can be represented as

$$Z = U\Lambda U', \qquad \Lambda = [\lambda_1, \ldots, \lambda_n], \quad \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0,$$

where U is a unitary matrix. Let $\dot{Z}$ be the volume element of the space of complex number symmetric matrices. The elements $[\pm 1, \ldots, \pm 1]$ form a subgroup of the unitary group U_n. We denote by $\{U_n\}$ the manifold of left cosets of U_n with respect to this subgroup. Let $\{\dot{U}\}$ be the volume element of $\{U_n\}$; then

$$\dot{Z} = 2^{\frac{1}{2}n(n+1)}\prod_{j<k}|\lambda_j^2 - \lambda_k^2|\lambda_1 \cdots \lambda_n d\lambda_1 \cdots d\lambda_n\{\dot{U}\}.$$

6. Under the transformation

$$S = (I + iT)(I - iT)^{-1}$$

the space of real symmetric matrices T and the space of symmetric unitary matrices S are in one-one correspondence except for lower dimensional manifolds. Moreover

$$\dot{S} = 2^{\frac{1}{2}n(n+1)}\det(I + T^2)^{-\frac{1}{2}(n+1)}\dot{T}.$$

7. Let K denote a skew-symmetric matrix which can be represented in the form

$$K = \Gamma F\Gamma',$$

where Γ is a real orthogonal matrix with determinant 1,

$$F = \begin{cases} \begin{pmatrix} 0 & e^{i\theta_1} \\ -e^{i\theta_1} & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & e^{i\theta_\nu} \\ -e^{i\theta_\nu} & 0 \end{pmatrix}, & \text{when } n \text{ is even,} \\ \begin{pmatrix} 0 & e^{i\theta_1} \\ -e^{i\theta_1} & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & e^{i\theta_\nu} \\ -e^{i\theta_\nu} & 0 \end{pmatrix} \dotplus 0, & \text{when } n \text{ is odd,} \end{cases}$$

$$2\pi \geqslant \theta_1 \geqslant \cdots \geqslant \theta_\nu \geqslant 0, \qquad \nu = [n/2].$$

These matrices form a group denoted by O_n^+.

Let Δ be the group of real orthogonal matrices given by

$$\begin{cases} \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} \cos\theta_\nu & \sin\theta_\nu \\ -\sin\theta_\nu & \cos\theta_\nu \end{pmatrix}, & \text{when } n \text{ is even,} \\ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} \cos\theta_\nu & \sin\theta_\nu \\ -\sin\theta_\nu & \cos\theta_\nu \end{pmatrix} \dotplus 1, & \text{when } n \text{ is odd,} \end{cases}$$

Σ the left coset of O_n^+ with respect to this group, and $\dot{\Sigma}$ its volume element; then

$$\dot{K} = a \prod_{1 \leqslant \alpha < \beta \leqslant \nu} \sin^2(\theta_\beta - \theta_\alpha) d\theta_1 \cdots d\theta_\nu \dot{\Sigma}$$

where

$$a = \begin{cases} 2^{2\nu(\nu-1)+\nu/2}, & \text{when } n \text{ is even,} \\ 2^{2\nu(\nu-1)+3\nu/2}, & \text{when } n \text{ is odd.} \end{cases}$$

8. Under the transformation

$$\Gamma = (I - K)(I + K)^{-1}$$

the space O_n^+ and the space of real symmetric matrices K are in one-one correspondence except for lower dimensional manifolds. Moreover,

$$\dot{\Gamma} = 2^{\frac{1}{2}n(n-1)} \det(I - K^2)^{-\frac{1}{2}(n-1)} \dot{K}.$$

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APPENDIX 3

HARMONIC ANALYSIS ON UNITARY GROUPS

by S. KUNG

In the study of the space of continuous functions on a finite compact group we usually have the following type of approximation theorem: There exists a complete orthonormal system of functions on the group such that every continuous function on the group can be approximated by a finite linear combination of the functions of the system. Such a type of theorem has its theoretical interests only. Generally speaking, a concrete "summability theorem" is much better than or more useful than an "approximation theorem" of the existence type. In §5.11 Professor Hua introduced Abel summability theorems for Fourier expansion of a function defined on the n-dimensional unitary group $\mathfrak{U}_n$. Consequently, we deduce the approximation theorems on any finite compact group (Peter-Weyl [1]) and the approximation theorems on any compact homogeneous space (Weyl [1]).

Inspired by the convergence theorem of Professor Hua, we (Kung [1], [2]) further studied various summability theorems. In this appendix we shall only mention the Dirichlet kernel, Fejér kernel and Cesàro kernel corresponding to ordinary convergence and mean convergence respectively. Surely, we may build the Fourier analysis on a characteristic manifold of all classical domains.

1. **Analytic expression for $\rho_h(\lambda)$.** First, we shall find the analytic expression for $\rho_h(\lambda)$ in §5.11. We have

$$\rho_h(\lambda)I^{(N(h))} = \frac{1}{\omega_n}\int\cdots\int_V \frac{(1-\lambda^2)^{n^2}}{|\det(I-\lambda V)|^{2n}}\, A_h(V)\dot{V}.$$

Calculating the trace, we get

$$\rho_h(\lambda)N(h) = \frac{1}{\omega_n}\int\cdots\int_V \frac{(1-\lambda^2)^{n^2}}{|\det(I-\lambda V)|^{2n}}\, \chi_h(V)\dot{V}. \tag{1.1}$$

For the purpose of evaluation of this definite integral, we introduce the notation

$$\binom{p}{q,\ t} \doteq \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{ip\theta}\,d\theta}{(1-\lambda e^{i\theta})^q(1-\lambda e^{-i\theta})^t}, \qquad 0<\lambda<1, \tag{1.2}$$

where q, t are nonnegative integers, p is an integer.

If $q = 0$, $p < 0$, since

$$e^{ip\theta}(1-\lambda e^{-i\theta})^t$$

is a Fourier series with only negative exponents, therefore

$$\binom{p}{0,\ t} = 0, \qquad \text{if } p < 0. \tag{1.3}$$

Similarly,

$$\binom{p}{q,\ 0} = 0, \qquad \text{if } p > 0. \tag{1.4}$$

From

$$\frac{e^{ip\theta}}{(1-\lambda e^{i\theta})^q(1-\lambda e^{-i\theta})^t} - \frac{e^{ip\theta}}{(1-\lambda e^{i\theta})^{q-1}(1-\lambda e^{-i\theta})^t} = \frac{\lambda e^{i(p+1)\theta}}{(1-\lambda e^{i\theta})^q(1-\lambda e^{-i\theta})^t},$$

it follows that

$$\binom{p}{q,\ t} = \lambda\binom{p+1}{q,\ t} + \binom{p}{q-1,\ t}. \tag{1.5}$$

Similarly, we get

$$\binom{p}{q,\ t} = \lambda\binom{p-1}{q,\ t} + \binom{p}{q,\ t-1}. \tag{1.6}$$

Further, from

$$\frac{1-\lambda^2}{(1-\lambda e^{i\theta})(1-\lambda e^{-i\theta})} = \frac{1}{1-\lambda e^{i\theta}} + \frac{1}{1-\lambda e^{-i\theta}} - 1,$$

we have

$$(1-\lambda^2)\binom{p}{q,\ t} = \binom{p}{q,\ t-1} + \binom{p}{q-1,\ t} - \binom{p}{q-1,\ t-1}. \tag{1.7}$$

(1.5), (1.6) and (1.7) are the fundamental algorithms for evaluating the integral (1.2). Applying (1.5) repeatedly, we obtain

$$\binom{p}{q,\, t} = \binom{p}{q-1,\, t} + \lambda\binom{p+1}{q,\, t}$$
$$= \binom{p}{q-2,\, t} + \lambda\binom{p+1}{q-1,\, t} + \lambda\binom{p+1}{q,\, t}$$
$$= \cdots = \lambda\sum_{j=0}^{q-1}\binom{p+1}{q-j,\, t} + \binom{p}{0,\, t}.$$

If $p < 0$, then from (1.3) we have

$$\binom{p}{q,\, t} = \lambda\sum_{j=0}^{q-1}\binom{p+1}{q-j,\, t}. \tag{1.8}$$

Applying (1.8) repeatedly, we get

$$\binom{p}{q,\, t} = \lambda^2\sum_{j=0}^{q-1}\sum_{k=0}^{q-j-1}\binom{p+2}{q-j-k,\, t} = \lambda^2\sum_{l=0}^{q-1}(l+1)\binom{p+2}{q-l,\, t}.$$

When $p < 0$,

$$\binom{p}{q,\, t} = \lambda^{-p}\sum_{k=0}^{q-1}\binom{k-p-1}{k}\binom{0}{q-k,\, t}. \tag{1.9}$$

Similarly, for $p > 0$, using (1.6), we get

$$\binom{p}{q,\, t} = \lambda^{p}\sum_{k=0}^{t-1}\binom{k+p-1}{k}\binom{0}{q,\, t-k}. \tag{1.10}$$

By (1.9) and (1.10), the problem of evaluating (1.2) is now reduced to the problem of evaluating $\binom{0}{p,\, q}$. And by (1.7), again it is reduced to the problem of evaluating $\binom{0}{0,\, q}$ and $\binom{0}{p,\, 0}$. Obviously, their values are

$$\binom{0}{p,\, 0} = \binom{0}{0,\, q} = 1. \tag{1.11}$$

We now evaluate the integral of (1.1).

Any unitary matrix can be expressed as

$$V = W\Lambda W^{-1}, \qquad \Lambda = [e^{i\theta_1}, \ldots, e^{i\theta_n}], \qquad 2\pi \geqslant \theta_1 \geqslant \cdots \geqslant \theta_n \geqslant 0$$

where W is also a unitary matrix. Since $e^{i\theta_\nu}$ $(1 \leqslant \nu \leqslant n)$ is a characteristic root of V, Λ is uniquely determined. Thus we have

$$N(h)\rho_n(\lambda) = \frac{1}{\omega_n} \int \cdots \int_V \frac{(1-\lambda^2)^{n^2}}{|\det(I-\lambda V)|^{2n}} \chi_h(V)\dot{V}$$

$$= \frac{1}{(2\pi)^n} \int \cdots \int_{2\pi \geqslant \theta_1 \geqslant \cdots \geqslant \theta_n \geqslant 0} \frac{(1-\lambda^2)^{n^2}}{|\det(I-\lambda V)|^{2n}} \chi_h(\Lambda)$$

$$\times |D(e^{i\theta_1}, \ldots, e^{i\theta_n})|^2 d\theta_1 \cdots d\theta_n$$

$$= \frac{1}{(2\pi)^n} \int \cdots \int_{2\pi \geqslant \theta_1 \geqslant \cdots \geqslant \theta_n \geqslant 0} \frac{(1-\lambda^2)^{n^2}}{|\prod_{\nu=1}^n (1-\lambda e^{i\theta_\nu})|^{2n}}$$

$$\times \begin{vmatrix} e^{il_1\theta_1} & \cdots & e^{il_1\theta_n} \\ \cdots & \cdots & \cdots \\ e^{il_n\theta_1} & \cdots & e^{il_n\theta_n} \end{vmatrix} D(e^{-i\theta_1}, \ldots, e^{-i\theta_n}) d\theta_1 \cdots d\theta_n$$

$$= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{(1-\lambda^2)^{n^2}}{|\prod_{\nu=1}^n (1-\lambda e^{i\theta_\nu})|^{2n}} e^{i(l_1\theta_1 + \cdots + l_n\theta_n)}$$

$$\times D(e^{-i\theta_1}, \ldots, e^{-i\theta_n}) d\theta_1 \cdots d\theta_n$$

$$= (-1)^{\frac{1}{2}n(n+1)} \sum_\delta \delta^{0,1,\ldots,n-1}_{k_0,k_1,\ldots,k_{n-1}} \frac{1}{(2\pi)^n}$$

$$\times \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{(1-\lambda^2)^{n^2}}{\prod_{\nu=1}^n |1-\lambda e^{i\theta_\nu}|^{2n}}$$

$$\times e^{i[(l_1-k_0)\theta_1 + \cdots + (l_n-k_{n-1})\theta_n]} d\theta_1 \cdots d\theta_n$$

$$= (-1)^{\frac{1}{2}n(n-1)} (1-\lambda^2)^{n^2} \sum_\delta \delta^{0,1,\ldots,n-1}_{k_0,k_1,\ldots,k_{n-1}}$$

$$\times \prod_{\nu=1}^n \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i(l_\nu - k_{\nu-1})\theta_\nu}}{|1-\lambda e^{i\theta_\nu}|^{2n}} d\theta_\nu \right) =$$

$$= (-1)^{\frac{1}{2}n(n-1)}(1-\lambda^2)^{n^2} \begin{vmatrix} \binom{l_1 - 0}{n,\, n} & \binom{l_2 - 0}{n,\, n} & \cdots & \binom{l_n - 0}{n,\, n} \\ \binom{l_1 - 1}{n,\, n} & \binom{l_2 - 1}{n,\, n} & \cdots & \binom{l_n - 1}{n,\, n} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{l_1 - (n-1)}{n,\, n} & \binom{l_2 - (n-1)}{n,\, n} & \cdots & \binom{l_n - (n-1)}{n,\, n} \end{vmatrix} \tag{1.12}$$

Using (1.5) repeatedly, such as

$$\lambda\binom{l_1 - 1}{n,\, n} = \binom{l_1}{n,\, n} - \binom{l_1}{n,\, n-1},$$

we get

$$\rho_h(\lambda)N(h) = (1-\lambda^2)^{n^2}\lambda^{-\frac{1}{2}n(n-1)} \begin{vmatrix} \binom{l_1}{n,\, n} & \cdots & \binom{l_n}{n,\, n} \\ \binom{l_1}{n,\, n-1} & \cdots & \binom{l_n}{n,\, n-1} \\ \cdots & \cdots & \cdots \\ \binom{l_1}{n,\, 1} & & \binom{l_n}{n,\, 1} \end{vmatrix}.$$

Using (1.5), (1.6) and (1.7) repeatedly, we can evaluate this determinant. Finally, we obtain the analytic expression of $\rho_h(\lambda)$ (Kung [1]). For $l_1 > l_2 > \cdots > l_s \geqslant 0 > l_{s+1} > \cdots > l_n$ $(n \geqslant s \geqslant 0)$ we have

$$\rho_h(\lambda) = \lambda^{h_1 + \cdots + h_s - h_{s+1} - \cdots - h_n}$$

$$\times \sum_{s \geqslant g_{s+1} \geqslant \cdots \geqslant g_n \geqslant 0} \frac{N_s(h, g)N_s(g, h)}{N(h)N(g)} \lambda^{2(g_{s+1} + \cdots + g_n)},$$

where $N_s(a, b) = N(a_1, \ldots, a_s, b_{s+1}, \ldots, b_n)$, and in $g = (g_1, \ldots, g_n)$, the set of integers $g_1 + n - 1, g_2 + n - 2, \ldots, g_n$ is a permutation of $0, 1, 2, \ldots, n-1$, and $0 \geqslant g_1 \geqslant g_2 \geqslant \cdots \geqslant g_s \geqslant s - n$.

Of course, we may write $\rho_h(\lambda)$ as a polynomial

$$\rho_h(\lambda) = \lambda^{\sum_{\nu=1}^{n} |h_\nu|} \sum_{\nu=0}^{s(n-s)} b_\nu \lambda^{2\nu}$$

with coefficients

$$b_\nu = \sum_{\substack{s \geqslant g_{s+1} \geqslant \cdots \geqslant g_n \geqslant 0 \\ g_{s+1} + \cdots + g_n = \nu}} \frac{N_s(h, g) N_s(g, h)}{N(h) N(g)}.$$

It is a polynomial of degree $\sum_{\nu=0}^{n} |h_\nu| + 2s(n-s)$. When $s = 0$ or n,

$$\rho_h(\lambda) = \lambda^{\sum_{\nu=1}^{n} |h_\nu|}.$$

2. Dirichlet kernel.

LEMMA. *Let*

$$g(\theta) = \sum_{-M \leqslant \nu \leqslant N} a_\nu e^{i\nu\theta}, \tag{2.1}$$

where M, N can be finite or infinite (in the case of infinity, we shall assume the convergence of (2.1)). *Then we have the following identity*:

$$\frac{(-i)^{\frac{1}{2}n(n-1)}}{1!2!\cdots(n-1)!} \begin{vmatrix} g(\theta_1) & \cdots & g(\theta_n) \\ g'(\theta_1) & \cdots & g'(\theta_n) \\ \cdots & \cdots & \cdots \\ g^{(n-1)}(\theta_1) & \cdots & g^{(n-1)}(\theta_n) \end{vmatrix} \tag{2.2}$$

$$= \sum_{N \geqslant l_1 > l_2 > \cdots > l_n \geqslant -M} a_{l_1} \cdots a_{l_n} N(f_1, \ldots, f_n)\, M_f(e^{i\theta_1}, \ldots, e^{i\theta_n}).$$

PROOF. Let

$$G(z) = \sum_{-M \leqslant \nu \leqslant N} a_\nu z^\nu, \tag{2.3}$$

where M, N can be finite or infinite; in the case of infinity, we should assume the convergence of (2.3).

By (1.2.4) and Theorem 1.2.4, we have

$$\frac{(-1)^{\frac{1}{2}n(n-1)}}{1!2!\cdots(n-1)!}\begin{vmatrix} G(x_n) & \cdots & G(x_n) \\ G'(x_1) & \cdots & G'(x_n) \\ \cdots & \cdots & \cdots \\ G^{(n-1)}(x_1) & \cdots & G^{(n-1)}(x_n) \end{vmatrix}$$

$$= \sum_{N \geqslant l_1 > l_2 > \cdots > l_n \geqslant -M} a_{l_1} \cdots a_{l_n} N(f_1, \ldots, f_n) M_f(x_1, \ldots, x_n).$$

Take $z = xe^{i\theta}$ and let $G(e^{i\theta}) = g(\theta)$; then

$$g'(\theta) = iG'(e^{i\theta})e^{i\theta} = i\frac{\partial}{\partial x}G(xe^{i\theta})|_{x=1},$$

$$g''(\theta) = i^2G''(e^{i\theta})e^{2i\theta} + i^2G'(e^{i\theta})e^{i\theta}$$

$$= i^2\frac{\partial^2}{\partial x^2}G(xe^{i\theta})|_{x=1} + i^2\frac{\partial}{\partial x}G(xe^{i\theta})|_{x=1},$$

$$g'''(\theta) = i^3G'''(e^{i\theta})e^{3i\theta} + 3i^3G''(e^{i\theta})e^{2i\theta} + i^3G'(e^{i\theta})e^{i\theta}$$

$$= i^3\frac{\partial^3}{\partial x^3}G(xe^{i\theta})|_{x=1} + 3i^3\frac{\partial^2}{\partial x^2}G(xe^{i\theta})|_{x=1} + i^3\frac{\partial}{\partial x}G(xe^{i\theta})|_{x=1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots;$$

that is, $g^{(l)}(\theta)$ is a linear combination of

$$\frac{\partial^l}{\partial x^l}G(xe^{i\theta})|_{x=1}, \ldots, \frac{\partial}{\partial x}G(xe^{i\theta})|_{x=1},$$

and the coefficient of $(\partial^l/\partial x^l)G(xe^{i\theta})|_{x=1}$ is i^l; hence

$$\left|\begin{matrix} G(x_1e^{i\theta_1}) & \cdots & G(x_ne^{i\theta_n}) \\ \frac{\partial}{\partial x_1}G(x_1e^{i\theta_1}) & \cdots & \frac{\partial}{\partial x_n}G(x_ne^{i\theta_n}) \\ \cdots & \cdots & \cdots \\ \frac{\partial^{n-1}}{\partial x_1^{n-1}}G(x_1e^{i\theta_1}) & \cdots & \frac{\partial^{n-1}}{\partial x_n^{n-1}}G(x_ne^{i\theta_n}) \end{matrix}\right|_{x_1=1,\ldots,x_n=1}$$

$$= i^{\frac{1}{2}n(n-1)}\left|\begin{matrix} g(\theta_1) & \cdots & g(\theta_n) \\ g'(\theta_1) & \cdots & g'(\theta_n) \\ \cdots & \cdots & \cdots \\ g^{(n-1)}(\theta_1) & \cdots & g^{(n-1)}(\theta_n) \end{matrix}\right|.$$

Now, we take

$$g(\theta) = \sum_{N+n-1\geqslant\nu\geqslant-N} e^{i\nu\theta} = e^{\frac{1}{2}i(n-1)\theta}\,\frac{\sin(N+\frac{1}{2}n)\theta}{\sin\frac{1}{2}\theta} = d_N(\theta)e^{\frac{1}{2}i(n-1)\theta}. \tag{2.4}$$

Since

$$\left|\begin{matrix} g(\theta_1) & \cdots & g(\theta_n) \\ g'(\theta_1) & \cdots & g'(\theta_n) \\ \cdots & \cdots & \cdots \\ g^{(n-1)}(\theta_1) & \cdots & g^{(n-1)}(\theta_n) \end{matrix}\right| \tag{2.5}$$

$$= e^{\frac{1}{2}i(n-1)(\theta_1+\cdots+\theta_n)}\left|\begin{matrix} d_N(\theta_1) & \cdots & d_N(\theta_n) \\ d_N'(\theta_1) & \cdots & d_N'(\theta_n) \\ \cdots & \cdots & \cdots \\ d_N^{(n-1)}(\theta_1) & \cdots & d_N^{(n-1)}(\theta_n) \end{matrix}\right|$$

and

$$D(e^{i\theta_1}, \ldots, e^{i\theta_n}) = \prod_{1 \leqslant \nu < \mu \leqslant n} (e^{i\theta_\nu} - e^{i\theta_\mu})$$

$$= \prod_{1 \leqslant \nu < \mu \leqslant n} (e^{\frac{1}{2}i(\theta_\nu - \theta_\mu)} - e^{\frac{1}{2}i(\theta_\nu - \theta_\mu)}) e^{\frac{1}{2}i(\theta_\nu + \theta_\mu)}$$

$$= (-2i)^{\frac{1}{2}n(n-1)} \prod_{\mu < \nu} \sin \tfrac{1}{2}(\theta_\nu - \theta_\mu) e^{\frac{1}{2}(n-1)i(\theta_1 + \cdots + \theta_n)}, \qquad (2.6)$$

by the lemma, we have

$$\frac{2^{-\frac{1}{2}n(n-1}}{1!2! \cdots (n-1)!} \begin{vmatrix} d_N(\theta_1) & \cdots & d_N(\theta_n) \\ d'_N(\theta_1) & \cdots & d'_N(\theta_n) \\ \cdots & \cdots & \cdots \\ d_N^{(n-1)}(\theta_1) & \cdots & d_N^{(n-1)}(\theta_n) \end{vmatrix}$$

$$\times \frac{1}{\Pi_{1 \leqslant \mu < \nu \leqslant n} \sin \frac{1}{2}(\theta_\nu - \theta_\mu)}$$

$$= \sum_{N+n-1 \geqslant l_1 > l_2 > \cdots > l_n \geqslant -N} N(f_1, \ldots, f_n)$$

$$\times M_f(e^{i\theta_1}, \ldots, e^{i\theta_n}) \Big/ D(e^{i\theta_1}, \ldots, e^{i\theta_n})$$

$$= \sum_{N \geqslant f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n \geqslant -N} N(f_1, \ldots, f_n) \chi_f(e^{i\theta_1}, \ldots, e^{i\theta_n}). \qquad (2.7)$$

This expression is called the Dirichlet kernel in the unitary group $\mathfrak{U}_n$, and is denoted by $\mathcal{D}_n(U)$.

Now, we turn to the partial sum of the Fourier series

$$S_N = \sum_{N \geqslant f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n \geqslant -N} c_{ij}^f \varphi_{ij}^f(U), \qquad (2.8)$$

where

$$c_{ij}^f = \frac{1}{\omega_n} \int \cdot \underset{V}{\cdot} \cdot \int f(V) \overline{\varphi_{ij}^f(V)} \dot{V}.$$

It follows that

$$S_N = \frac{1}{\omega_n} \sum_{N \geq f_1 \geq \cdots \geq f_n \geq -N} \int \cdots \int_V f(V) \varphi_{ij}^f(U) \overline{\varphi_{ij}^f(V)} \dot{V}$$

$$= \frac{1}{\omega_n} \int \cdots \int_V f(V) \mathcal{D}_N(U\bar{V}') \dot{V}.$$

This method shows that corresponding to a method of summability for Fourier series of a single variable, we have a method of summability on $\mathfrak{U}_n$. Also, in the following section, another idea (Kung [2]) will be introduced. In short, the method of this section can be described as follows: given a method of summation, a corresponding kernel can be obtained. The method in the next section goes in the opposite direction: given a kernel and its Fourier expansion, a method of summation can be obtained.

3. Fejér kernel. We have also introduced various types of methods of summability (Kung [1], [2]). As an example, we give a generalization of the Fejér kernel:

$$F_N(V) = \frac{1}{B_N(N+1)^{n^2}} \left| \frac{\det(I - V^{n+1})}{\det(I - V)} \right|^{2n}, \tag{3.1}$$

where B_N is chosen so that $\int \cdots_U \int F_N(U)\dot{U} = 1$.

The Fejér sum of the Fourier series of the function $f(U)$ is defined as

$$\sum_{nN \geq f_1 \geq f_2 \geq \cdots \geq f_n \geq -nN} B_{f_1,\ldots,f_n} \operatorname{Sp}(c_{f_1,\ldots,f_n} A'_{f_1,\ldots,f_n}(U)), \tag{3.2}$$

where $c_f = (c_{ij}^f)_{1 \leq ij \leq N(f)}$ is the matrix formed by the Fourier coefficients c_{ij}^f, and

$$B_{f_1,\ldots,f_n} = \int \cdots_V \int \chi_f(\bar{V}) F_N(V) V.$$

We proved that, for any continuous function $f(U)$, the Fejér sum tends to $f(U)$. Also we found explicitly that

$$B_N = \frac{(n!)^{n-1}((2n)!)^{n-1} 2^{n(n-1)}}{((2n-1)! \cdots (n+1)!)^2} \left(1 - \frac{1}{N+1}\right)^{\frac{1}{2}n(n-1)} \tag{3.3}$$

and

$$B_{f_1,\dots,f_n} = \frac{(2n)! \cdot \cdot \cdot 2!1!}{N(f)\,((2n)!)^2 n!\; 2^{n(n-1)}(N+1)^{\frac{1}{2}n(n+1)} N^{\frac{1}{2}n(n-1)}}$$

$$\times \sum_{\substack{s_1=0\\k_1\geqslant 0}}^{2n} \cdots \sum_{\substack{s_n=0\\k_n\geqslant 0}}^{2n} \binom{2n}{s_1}\binom{n+k_1}{n} \cdots \binom{2n}{s_n}\binom{n+k_n}{n}$$

$$\times N((N+1)s_1 - f_1, \dots, (N+1)s_n - f_n). \tag{3.4}$$

Moreover, we defined the Cesàro (c, α) sum of the Fourier series of the function $f(U)$ as

$$\sum_{nN\geqslant f_1\geqslant f_2\geqslant\cdots\geqslant f_n\geqslant -nN} \beta^{\alpha}_{f_1,\dots,f_n} \mathrm{Sp}(c_{f_1,\dots,f_n} A'_{f_1,\dots,f_n}(U)), \tag{3.5}$$

where $\alpha > -1$, and

$$\beta^{\alpha}_{f_1,\dots,f_n} = \frac{1}{\omega_n N(f)} \int \cdot \underset{V}{\cdot} \cdot \int \chi_{f_1,\dots,f_n}(\bar{V}) K^{\alpha}_N(V)\dot{V};$$

$K^{\alpha}_N(v)$ is the Cesàro (c, α) kernel, which is equal to

$$\frac{\det^n \left[\sum_{k=0}^{n} \binom{N+\alpha-k-1}{N-k} V^k(I - \bar{V}'^{k+1}) \right]}{B^{\alpha}_N \left(2\binom{N+\alpha}{N} \right)^{n^2} \det^n(I - \bar{V}')},$$

where B^{α}_N is chosen so that $\int \cdot \underset{U}{\cdot} \cdot \int K^{\alpha}_N(U)\dot{U} = 1$.

We have proved that the Fourier series of a function which is continuous on the unitary group $\mathfrak{U}_n$ is (c, α) summable to this function when $\alpha > (n-1)/n$.

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