

Translations of  
**MATHEMATICAL  
MONOGRAPHS**

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Volume 7

**Applications  
of Functional Analysis  
in Mathematical Physics**

S. L. Sobolev



**American Mathematical Society**



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Providence, Rhode Island

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## AUTHOR'S PREFACE

The present book arose as a result of revising a course of lectures given by the writer at the Leningrad State University. The notes for the lectures were taken and revised by H. L. Smolicki and I. A. Jakovlev, who contributed to them a series of valuable remarks and additions. Several additions, arising naturally during the lectures, were also made by the author himself.

In this fashion there came into being this monograph, a unifying treatment from a single point of view of a number of problems in the theory of partial differential equations. There are considered in it variational methods with applications to the Laplace equation and the polyharmonic equations as well as the Cauchy problem for linear and quasi-linear hyperbolic equations. The presentation of the problems of mathematical physics demands a suitable consideration of some new results and methods in functional analysis, which constitute in themselves the basis of all the later material. The first part is concerned with this basis. The material indicated above, the particular problems posed, and the methods for their investigation are not to be found in the ordinary course in mathematical physics and in particular, they are not in my book *Equations of mathematical physics*. The present book is of value for graduate students and research workers.

The author warmly thanks his assistants, H. L. Smolicki and I. A. Jakovlev, without whose assistance this book could not have been written in such a short time.

S. Sobolev

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