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Probabilistic Methods in the Theory of Numbers

J. Kubilius

American Mathematical Society

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ВЕРОЯТНОСТНЫЕ МЕТОДЫ В ТЕОРИИ ЧИСЕЛ

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PREFACE

For a long time the role of probabilistic arguments in number theory was almost exclusively that of heuristic device. In the course of the past twenty years, however, probabilistic methods have been firmly established on the same footing with the other methods of number theory. This is but another instance of the penetration of the problems and methods of one branch of knowledge into another, which is characteristic of modern science. The present book is devoted to applications of methods of probability theory to number theory. It was not possible, within the limits of the book, to cover more than a part of such applications. Many deep and important results have not been touched upon. The choice of material was determined not only by such subjective factors as the personal tastes and interests of the author, but also by the fact that, as noted in the Introduction, it is still difficult to give a systematic exposition of all the various applications of probability theory to number theory. For that reason the author decided to limit himself to the probabilistic theory of the distribution of additive number-theoretic functions.

The first variant of the present book was published in 1959. In 1962 appeared a second edition which had been considerably revised and augmented with results obtained since the publication of the first edition. I am very grateful to the American Mathematical Society for deciding to publish an English translation of my book. This is especially gratifying to me in view of the important contribution of American mathematicians to the probabilistic theory of numbers. Several of their results, as the reader will see, are reflected in the book. The translation was made from the second edition, but is not identical with it. In addition to corrections of an editorial nature, there are two substantial changes. In Chapter IV I have added a proof of the Erdös-Wintner theorem, and the part of Chapter IX having to do with the theorem on large deviations has been almost completely rewritten. A difficult result was replaced by one which is simpler and more transparent, although somewhat less precise.

It is my pleasant duty to thank Professor W. J. LeVeque for his original suggestion that a translation should be published and for his general supervision of the work.

I am especially grateful to Mrs. Susan Schuur and Miss Gretchen Burgie for their careful translation.

J. Kubilius

NOTATION

The following notation and definitions are used throughout the book, except in Chapter X.

m is a positive integer;

n is a large positive integer;

s is a fixed positive integer;

$(m_1, \cdots, m_s)$ and $[m_1, \cdots, m_s]$ are, respectively, the greatest common divisor and the least (positive) common multiple of the numbers $m_1, \cdots, m_s$;

$N_n\{\cdots\}$ is the number of positive integers $m \leqq n$ satisfying the conditions which appear in the brackets;

$\nu_n\{\cdots\} = N_n\{\cdots\}/n$ is the frequency of positive integers $m \leqq n$ satisfying the conditions given in backets;

$-p$ and q are primes;

$\sum_p$, $\sum_{p^\alpha}$, $\prod_p$, $\prod_{p^\alpha}$, denote that the sum or product is taken over all primes or, respectively, over all prime powers;

$p^\alpha \parallel m$ means that $p^\alpha \mid m$, $p^{\alpha+1} \nmid m$;

$r = r(n)$ is some function of n, which in each instance is defined more precisely;

$$\gamma_p = \left[\frac{\ln r}{\ln p}\right];$$

$$\pi(p^\alpha) = \begin{cases} \dfrac{1}{p^\alpha}\left(1 - \dfrac{1}{p}\right) & \text{if} \quad 0 \leqslant \alpha < \gamma_p, \\ \dfrac{1}{p^\alpha}, & \text{if} \quad \alpha = \gamma_p; \end{cases}$$

$$\rho(p^\alpha) = \frac{1}{p^\alpha}\left(1 - \frac{1}{p}\right);$$

$\alpha_p(m)$ is the non-negative integer such that $p^{\alpha_p(m)} \parallel m$;

$$\beta_p(m) = \min\left(\alpha_p(m),\ \gamma_p\right);$$

$$\delta_p(m) = \begin{cases} 1, & \text{if} \quad p \mid m, \\ 0, & \text{if} \quad p \nmid m; \end{cases}$$

$$\eta_p(m) = \begin{cases} \frac{1}{p}, & \text{if} \quad p \mid m, \\ 1 - \frac{s}{p}, & \text{if} \quad p \nmid m; \end{cases}$$

$f(m)$, $f_1(m), \cdots, f_s(m)$ are additive number-theoretic functions;

$$f^{(p)}(m) = f(p^{\beta_p(m)});$$

$$f(m)_u = \sum_{p \leq u} f^{(p)}(m);$$

$$A(u) = \sum_{p \leq u} \frac{f(p)}{p}, \quad A_k(u) = \sum_{p \leq u} \frac{f_k(p)}{p} \quad (k = 1, \ldots, s);$$

$$B(u) = \left(\sum_{p \leq u} \frac{|f^2(p)|}{p} \right)^{\frac{1}{2}},$$

$$B_k(u) = \left(\sum_{p \leq u} \frac{|f_k^2(p)|}{p} \right)^{\frac{1}{2}} \quad (k = 1, \ldots, s);$$

$$D(u) = \left(\sum_{p^\alpha \leq u} \frac{|f^2(p^\alpha)|}{p^\alpha} \right)^{\frac{1}{2}},$$

$$D_k(u) = \left(\sum_{p^\alpha \leq u} \frac{|f_k^2(p^\alpha)|}{p^\alpha} \right)^{\frac{1}{2}} \quad (k = 1, \ldots, s);$$

$\omega(m)$ is the number of distinct prime divisors of m;

$\Omega(m)$ is the total number of prime divisors of m, i.e., multiple divisors are counted according to their multiplicity;

$\omega_1(m)$ is the number of distinct prime divisors of m of the form $4k+1$;

$\omega_2(m)$ is the number of distinct prime divisors of m of the form $4k-1$;

$\tau_k(m)$ is the number of representations of m as a product of k factors, taking into account the order of the factors;

$\tau(m) = \tau_2(m)$ is the number of divisors of m;

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} \, du;$$

$c, c_1, c_2, \cdots$ are absolute constants.

B is a number (not always the same one) which is bounded in absolute value by a constant. This constant is either absolute or depends on given parameters. In proofs, the constant may depend on the parameters which appear in the statements of the lemmas or theorems.

Estimates involving the symbols O, o, $\sim$, $\asymp$ are for the most part relative to n.

By the asymptotic density of a set of positive integers E is meant the limit $\lim_{n\to\infty} \nu_n\{m \in E\}$ whenever it exists.

The remaining notation is either generally accepted or explained in the text. Any departures from the notation given here will be mentioned when they occur.

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INTRODUCTION

Long ago it was noticed that certain results of number theory can be interpreted probabilistically. Therefore it seemed natural to try to use methods of probability theory in order to solve number-theoretic problems. The axiomatization of probability theory ensured general acceptance for the use of such methods in the theory of numbers. There is now a number of results in all of the major branches of arithmetic whose proofs are strictly dependent on probability-theoretic concepts. The introduction in number theory of ideas and methods of probability theory has made possible a new approach to several number-theoretic problems; it has supplied number theory with a new arsenal of refined methods of study, and has led to new, and sometimes very unexpected, results.

At the present time it is still difficult to give a systematic exposition of the various applications of probability theory to number theory. We shall restrict ourselves to one field of such applications: to the theory of the distribution of the values of additive and multiplicative functions. The values of these functions are closely related to the properties of the distribution of primes in the sequence of positive integers. For other applications of probability theory to arithmetic we refer the reader to Ju. V. Linnik's book [**24**], containing profound results on binary additive problems, and also to the books of A. G. Postnikov [**26**] and M. Kac [**76**], and to survey articles by M. Kac [**74**], P. Erdös [**61**; **62**], A. Rényi [**87**], and the author [**19**].

A sequence of real or complex numbers $f(m)$ (or $g(m)$) $(m=1,2,\cdots)$ is called an *additive* (*multiplicative*) number-theoretic function, if for any pair of relatively prime integers m_1, m_2,

$$f(m_1 m_2)=f(m_1)+f(m_2) \quad (g(m_1 m_2)=g(m_1)\,g(m_2)).$$

From the definition it follows immediately that $f(1)=0$, and if $g(m)$ is not identically zero (only such multiplicative functions will be considered below), $g(1)=1$. Furthermore, it is obvious that additive and multiplicative functions can be represented in the form

$$f(m)=\sum_{p^\alpha \| m} f(p^\alpha)=\sum_p f(p^{\alpha_p(m)}),$$

$$g(m)=\prod_{p^\alpha \| m} g(p^\alpha)=\prod_p g(p^{\alpha_p(m)}).$$

From these "canonical" representations it follows that additive and multiplicative functions are completely determined by giving their values $f(p^\alpha)$, $g(p^\alpha)$ for prime powers p^α. If the values of $f(m)$ coincide for all powers of a prime, and similarly for $g(m)$, i.e., $f(p^\alpha)=f(p)$, $g(p^\alpha)=g(p)$ for all p and all $\alpha=2,3,\cdots$, then they are called *strongly additive* and *strongly multiplicative*, respectively. If $f(m_1m_2)=f(m_1)+f(m_2)$, $g(m_1m_2)=g(m_1)\,g(m_2)$ for any positive integers m_1, m_2, not necessarily relatively prime, then they are said to be *completely additive* and *completely multiplicative*, respectively. In this case $f(p^\alpha)=\alpha f(p)$, $g(p^\alpha)=g^\alpha(p)$ for all primes p and $\alpha=2,3,\cdots$.

The concept of additive and multiplicative functions can be generalized by defining them on any multiplicative semi-group of integers, or even on multiplicative semi-groups of an arbitrary nature.

As is well known, most of the classical functions of number theory are additive or multiplicative, and many classical problems of arithmetic are closely connected with the behavior of these functions. Therefore, the study of such functions occupies a significant place in the problems of number theory.

In general, the values of both additive and multiplicative functions are distributed very erratically. If we follow the change in the values of these functions as the argument runs through the positive integers in order, we obtain a very chaotic picture, which is usually observed when considering jointly the additive and multiplicative properties of the integers. Nevertheless it turns out that, in the large, the distribution of the values of many of these functions is subject to certain simple laws, which can be formulated and proved by using ideas and methods of probability theory. In addition, the study of the distribution of the values of multiplicative functions leads, in many instances, to the study of additive functions. This happens, for example, when the multiplicative function $g(m)$ is everywhere positive, since in that case the principal value of the logarithm of $g(m)$ is a real additive function. Hence in what follows we shall study primarily additive number-theoretic functions.

In classical and in more recent research, in studying the distribution of the values of number-theoretic functions $f(m)$, mathematicians usually limited themselves to consideration of the sum

$$A_n=\frac{1}{n}\sum_{m=1}^{n} f(m),$$

which is the mean value of the function $f(m)$ on the segment $\{1,\cdots,n\}$ of the sequence of positive integers, and sought asymptotic approximating expressions for it in terms of the simplest possible functions of n. It is evident, however, that the values of the function $f(m)$ can oscillate about the mean value A_n within very wide bounds, even in the case of relatively simple classical number-theoretic functions. Thus, in the case of the function $\omega(m)$, a simple calculation shows that

$$A_n \sim \ln\ln n,$$

while

$$\underline{\lim}_{n\to\infty} \omega(m) = 1,$$

$$\overline{\lim}_{n\to\infty} \frac{\omega(m) \ln\ln m}{\ln m} = 1.$$

However, it turns out that very large or very small values of additive functions occur rather seldom on the whole. Therefore it is natural to raise the question as to how much the values of these functions deviate from the mean value for the great majority of values of the argument.

The first nontrivial result in this direction for the functions in which we are interested is due to Hardy and Ramanujan [71], who showed in 1917 that for any positive function $\psi(n)$ which increases without bound as $n \to \infty$, the frequency

$$\nu_n\left\{ \left| \omega(m) - \ln\ln n \right| \leqslant \psi(n) \sqrt{\ln\ln n} \right\}$$

approaches unity as $n \to \infty$. In other words, for "almost all" positive integers $m \leqq n$, the values of the function $\omega(m)$ deviate from $\ln\ln n$ by an amount not exceeding $\psi(n)\sqrt{\ln\ln n}$ in absolute value.

The proof which Hardy and Ramanujan proposed is arithmetical and rather complicated. It is based on a very precise upper bound for $N_n\{\omega(m) = k\}$ (k is a positive integer):

$$N_n\left\{ \omega(m) = k \right\} < \frac{c_1 n (\ln\ln n + c_2)^{k-1}}{(k-1)! \ln n}.$$

On the other hand, the Hardy-Ramanujan theory is an analogue of the probability-theoretic law of large numbers. Therefore it naturally occurs to one to attempt to use in its proof concepts analogous to those used in the proof of the law of large numbers. Such a proof was found by P. Turán [95] in 1934. It depends on the easily obtained estimate

$$\sum_{m=1}^{n} \left(\omega(m) - \ln\ln n \right)^2 = Bn \ln\ln n,$$

which implies

$$\psi^2(n)\ln\ln n\cdot N_n\left\{\left|\omega(m)-\ln\ln n\right|>\psi(n)\sqrt{\ln\ln n}\right\}=Bn\ln\ln n.$$

The Hardy-Ramanujan theorem follows from this. In Turán's proof it is easy to see the analogue of Čebyšev's method for the proof of the law of large numbers.

Using the same device, Turán [**96**] generalized this theorem to a larger class of functions: if for all primes p a strongly additive number-theoretic function $f(m)$ satisfies

$$0\leqslant f(p)<c_3$$

and if $A(n)\to\infty$ as $n\to\infty$, then

$$\nu_n\left\{\left|f(m)-A(n)\right|\leqslant\psi(n)\sqrt{A(n)}\right\}\to 1.$$

Wishing to characterize more precisely the distribution of the values of number-theoretic functions $f(m)$ on the segment $\{1,\cdots,n\}$ of the sequence of positive integers, we naturally arrive at the concepts of asymptotic local and integral distribution laws. In the first case it is a matter of finding an asymptotic expression for the number of positive integers $m\leqq n$ for which $f(m)$ assumes a given value. At present, such asymptotic expressions are known for only a few concrete number-theoretic functions. Thus, it follows with relatively little difficulty from the prime number theorem that for any fixed positive integer k,

$$\nu_n\left\{\omega(m)=k\right\}\sim\frac{(\ln\ln n)^{k-1}}{(k-1)!\ \ln n}.$$

Thus the function $\omega(m)$ is distributed on the set $\{1,\cdots,n\}$ approximately according to the Poisson law with parameter $\ln\ln n$. An analogous formula has also been obtained for the case when k increases not too rapidly with n, namely, for all $k<c_4\ln\ln n$, where c_4 is a constant <2. The known proofs of local theorems of this kind require rather refined arithmetic arguments; usually they are deduced from the prime number theorem or involve the same ideas as lead to the prime number theorem.

In the case of integral asymptotic laws we seek asymptotic expressions for $\nu_n\{f(m)<x\}$, where x is any real number. Since $\nu_n\{f(m)<x\}$ is a distribution function in the probability-theoretic sense, it is natural to consider the convergence of the sequence of distribution functions $\nu_n\{f(m)<x\}$ $(n=1,2,\cdots)$, where $f(m)$ is "centered and normalized" in an appropriate

manner, to a limiting distribution function at each of its points of continuity.

Very general results are due to P. Erdös and A. Wintner. By generalizing a number of previous results [37; 44; **54**$^{\mathrm{I,II}}$; **90**], Erdös [**54**$^{\mathrm{III}}$] proved in 1937 that for any real additive function $f(m)$, the convergence of the series

$$\sum_p \frac{\| f(p) \|}{p}, \tag{1}$$

$$\sum_p \frac{\| f(p) \|^2}{p}, \tag{2}$$

where

$$\| f(p) \| = \begin{cases} f(p), & \text{if} \quad |f(p)| \leqslant 1, \\ 1, & \text{if} \quad |f(p)| > 1, \end{cases}$$

is a sufficient condition for the convergence of the distribution function $\nu_n\{f(m) < x\}$, as $n \to \infty$, to a limiting distribution function. This condition also proved to be necessary [**65**].

The set of points of increase of the limiting distribution function coincides with the closure of the set of values of the function $f(m)$. In this connection, if the series

$$\sum_{f(p) \neq 0} \frac{1}{p}$$

converges, then the limiting distribution function is discrete; if this series diverges, then the limiting distribution function is continuous: absolutely continuous or singular. All of these cases actually occur [**57**].

If series (1) diverges while series (2) converges, then [**58**]

$$\nu_n \left\{ f(m) - \sum_{p \leqslant n} \frac{\| f(p) \|}{p} < x \right\}$$

converges to a limiting distribution function.

On the whole, these results were obtained by elementary methods. It is much more difficult to demonstrate the existence of asymptotic integral laws for additive functions with diverging series (2). This occurs, for example, for the function $\omega(m)$. Those devices which are useful in the study of additive functions in the cases considered above are not, in general, applicable here. If we had sufficiently precise local theorems, we could use them for the solution of this problem. Thus, from the aforementioned asymptotic formulas for $\omega(m)$ it is possible to obtain the following: for $(1+o(1))\ln\ln n<x<c_4\ln\ln n$, $c_4<2$,

$$\nu_n\left\{\omega(m)<x\right\}\sim\frac{1}{\ln n}\sum_{1\leq k<x}\frac{(\ln\ln n)^{k-1}}{(k-1)!}.$$

Hence, using the central limit theorem applied to a sum of independent random variables, distributed according to the Poisson law with parameter $\ln\ln n$, it can be shown, in particular, that for any fixed x,

$$\nu_n\left\{\frac{\omega(m)-\ln\ln n}{\sqrt{\ln\ln n}}<x\right\}\sim\frac{1}{\ln n}\sum{}'\frac{(\ln\ln n)^{k-1}}{(k-1)!}\sim G(x). \tag{3}$$

The accent on the summation sign denotes that the summation is over all positive integers $k<\ln\ln n+x\sqrt{\ln\ln n}$. Consequently, the values of the function $\omega(m)$ are distributed asymptotically according to the normal law.

A simpler proof of (3), for the particular case $x=0$, was proposed in 1936 by P. Erdös [**56**]; it uses the method of the sieve of Eratosthenes instead of the prime number theorem. In 1947 W. LeVeque [**79**], using the same method, proved (3) in the general case; he also obtained an estimate of the remainder term.

The arithmetic method of Erdös and LeVeque is based on specific properties of the function $\omega(m)$ and is difficult to generalize. However, in 1939 [**63**; **64**] Erdös and M. Kac used the central limit theorem of probability theory, together with elementary arithmetic methods like the sieve of Eratosthenes, to obtain the following more general theorem.

If a real, strongly additive number-theoretic function $f(m)$ has the properties: $B(n)\to\infty$ as $n\to\infty$ and $|f(p)|\leqq 1$ for all primes p, then

$$\nu_n\left\{\frac{f(m)-A(n)}{B(n)}<x\right\}\to G(x)$$

as $n\to\infty$.

Another subject of study, as interesting as the behavior of the additive function $f(m)$, is the joint distribution of the functions $f(m)$ and $f(m+1)$. The first result in this direction is due to W. LeVeque, who proved in 1947 [**79**] that for a function $f(m)$ satisfying the conditions of the Erdös-Kac theorem,

$$\nu_n\left\{\frac{f(m)-A(n)}{B(n)}<x,\quad\frac{f(m+1)-A(n)}{B(n)}<y\right\}\to G(x)\,G(y);$$

from this he deduced that

$$\nu_n \left\{ \frac{\omega(m) - \omega(m+1)}{\sqrt{2 \ln\ln n}} < x \right\} \to G(x).$$

Earlier, Erdös [**55**; **58**] had studied relations of the latter type for $x=0$, using elementary methods.

In 1954-1955, proceeding from the Kolmogorov axioms for probability theory, the present author succeeded in giving a probability-theoretic interpretation of additive functions, and thus in reducing questions on the distribution of the values of these functions to the corresponding problems of the theory of sums of independent random variables.

During the last few years, the probabilistic theory of the distribution of additive functions has been studied by many authors: M. B. Barban, E. Vilkas, H. Delange, P. Erdös, M. Kac, J. Kubilius, K. Prachar, A. Rényi, G. Rieger, M. Tanaka, P. Turán, R. Uzdavinis, H. Halberstam, H. Shapiro. The methods of this theory have been worked out, previous results have been strengthened and generalized in different directions, and many new results have been obtained.

The theorems proved in this book include almost all of the basic known results in the probabilistic theory of the distribution of additive and multiplicative functions. Several of them are published for the first time here.

In the first two chapters, the method of proof is developed. The lemmas of the first chapter constitute the arithmetic nucleus of our further arguments. The probability-theoretic interpretation of additive number-theoretic functions is presented in the second chapter.

Chapter III contains the proof of an analogue of the law of large numbers for arbitrary additive functions. In Chapters IV-V we consider one-dimensional integral and local asymptotic laws for additive functions and their sums. The results of Chapters I and II, together with well-known limit theorems of the theory of sums of independent random variables, enable us to give necessary and sufficient conditions for the existence of asymptotic integral laws for a relatively large class of additive functions.

In Chapter VI we estimate the rate of convergence to the normal law of the laws of distribution of the values of additive functions on finite segments of the sequence of positive integers.

Several analogues of well-known limit theorems on the behavior in the large of a sequence of sums of independent random variables are proved in Chapter VII.

In Chapter VIII many-dimensional integral asymptotic laws are considered.

Chapter IX is devoted to the asymptotic expansion of the distribution laws of a certain class of additive number-theoretic functions, and to theorems on large deviations.

In Chapter X we consider additive number-theoretic functions in the Gaussian number field.

In the bibliography are included all basic works on the probabilistic theory of additive number-theoretic functions, as well as those to which reference is made.

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