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Volume 13

Additive Theory of Prime Numbers

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American Mathematical Society

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Institute of Mathematics, Chinese Academy of Sciences, Peking, 1957

Translated from the Chinese by N. H. Ng

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FOREWORD

This book is a summing up of the methods of study of the additive theory of prime numbers by Academician Vinogradov of Soviet Russia and the author, with discussions centering on Vinogradov's mean-value theorem and its improvement by the author.

The author combines the methods of study of the Goldbach problem and the Waring problem, the latter being extended by letting the summands be polynomials with integral coefficients in which the variables are restricted to prime values. He also limits the variables in Tarry's problem to assuming only prime values, and at the same time carries out broader discussions on indeterminate equations in unknown primes.

In its original form this book was first published in the Russian language [1] in the Soviet Union in 1947. In 1953 the Chinese edition [2] was published by the Chinese Academy of Sciences. In its present form the book is completely revised and supplemented, with great changes made in the contents, as in §5 of Chapter IV, §§6 and 8 of Chapter V, and the entire Chapter IX, which have been completely rewritten. Large parts of a number of other sections have also been rewritten, and an Appendix is added.

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1) In the Trudy of the Steklov Mathematical Institute 22 (1947).

2) Editor's note. A German translation, based on the Russian edition of 1947 and the Chinese edition of 1953, was published, under the title *Additive Zahlentheorie*, by B. G. Teubner, Leipzig, 1959.

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PREFACE TO THE PRESENT EDITION

The publication of a revised edition has enabled the auther to make a number of improvements and to add supplements to the book. §5 of Chapter IV, §§6 and 8 of Chapter V, and the entire Chapter IX were rewritten, and many other chapters and sections were partly rewritten.

After reading this book and a few other contemporary publications, such as the author's "Certain results in the additive theory of numbers," which will soon be published in Acta Mathematica Sinica, the reader will understand the basic principles of the modern additive theory of numbers and will be able to embark upon research. But it must be remembered that this book is merely an individual piece of work designed to describe one branch of the theory. To improve his comprehension of the subject, the reader should also acquaint himself with a more general treatment (for example, the author's "Introduction to number theory" (Chinese), Science Publishing Co., Peking, 1957; MR 20 #829).

The author takes this opportunity to express his thanks to his colleagues Yue Min-yi (Yüeh Min-i), Wang Yuan (Wang Yuan), Wu Fang (Wu Fang), Wei Dao-zheng (Wei Tao-cheng) and Chen Jing-run (Ch'en Ching-jun), who have pointed out errors or have rendered other assistance. Without their cooperation, this edition could not have been published at this early date.

L. K. Hua
July 7, 1957, Peking

PREFACE TO THE 1953 CHINESE EDITION

It has been twelve years since the completion of the first draft and six years since the publication of the Russian edition. During these years the problem of publishing the book has been such that the protracted delay finally caused the original draft [1] to disappear without leaving a trace. To-day, after being urged to

1) Note by the editor of the Russian edition. In this original draft, which was written in English, the term "exponential sum" was used instead of "trigonometric sum."

do so by the Chinese Academy of Sciences, I had to translate the book from the Russian edition in order to send it to the publishers.

The twelve years were not idle; the scientific developments within these years were very great, and therefore to present only the Russian translation would have been inadequate to deal with the current situation. I rewrote several chapters, especially Chapter V, in which I included the creative work of Academician Vinogradov between 1942 and 1947 and my work in 1947.

Upon completion of my work, I was indeed well content. This was not only because my book could be published in my mother country, but especially because the unbreakable Sino-Soviet friendship which we had longed for in the past was realized. Without the encouragement of the Chinese Academy of Sciences, this book could not have been reprinted, and I extend my sincere gratitude to them. The members of the Institute of Mathematics all shared the responsibility of proof reading and editing, and the great number of persons who participated in various ways makes it impossible for me to thank them individually here.

L. K. Hua
May 1953, Peking

PREFACE TO THE RUSSIAN EDITION

After several years of war the author, who had been invited by Academician Vinogradov to make a tour of Soviet Russia, was delighted to learn that a Russian translation of the essay he had written in the years 1940 and 1941 had been sent to the publishers. In 1942, Academician Vinogradov had already made his method more precise, a fact of which the author was unaware before his arrival in Moscow. This precision strengthened the theorem on the mean value (Theorem 7, this book) and thus made it possible to improve Theorems 8, 9, 11, 13, 17 and others. For example, Theorem 11 is also true for $s \geq 10k^2 \log k$, while Theorem 13 also holds for $s \geq s_0 \sim 4k \log k$, and so forth.

In conclusion, the author extends his gratitude to Professors B. I. Segal and D. A. Vasil'kov for their translation of this book into Russian.

L. K. Hua
April 17, 1946, Moscow

PREFACE ORIGINALLY INTENDED FOR THE RUSSIAN EDITION [1)]

This book contains a description of new findings in the additive theory of prime numbers. The basic principles of this field of study were established by Academician I. M. Vinogradov and further developed by the author. The work of Vinogradov, which has opened up new paths for development, is simplified, changed and redescribed in Chapters V and VI. Apart from Lemma 7.14, no specialized knowledge is necessary for reading this book.

The greater part of the book is the work of the author, systematically described here for the first time.

The author wishes to express his profound gratitude to Academician Vinogradov.

Assistance rendered by his colleagues Min Si-hao (Min Szu-hao) and Zhong Kai-lai (Chung K'ai-lai) in preparing the draft is gratefully acknowledged by the author.

In conclusion, the author wishes to express his deep gratitude to the Soviet Academy of Sciences for its favorable reception of his work. In these difficult days we are especially happy to see the results of our scientific study win the approval of the highest authority of a people to whom we are bound by the closest ties of friendship. This kind of cultural cooperation will always be treasured and is especially meaningful at the present time. It is respectfully hoped that the publication of this book will strengthen the true friendship and mutual affection of our two great peoples.

National Tsinghua University

L. K. Hua

Feb. 18, 1941, Kunming, China

1) This book was sent to the editorial department of special publications of the Institute of Mathematics in 1941, but the publication was delayed as a result of the war.

EXPLANATORY REMARKS

The book has no general introduction but the first section of each chapter gives a description of the principal results of that chapter. The following symbols are frequently used:

z is a real number, $[z]$ is the greatest integer in z, and $\{z\}$ is the distance from z to the nearest integer.

$$e(x) = e^{2\pi i x}. \qquad e_q(x) = e^{2\pi i x/q}.$$

k denotes a positive integer, P is a "sufficiently large" positive number, and $L = \log P$.

$\max(a, b, \cdots, g)$ denotes the largest number among $a, b, \cdots, g$ and $\min(a, b, \cdots, g)$ denotes the smallest.

As usual, $a|b$ means that a divides b, and $a \nmid b$ means that a does not divide b. In this book, p is frequently used for a prime number, and $p^l \| n$ means that $p^l | n$ but $p^{l+1} \nmid n$.

The expression $c(a, b, \cdots, g)$ denotes a positive number which depends on $a, b, \cdots, g$, and ϵ denotes an arbitrary small positive number; these numbers are not necessarily the same every time they occur.

The expression $f(x) = O(\phi(x))$ or $f(x) \ll \phi(x)$ means that $|f(x)| \leq c(a, b, \cdots, g)\phi(x)$.

In stating the theorems, we shall not use the symbols $\ll$ and O but an inequality of the above form. In the proofs and lemmas, if the symbol $\ll$ or O is used, the constant implied in it depends only on the $a, b, \cdots, g$ involved in the description of the theorem.

The meaning of any particular symbol will not necessarily be the same in all parts of the book.

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