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Volume 18

Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space

I. C. Gohberg
M. G. Kreĭn

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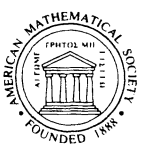

American Mathematical Society
Providence, Rhode Island

И. Ц. ГОХБЕРГ
М. Г. КРЕЙН

ВВЕДЕНИЕ В ТЕОРИЮ ЛИНЕЙНЫХ НЕСАМОСОПРЯЖЕННЫХ ОПЕРАТОРОВ В ГИЛЬБЕРТОВОМ ПРОСТРАНСТВЕ

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AUTHORS' PREFACE TO THE ENGLISH EDITION

We were happy to learn that immediately after the publication of our book, the American Mathematical Society had undertaken the publication of an English edition.

We wish to thank the translation editor, Dr. S. H. Gould, for the energy and initiative which he has displayed.

We wish to express our gratitude to Dr. A. Feinstein, who took upon himself the task of translating this book and who has fulfilled it with considerable responsibility in the shortest time. We owe to him, as well as to Professor T. Ando, the correction of a number of inaccuracies, a list of which would, to our surprise, prove to be rather long.

February 23, 1967. I.C. Gohberg, M.G. Kreĭn

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PREFACE

The history of the writing of this book is not altogether usual. However, if all authors shared with their readers the history of the origins of their books, then in all probability we would have to deny this statement.

In the spring of 1959 we decided to revise our paper [1] on the theory of defect numbers, root numbers and indices of linear operators, attempting mainly to include in it various results from the theory of perturbations of selfadjoint operators. In particular, we had decided to complete it with the theory of perturbation determinants.

At this point it was discovered that the method of perturbation determinants also made it possible to obtain new results for nonselfadjoint operators. We decided to discuss these results, in connection with which we began to think about what to do in this area, which was new to us. Thus step by step we changed our original plan.... In roughly a year we sent to the editors of the journal "Uspehi Matematičeskih Nauk" a review paper (in it, however, no space was found for perturbation determinants), in which we attempted to give something of a "bird's eye view" of the current state of investigation in the theory of nonselfadjoint operators. The editors of the journal "voted down" the paper, for which we wish to express our deep and sincere gratitude.

The fact is that the size of the original paper (twelve printer's sheets) exceeded by three times the norm established for review papers, and the editors thought it advisable to recommend the publication of the paper as a separate monograph in the series "Modern problems of mathematics".

Naturally, this suggestion induced us to extend the paper somewhat and, in particular, to restore to its rights the theory of perturbation determinants. We thought that all of this would require not more than two or three months. In actuality the work dragged out for three years. It should be said that even while we were working on the paper we had already established close scientific contact with Soviet mathematicians active in this field. Our colleagues gave us help and information concerning their results (before publication), and thus guarded us against the illusion of completeness. If to this one adds our desire to reflect in this book the investigations into which we ourselves were drawn by this work, one can understand our naïve delusion with respect to dates.

We hope that we are properly understood: we bear no grudge towards those who have given us copies of their papers before publication, who have communicated new, improved versions of proofs and have furnished us with all sorts of oral and written information. On the contrary, we are deeply grateful to all those with whose help we have been able, we hope, to achieve a certain degree of harmony in discussing the wide range of investigations which have come before our eyes, concerned with a large new domain of functional analysis—the theory of linear nonselfadjoint operators. Among all our colleagues who have helped us we wish particularly to mention M. S. Brodskiĭ, S. G. Kreĭn, B. Ja. Levin, V. B. Lidskiĭ, Ju. I. Ljubič, A. S. Markus, V. I. Macaev, and L. A. Sahnovič.

However, we have still not completed our account of how this book came to be written. We hope to continue this account in our monograph on the theory of abstract triangular representations of linear operators and its applications to the theory of canonical differential equations.

The editor of this book, F. V. Širokov, time and again "intruded" with his criticism in depth of the presentation of one topic or another. It often turned out that following such an intrusion we began to better understand even our own researches, and as a result the presentation of them gained in simplicity and clarity.

We wish to express our deep gratitude to F. V. Širokov for his considerable efforts and bold "excesses"of editorial authority.

Odessa, Arcadia I. C. GOHBERG
September 1, 1964 M. G. KREĬN

INTRODUCTION

The theory of nonselfadjoint operators in Hilbert space is a young branch of functional analysis. In recent times it has attracted the ever increasing attention of mathematicians and physicists, and sometimes of engineers also. The aim of this book is to present a number of achievements in this field, most of them related to the theory of completely continuous operators.

For a long period the investigations of D. Hilbert and F. Riesz, generalizing Fredholm's theory of integral equations, remained the most essential part of this theory.

In contrast with the theory of selfadjoint operators, no spectral decomposition nor even any theorems on the completeness of the system of root vectors had been obtained until recently in the abstract theory of nonselfadjoint operators.

At the same time, important progress was made in the theory of nonselfadjoint boundary value problems for ordinary differential equations by Birkhoff (in 1908) and J. D. Tamarkin (in 1911). These authors started from the methods of Cauchy and Poincaré, based on the study of the analytic properties of the resolvent of the problem.

But as for boundary value problems for partial differential equations, in view of the difficulty in constructing the resolvent, there was for a long time no similar progress.

This situation changed only in 1951, thanks to progress in the abstract theory of nonselfadjoint operators. In this year the work of M. V. Keldyš appeared, in which he established theorems on the completeness of the eigenvectors and associated vectors and theorems concerning the asymptotic properties of the eigenvalues for a wide class of polynomial bundles of nonselfadjoint operators. These theorems made it possible to obtain important results in boundary value problems for partial differential equations; they also led to strong new results for ordinary differential equations.

At the same time, the asymptotic theorems of M. V. Keldyš for abstract operators made it possible to generalize the original investigations of T. Carleman on the asymptotic behavior of the eigenvalues for boundary value problems for second order elliptic differential operators.

In the same period (1951—1956) M. S. Livšic and a group of his coworkers began the development of a new and profound technique in the theory of nonselfadjoint operators.

The guiding light of these investigations was the idea of generalizing to abstract operators the algebraic result of I. Schur on the reduction of a matrix to triangular form by a unitary transformation. These investigations led M. S. Livšic to the creation of a theory of characteristic matrix-functions and triangular models of a nonselfadjoint operator. The realization of this idea in the abstract theory of operators in infinite-dimensional space entailed many difficulties. An essential role in overcoming these difficulties was played by the deep work of V. P. Potapov in the theory of analytic matrix-functions, which in turn was stimulated by investigations on nonselfadjoint operators.

New progress in the theory of the triangular representation of an operator has taken place in recent years (1958—1962).

Already in 1954 N. Aronszajn and K. Smith established the existence of a nontrivial invariant subspace for every completely continuous operator in a Banach space. Later, in 1958, L. A. Sahnovič first deduced from this geometric fact the existence of triangular models for a wider class of operators, and with the help of these models obtained new analytic results in the theory of nonselfadjoint operators.

At approximately the same period, there appeared in papers of M. S. Brodskiĭ an integral giving an abstract triangular representation of a nonselfadjoint operator. The naturalness and transparency of this representation attracted to it the attention of a new group of mathematicians. In a short time the theory of the new integral and its applications was substantially developed (M. S. Brodskiĭ, I. C. Gohberg and M. G. Kreĭn, V. I. Macaev). This integral led in a natural way to the singling out of remarkable classes of Banach spaces, formed from completely continuous operators and representing ideals of the ring of bounded operators. Certain of these spaces were introduced earlier by J. von Neumann and R. Schatten, and many arose precisely in connection with the theory of the integral giving a triangular representation of operators.

Let us mention, finally, that the "abstract triangular integral" made it possible to solve, for a wide class of operators, the problem of their "factorization" (Gohberg and Kreĭn).

Although geometric and algebraic ideas lie at the foundation of the theory of the abstract triangular integral, the solution of a number of fundamental questions of this theory was achieved only thanks to the

application of various (and sometimes new) tools from the theory of analytic functions.

It should be generally emphasized that the development of the theory of linear operators has drawn more and more widely on the theory of functions and, in particular, analytic functions (the theory of the growth and the distribution of the zeros and poles of entire and meromorphic functions, various generalizations and refinements of the Phragmén-Lindelöf theorem (for example the theorem of Levinson-Sjeberg), the theory of subharmonic functions, Tauberian theorems and others).

The basic channel for the use of the methods of the theory of functions is the natural and by now classical tradition of studying the spectral properties of a linear operator by studying its resolvent as an analytic operator-function. Fairly recently another channel appeared for the use of these methods; it is connected with the perturbation determinants of linear operators (M. G. Kreĭn, B. Ja. Levin, V. I. Macaev).

On the basis of the study of the resolvent, it has been possible in recent years to obtain strong tests for the existence of a "sufficiently complete" set of invariant subspaces of a (not necessarily bounded) operator (J. Wermer, F. Wolf, E. Bishop, V. I. Macaev and Ju. I. Ljubič). This, in turn, made it possible to develop the theory of abstract triangular representations, extending it to a wide class of bounded and even unbounded operators (L. A. Sahnovič, M. S. Brodskiĭ, Gohberg and Kreĭn).

The study of the resolvent has also made it possible to establish theorems concerning the formation of bases from the root vectors of a nonselfadjoint operator and theorems on the summation of expansions in root vectors (M. A. Naĭmark, V. B. Lidskiĭ, A. S. Markus). Moreover, a justification has been obtained for the Fourier method for equations of evolution with a nonselfadjoint operator of one class or another (Lidskiĭ).

Let us mention that the technique for obtaining bounds for the resolvent of a nonselfadjoint operator, developed by T. Carleman, E. Hille and J. D. Tamarkin, M. V. Keldyš, V. B. Lidskiĭ and others, has been substantially improved in recent papers of V. I. Macaev.

In this brief review, as in the book itself, we have not touched upon the interesting direction in the spectral theory of nonselfadjoint operators developed by N. Dunford [1] and a group of his students and followers.

The theory of the abstract triangular representation of operators and the problem of the factorization of abstract operators, and also their applications, will be discussed in another monograph by the authors.

In the present book these problems are left almost untouched, although in various places we allude to isolated facts from the theory of the triangular representation. The fact is that complete results are sometimes attained only by the interaction of three types of methods: methods based on the study of the growth of the resolvent, methods of the theory of perturbation determinants and, lastly, methods of the theory of the triangular representation of an operator.

In recent times appreciable development has taken place in the theory of special operators which become selfadjoint or unitary upon the introduction of a certain indefinite metric in the Hilbert space. The investigations in this theory deserve discussion in a separate monograph; in the present book we have not touched upon them, except for a few cases related to them (Chapter V, §12).

Going on to a brief indication of the contents of the individual chapters, we mention, first of all, that a substantial part of the material is discussed here, with adequate proofs, for the first time. The material as a whole is systematized for the first time in this monograph.

In the first chapter, well-known results of the general theory of bounded nonselfadjoint operators are recalled. As a rule, these results are not specific for Hilbert spaces; they could have been formulated for operators in a Banach space.

In the third chapter we discuss the theory of symmetrically-normed ideals of the ring of bounded operators in a Hilbert space. This chapter includes the basic content of the elegant book of R. Schatten [2], devoted to the von Neumann-Schatten theory of "norm ideals" and "cross-norms". Together with the "veteran ideals" (the ideals of the nuclear operators, Hilbert-Schmidt operators and others), we single out and study new ideals which play important roles in various questions of the theory of nonselfadjoint operators. We are able to develop a treatment of the theory of symmetrically-normed ideals, starting from higher principles, thanks to the systematic use of the theory of the s-numbers of completely continuous operators worked out by H. Weyl, Ky Fan, A. Horn and others. The theory of s-numbers forms the basic content of the second chapter. General theorems on s-numbers of linear operators are also used systematically in the following two chapters.

The fourth chapter is devoted to the theory of perturbation determinants and some of its applications. The use of the extensive apparatus of the modern theory of analytic functions distinguishes this chapter from the others. This apparatus is regarded as auxiliary, and in most cases results of the theory of functions are presented without proof.

In the fifth chapter we discuss various theorems on the completeness of the system of root (eigen- and associated) vectors of a completely continuous operator (operator bundle). In the selection of the material the authors have attempted, on the one hand, to present sufficiently original and strong results and, on the other, to present as fully as possible the various existing methods.

Here we also discuss various results on the growth of the resolvent and theorems on the asymptotic properties of the spectrum (of operators of various classes).

In this chapter, apparently, the fundamental results of M. V. Keldyš are for the first time treated in a sufficiently detailed form.

Considerable attention has also been devoted to theorems on the completeness of the system of root vectors of a dissipative operator.

The last section is devoted to the study of the spectral properties of a selfadjoint quadratic bundle. Here we use the results of almost all the other sections of this chapter. However, as recent investigations have shown (M. G. Kreĭn and H. Langer), for the construction of a complete theory of quadratic selfadjoint bundles it is natural to use various theorems of the theory of operators in spaces with an indefinite metric. In view of this, certain results are presented here without proof.

As a supplement to the theorems on completeness, the authors have thought it appropriate to discuss in Chapter VI the simplest tests for the existence of a basis (of one kind or another) made up of the root vectors of a given linear operator (results of B. R. Mukminov, I. M. Glazman and A. S. Markus). The theory of bases in a Hilbert space is not discussed in texts on functional analysis; for this reason it too is presented in this chapter.

The presentation in this book is carried out in the spirit of the abstract theory of operators: it is illustrated by various applications to the theory of integral equations.

The reader who has some experience in the theory of boundary value problems for differential equations, or an acquaintance with the theory of linear vibrating systems with a finite or infinite number of degrees of freedom, will easily discover how many of the results discussed here find immediate application in each of these fields. A clearer presentation of the possibilities and prospects existing here can be obtained from the interesting survey article by M. V. Keldyš and V. B. Lidskiĭ [1] (cf. also the survey by C. L. Dolph [1] and the report by M. G. Kreĭn and H. Langer [2]).

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