Constructive Real Numbers and Function Spaces

by

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КОНСТРУКТИВНЫЕ ВЕЩЕСТВЕННЫЕ ЧИСЛА И КОНСТРУКТИВНЫЕ ФУНКЦИОНАЛЬНЫЕ ПРОСТРАНСТВА

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APPENDIX

ON CRITICISM OF CLASSICAL MATHEMATICS

This appendix is an extension of §0.5, in which the basic content of the criticism of classical mathematics set forth by Brouwer, Weyl, Markov and some other mathematicians was briefly described. The general formulation in §0.5 is lacking in precision and concreteness. This appendix is devoted to a consideration, more detailed and more concrete than in §0.5, of the situation which has stimulated criticism with respect to the foundations of classical mathematics. Here we shall try to present a summary of the most important aspects of the critical analysis of classical mathematics carried out by Brouwer, Weyl, Hilbert, Markov, and other mathematicians, and we shall also attempt a detailed presentation of some components of this critical analysis.

I. First of all, we shall go into some aspects of the processes involved on the formulation of mathematical concepts.

In the formulation of mathematical concepts mental acts of various kinds are carried out. Let us note certain kinds of such mental acts.

1) Mental acts of "pure" abstraction. These mental acts consist in the conceptual selection of certain properties out of all the properties applicable to all the objects included at the given moment in our field of attention, and, with respect to the existence or nonexistence of any other properties, our consciousness remains completely indifferent.

If we associate with the chosen properties some terms not being used for other purposes, then we obtain some general concepts.

2) Mental acts of idealization. These acts consist of the generation by our imagination of certain ideas or concepts, considered by our consciousness as objects of study endowed by our imagination not only with those properties which were selected by acts of "pure" abstraction with respect to the objects forming the initial material for the given mental operations, but also with conceptual properties which are completely absent from the initial objects or which reflect properties of the initial objects in a considerably distorted form.

3) Multi-stage levels of various acts of "pure" abstraction and acts of idealization. At every stage of the mental processes of this kind, as initial objects for acts of "pure" abstraction and acts of idealization one uses not only objects which are the initial material in the whole chain of mental operations but also concepts and ideas which have been formed at preceding stages.

Mental acts of the kinds mentioned above are acts of abstraction.[1)]

The presence of acts of idealization in almost all the processes of formation of mathematical concepts is the result of the tendency of people, in studying new objects or new connections between certain objects or, speaking more generally, in studying new situations, to look for support from previously accumulated knowledge and previously developed tools relating to different initially given objects. The complexity of the situation being studied often forces people to look for conceptual support by acts of imagination which bring our presentation of the situation under study closer to a situation which has been previously studied and for which there exists some (sometimes only partial or remote) resemblance to the given situation and for which a sufficiently simple apparatus has been worked out. Often it is necessary to proceed in this way simply because no other realizable possibility is found to be successful.

If in the course of studying some situation we succeed in finding an idealization which permits us to solve a certain problem with the required precision, then this idealization may be fixed and turned into a tradition, to which people automatically turn even in those cases where it is necessary to consider problems of new kinds or to meet more strenuous requirements in the formulation and solution of problems of the initial type. The attractive aspect of such traditions is the development of and familiarity with a certain apparatus. This attractive aspect explains the fact that many such traditions do not encounter opposition for a long time. One of these traditional idealizations is the abstraction of actual infinity.

II. The systematic application in contemporary mathematics of the abstraction of actual infinity and of ideas generated by this abstraction can explain to a considerable extent the tendency to rely, in various mathematical considerations, on those methods of thought which are customary for people with academic background, and which classical logic puts at our disposal.

1) Sometimes by abstractions we mean only mental acts of "pure" abstraction. In this appendix the term "abstraction" will be used in a much wider sense, including the mental acts of the kinds mentioned above. This interpretation of the term "abstraction" has many precedents in the application of this term in the literature devoted to the foundations of mathematics.

It was remarked above that in acts of idealization our imagination generates ideas and concepts in which are combined, first, properties chosen by acts of "pure" abstraction in the study of certain objects, and, second, those conceptual properties which distort the picture of our knowledge about the original situation and become objects of study only as a result of a conceptual approximation of the situation of interest to us with a situation which has already been studied, and which are produced so that one may try to use, even partially, some previously worked out apparatus.

After an idea or concept arising as a result of an act of idealization enters into the fabric of mathematical theory, mathematicians usually forget the mechanism of the origin of the given idea or concept and fail to notice the difference between properties of the first kind and of the second kind. Properties of both kinds become in equal measure the initial basis for new acts of abstraction for processes of logical inference. As a result of new acts of idealization generated on this basis, new ideas or concepts arise in which it is even more difficult to separate properties which can be considered satisfactory reflections of properties of the initial objects lying at the base of the formation of the whole chain of ideas and concepts being considered from those properties which are only products of a method. This indefiniteness grows still larger in the transition to higher levels of the processes of abstraction occurring in mathematics.

An analogous situation exists when we consider processes of logical deduction. In such processes we use properties of the second kind with the same right as properties of the first kind. In addition, in applying a rule of inference, mathematicians proceed from the belief that the rule being applied is admissible in the cases being considered. However, the justification for the admissibility of rules of inference includes within itself not only acts of "pure" abstraction but also acts of idealization, in many cases even multi-staged acts.

Hence every logical inference places before us a series of problems. In what way can one interpret the proposition obtained as a result of the given logical inference, i.e. in what way can one transform this proposition into a proposition about the initial objects which serve as a basis of the formation of the whole chain of ideas and concepts being considered? If a concrete condition suggests to us some interpretation, then the problem arises: should one consider the proposition obtained as a result of the given logical inference a satisfactory reflection of the properties of the initial objects and the connections between them? Does it not happen that the essential use of various acts of idealization of various conceptual properties and situations on the way between the initial objects and the given result of logical inference so radically isolates the conceptual processes in our consciousness from the initial situation that the final proposition resulting from the given logical inference gives a substantially distorted idea of the initial objects?

No general recipes for answering these questions exist. However, it is necessary to note the following.

The difficulty mentioned above of returning from ideas, concepts, and arguments to the initial objects which serve as a basis for the formation of the whole chain of ideas and concepts of the given theory always arises when acts of idealization are involved in the process of forming ideas and concepts. But this difficulty of restoration can occur in essentially different ways in different cases; it varies in dependence on the type of idealization used, on the nature of the level of the idealization, and on those aspects of the given theory which are of interest to us in the given case. In some cases the restoration can be realized much more easily than in others. (Intuitively speaking, in some cases mathematical concepts and arguments are significantly more "tangible" than in others.) In some cases logical inferences, considered together with some way of interpreting statements, give a much greater basis for recognizing their cognitive value than in other cases. Idealizations used in mathematics turn out to be nonuniform with respect to the point of view of interest to us now.

An important circumstance inducing criticism with respect to the foundations of classical mathematics is the fact that the abstraction of actual infinity and the ideas and concepts raised on its basis are very remote idealizations, i.e. idealizations for which the connection between the ideas, concepts and arguments, on the one hand, and the real objects forming the initial material for the whole chain of acts of abstraction under consideration, on the other hand, in many cases turns out to be very indirect and vague, or even do not exist at all. The difficulty of returning from the ideas, concepts and arguments, in the formation of which the abstraction of actual infinity has taken part, to the initial real objects and the connections between them turns out in many cases to be very considerable.

III. In classical mathematics a fundamental role is played by the general concept of set. This concept is not connected with the fixing of some definite method for individually describing those objects which are covered by this concept.

It is well known that the general concept of set is not definable in terms of simpler concepts (it is clarified only by means of some examples) and the methods of working with this concept are introduced on the basis of the system of ideas generated by the abstraction of actual infinity. The initial material for the formation of the general concept of set consists in the first place of various collections of real objects. (When we apply the expression "collection of real objects" we have in mind that the objects of which the collection consists are conceptually chosen from the surrounding environment by means of direct exhibition or by means of a clear characterization of their type, that the objects occurring in the collection exist simultaneously and steadily during some interval of time, and that they clearly differ from one another.)

It is necessary to emphasize that the experimental investigation of nature has not given any example of an infinite collection of real objects (see [10]). However, in the classical theory of sets, there occur as objects of study not only finite sets but also concepts with which are connected ideas about collections for which the process of counting never at any step exhausts all their elements, and with which the term "infinite set" is connected. Therefore reference to collections of real objects existing in nature as the source of the formation of the general concept of set is insufficient as an explanation of the mechanism for forming this concept.

Everyday experience and constructive human activity bring us together not only with objects existing in nature at that given interval of time but also with processes of construction of some new objects. Here we have in mind processes of construction carried out by both people and mechanisms.

For a long time mathematicians have used certain methods of introducing general concepts which cover not only objects existing in nature but also possible results of processes of construction. If we choose and fix some finite sequence of initial objects and some finite sequence of constructive operations, and we introduce some new term (denoted by the latter T), then we obtain all the necessary ingredients for a genetic definition (more concretely, for a constructive definition) of a new concept. Every object covered by this new concept receives the name "object of type T". Objects of type T are called possible (more precisely, potentially realizable[1]) results of the processes of construction realized by a

1) This term will be explained below.

sequential performance of constructive operations occurring in the fixed finite sequence. It is assumed that, before each step of the construction process, the next constructive operation can be chosen arbitrarily from among those constructive operations which serve as a basis of the given definition and the description of which permits their application to any results of the process obtained at preceding steps or to any initial objects.

It is possible (and in many cases also necessary) to introduce constructive definitions of a more general form, namely definitions in which one fixes: 1) a finite sequence of initial objects; 2) a finite sequence of constructive operations; 3) a term for the defined concept; and, in addition, 4) some condition clearly formulated in a suitable language. As objects of the defined type we count in this case those possible (potentially realizable) results of constructive processes, based on the given finite sequence of constructive operations, which satisfy the given condition.

If some general concept is introduced by means of a definition of one of the two types just mentioned, then in constructive mathematics one connects with the definition of this general concept (more precisely, with the text of the definition, along with which one assumes a definite method of interpretation) the name "set of objects of such-and-such a type".[1] In constructive mathematics the term "sets" is connected with individually given objects, each of which is a definition, written in a suitable precise language, of constructive objects of a concrete type. Applying the term "set" in this sense we can state that there is no justification for considering all elements of the given set as existing simultaneously. It can happen that at a given moment some elements of the set have already been constructed. We are justified for talking about these elements as if they exist simultaneously. But there can be only a finite number of such elements. Therefore, generally speaking, elements of a set are determined only as possible objects, and we are able to investigate them only on the basis of given constructive operations and a given selecting condition. One can sat that a general concept covering not only existing but also possible objects arises only as a result of fixing a

1) We remark that the concepts "natural number", "integer", "rational number" can be introduced by definitions of the first type (§1). The definitions of these concepts are, respectively, the set of natural numbers, the set of integers, and the set of rational numbers.

As examples of concepts introduced by definitions of the second type, one can cite the concepts of real *F*-number and real *FR*-number (§3.5). The definitions of these concepts are, respectively, the set of real *F*-numbers and the set of real *FR*-numbers.

finite sequence of constructive operations and a selecting condition.[1)] In other words, a set of the type just considered arises only as a result of fixing a finite sequence of constructive operations and a selecting condition. This fact underlines the exceptional role of the individual presentation of sets in the problem of the "tangibility" of mathematical concepts.

Both in classical and constructive mathematics, in considering objects characterized by definitions of the types described above, one assumes a certain idealization, called the abstraction of potential realizability (see [1]). It consists of the conceptual assumption that, in carrying out constructive operations on constructive objects, there do not arise obstructions of a material nature, caused by a limitation of the constructive possibilities of men and machines in space and time, by limited resources, etc. Applying this idealization, we treat as objects of study not only those objects which already exist or can be constructed in a really possible number of steps, but also imagined objects the construction of which, on the basis of the given constructive operations, could be realized if, in carrying out the given operations, no obstructions of a material nature arose. As a name for this realizability (stipulated by the indicated assumption) of construction one uses the term "potential realizability" (see [1]).

Applying the abstraction of potential realizability and starting, for example, from the definition of natural numbers, we arrive at the idea of potential infinity, i.e. at the idea of the possibility of extending the sequence of natural numbers endlessly. In this sense one can say that the set of natural numbers is potentially infinite.

From what has been said it follows that, using as objects of study only collections of real objects, we do not go beyond the concept of "finite set", and using as objects of study finite sequences of constructive operations and processes of construction realizable on their basis and applying the abstraction of potential realizability, we do go beyond the concept of "finite set" and we arrive at the concept of potentially infinite sets, but we do not all reach that idea with which the term "infinite set" is understood to be connected in classical mathematics. Classical mathematics does not give any indication as to how one can connect the idea used by it and denoted by this term with any real objects (collections

1) One can assume that a selecting condition figures in every constructive definition. Those definitions which do not include a selecting condition can be supplemented by a condition expressing the equality of an object with itself.

of real objects and finite sequences of constructive operations are not adequate for this purpose). In every case it is indisputable that this idea arises in the mind of every mathematician thanks to an act of idealization which is essentially different from the abstraction of potential realizability.

IV. Constructive mathematics limits itself to the consideration of sets individually given by one of the two methods indicated above.[1] Classical mathematics does not limit itself in this way and extends the sequence of ideas and concepts to be studied by introducing another idealization, called the abstraction of actual infinity.

This idealization can be characterized in the following manner. 1) Considering any fixed finite sequence of constructive operations, we begin by imagining as not only potentially realizable but as actually carried out all possible processes of construction admitted by the given finite sequence of constructive operations, and we conceive all the results of these operations as existing simultaneously. 2) We conceptually equate this imaginary picture with the situation with which we have to deal in considering collections of real objects, and, in particular, we begin to reason about the imaginary "collections" of all these results in the same way that we reason about collections of real objects, i.e. by the methods of classical logic. 3) We begin to conceive of these imaginary collections as existing independently of the finite sequence of constructive operations. 4) After this we give our imagination even greater scope and begin to conceive of infinite collections of simultaneously existing objects not connected with any constructive operations even by their "origin", meeting expressions of mental processes only in the introduction of certain axioms and the development of the process of logical deduction on the basis of classical logic; moreover, we disregard the problem of the possibility of a clear semantics which would permit us to connect the concepts and arguments of such a formal-deductive theory[2] with any real or potentially realizable objects or processes.

In classical mathematics not only the general concept of set but also the concept of subset of a given set is not connected with the fixing by any definite methods of an individual presentation of those objects which are covered by this

1) Here we have in mind suitably precise formulations of these methods (see [3, §7]).

2) In such a formal-deductive theory the general concept of set and the concepts based upon it figure only as terms, and the ideas (imaginary pictures) connected with these terms in the process of forming the theory (in the choice of the axioms and the apparatus of logical deduction) lie outside the theory itself.

concept. The same is true when, as the set about whose subsets we are talking, a set occurs which has a simple constructive presentation (for example, the set of all natural numbers, the set of all rational numbers, etc.). The concept of a mapping of one set into another and its special case, the concept of a sequence of elements of a given set, are also not connected with the fixing by any definite methods of an individual presentation of the objects covered by these concepts.

The nonconstructive concepts of subset, mapping, and sequence of elements do not contain any "tangible" initial objects for their contensive interpretation.[1)] The difficulty, caused by this situation, of interpreting the indicated concepts carries over also to those concepts which are defined in terms of them. In particular, these difficulties carry over to the concept of real number in the sense of Dedekind, at the basis of which lies the nonconstructive concept of subset of the set of rational numbers, and to the concept of real number in the sense of Meray-Cantor, at the basis of which lies the nonconstructive concept of a sequence of rational numbers.

V. In applying mathematics to science, engineering and other domains of human activity the concept of real number usually figures as a means of expressing concrete information about physical or other quantities, determined by indication of methods for matching objects or processes of certain types with given standards of measurement. However, there is a significant gap between these practical assignments of the concept of real number, on the one hand, and the content of this concept, as well as the theory of this concept, on the other hand. (Here we have in mind the concept of real number which is used in classical mathematics.) This gap can be described in the following way.

In mathematics there are no means for expressing concrete information about physical quantities other than groups of symbols introduced by means of constructive definitions. Therefore, in mathematics, only constructive objects can be considered as possible carriers of concrete information about physical quantities. However, in classical mathematics the theory of real numbers is constructed not as a theory of constructive objects of a certain type, intended for the expression of concrete information about physical quantities, but rather as a theory the objects of which are certain ideas formed in the imagination of mathematicians as a result of complicated processes of idealization. On the path between the

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1) What has been said above about the general concept of set extends (with suitable changes in details) to these nonconstructive concepts.

processes of measurement of physical quantities and those ideas which are connected with the term "real number" lies the abstraction of actual infinity.

VI. If we are to speak not about ideas originating in the imagination of mathematicians but about mathematical tools, then one can state that the assumption of the abstraction of actual infinity as a method of thought means the assumption without any limitation of the tools of logical deduction of classical logic. However, it is well known that among the propositions deducible by means of these logical tools there are some which are false from the point of view of the construction intepretation of propositions. (see below; see [3, §4.5]). In this sense the tools of logical deduction furnished by classical logic are not suitable in those theories in which the objects of investigation are constructive objects and the study of these objects is carried out by taking into account their method of definition.

From what has been said it follows that theorems about real numbers deducible in classical mathematical analysis are not statements directly concerning real or potentially realizable objects. In particular, there is no foundation for assuming that every theorem of the form "There exists a real number α satisfying condition S" permits an interpretation in the form of a statement about the existence or potential realizability of a constructive object falling under the concept "real number" and satisfying condition S.

For many theorems of the classical theory of real numbers one can find similarly formulated assertions which are formulable and provable within the scope of constructive mathematics. These constructive analogues of theorems of the classical theory of real numbers give a clearer opportunity for translation into assertions about physical or other quantities. The existence of constructive analogues of many theorems of this theory explains the fact that in spite of everything said above about the classical theory of real numbers, some theorems of this theory make it possible to answer, with a definiteness satisfying certain practical requirements, certain problems about physical or other quantities.

At the same time, for some theorems of the classical theory of real numbers which play an important role in this theory, it is necessary to state that among those assertions which are formulable within constructive mathematics and which have formulations close to the formulations of the given theorems of classical mathematics one cannot discover an assertion provable in constructive mathematics.

From what has been said it follows that the problem of interpreting the theorems of the classical theory of real numbers meets considerable difficulty, having its source in the nonconstructive concept of real number used in classical mathematics. The difficulty of interpretation connected with the concept of real number induces difficulties in all the superstructures based on this concept, and are supplemented in these superstructures by new difficulties caused by the idealizations which are introduced in the conceptual formation of these superstructures.

Very great difficulty of the type just considered is caused by the general concept of transfinite number (not connected in classical mathematics with the fixing of any definite methods for the individual presentation of the objects covered by this concept) and the idea of the method of transfinite induction based upon this concept.

Concepts with which considerable difficulty of interpretation is connected often form the basis for entire branches of classical mathematics. One can exhibit a large number of examples of such concepts. We shall limit ourselves to the examples cited above.

VII. An assertion of the form "There exist an object α_1 of type $P_1, \cdots,$ and an object α_k of type P_k such that condition S is fulfilled" is customarily called an existence assertion. In classical mathematics there occur various kinds of proofs of existence assertions. These include proofs which contain methods for constructing certain constructive objects satisfying the condition which figures in the given existence assertion. At the same time, there also occur proofs which are strictly regulated by the rules of inference of classical logic but do not give any means for obtaining some constructive and therefore concretely defined objects satisfying the given condition. In classical mathematics proofs of the second kind are considered completely acceptable. However, in spite of this, even in the presence of proofs of the second kind, considerable attention is given to searching for proofs of the first kind. For any existence theorem, the construction of a proof of the first kind is often considered to be a result of considerably greater value than the construction of a proof of the second kind.

This point of view on the relation between proofs of the first and second kinds is expressed in a particularly clear way when one considers existence assertions closely connected with applied mathematics. This point of view and the considerable attention to constructive objects connected with it are results of extensive experience in the application of mathematics in science and engineering. This experience testifies to the fact that constructive objects are the most "tangible" and the most adaptable for the interpretation of parts of classical mathematics.

However, in classical mathematics constructive objects are considered on a par with nonconstructive objects. Most often they are considered as concrete representations of some general concept the definition of which is not connected with the fixing of some definite methods for individually presenting the objects covered by this concept and, thanks to this freedom, permits us to assume that, in addition to constructive objects, this concept also covers some other objects. The root property of constructive objects, consisting of the fact that they are by definition potentially realizable results of constructive processes, is completely disregarded in classical mathematics. This disregard shows itself, in particular, in the fact that in all reasoning, including that in which constructive objects occur, the rules of inference of classical logic are applied without limitation.

The original source of the apparatus of logical deduction of classical logic is elementary logic, which has to do with the processes of thought arising under certain "elementary" conditions. These conditions are characterized by the following features.

1) The objects of study are invariable and form a fixed finite collection.[1]

2) For every given initial concept one can form a two-valued characteristic table selecting those objects of the given collection which belong to the given concept.

3) For every given initial relation one can form a two-valued characteristic table selecting those ordered groups of objects the terms of which are in the given relation.

For statements formulated with respect to the indicated conditions there is a precise semantics, subconsciously applied by people in appropriate cases in everyday practice for many thousands of years, but formulated clearly and in systematic form only at the end of the nineteenth century. The existence of a precise semantics gives us the opportunity of putting in a clear form and solving in an affirmative sense the problem of justifying, under the initially given conditions indicated above, the rules and methods of logical deduction of elementary

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1) We have in mind, of course, not absolute invariability but invariability only in a chosen interval of time of those aspects of the objects of study which are of interest in the given investigation and only to that degree of precision which suffices for the stated purpose of the given study.

logic.[1] It has been established that upon fulfillment of the conditions indicated above, every statement derivable by the methods which are placed at our disposal by the apparatus of logical deduction of elementary logic, including in derivations some true statements as initial statements or without inclusion of such additional initially given statements, are true statements.[2] This justification of the apparatus of logical deduction reveals the cognitive meaning of the processes of logical deduction which flow from the initially given conditions indicated above on the basis of the apparatus of elementary logic.

In the course of the historical development of the apparatus of logical deduction belonging to elementary logic, a displacement into a mathematical theory having to do with the concept of infinite set automatically occurred. This process of automatic displacement of the apparatus of elementary logic into a considerably more complex condition represents an even more remote idealization. This idealization is one of the chief aspects of the abstraction of actual infinity. Applying this idealization, we agree in the case of infinite sets of objects of study to permit (without any justification) the methods of logical deduction worked out for perfectly concrete initially given objects: for the case where we have a finite domain of objects of study and where all the conditions which were called "elementary" above are fulfilled. Hence we have agreed to treat every situation in the case of an infinite domain of objects of study as completely analogous to the situation which holds under the "elementary" conditions described above.

There is no justification for asserting a priori that the apparatus of logical deduction worked out for certain initially given objects will be suitable for other initially given objects. This observation supports, in particular, those mathematical theories in which the objects of study are constructive objects and the starting point for the investigation of these objects is the root property of constructive objects mentioned above.

VIII. It is well known that the interpretation of mathematical statements about

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1) Here we have in mind the contemporary apparatus of logical deduction of elementary logic. This apparatus has been formed gradually in the course of the history of human thought. Until the rise of mathematical logic, it was formed mainly in a heuristic way. The rise of mathematical logic was to a considerable degree a process of making precise, partially revising, and significantly extending the previously formed parts of this apparatus on the basis of the precise semantics mentioned above.

2) We remark that, upon fulfillment of the conditions indicated above, justification is obtained, in particular, for the law of the excluded middle.

constructive objects, based on the root property of constructive objects and on the so-called constructive interpretation of mathematical propositions (see [3])[1]) reveals the inadmissibility of certain rules of inference of classical logic. The rules of inference which we have in mind here can lead to false statements. Not all mathematical propositions about constructive objects provable by means of the rules of classical logic (i.e. by means of those rules of inference which are considered in classical mathematics to be completely admissible) can be considered as true under the interpretation of propositions which starts from the root property mentioned above of constructive objects. From what has been said it follows that classical mathematics as a whole and classical mathematical analysis in particular are not adapted to the exposure of those connections between constructive objects for which the root property mentioned above of these objects is essential.

Evidence of this unsuitability is manifold. Let us go into some of this evidence. The first manifestation is the fact that the consideration of constructive objects and nonconstructive concepts on a par generates a situation in which it is difficult to distinguish the theorems about constructive objects provable (even by means of classical logic) without drawing upon nonconstructive concepts. But now we turn out attention to another aspect of the matter, namely to the practical consequences which are caused by the use of the complete range of logical methods of classical logic.

Let us assume that we are given a constructive problem which presupposes some constructive initially given objects [for example, words in some alphabet, algorithm schemas (programs for computing machines), etc.] and consists of finding a method of constructing, for arbitrary admissible values of the initially given objects, some constructive objects satisfying a definite condition formulated in terms of constructive mathematics. Let us further assume that mathematicians, in trying to solve this problem, in their investigation subconsciously limit their field of attention only to constructive objects, but are inclined to believe that, for the problems of interest to them, the type of theorem provable by means of classical logic plays a satisfactory guiding role. Let us pose the question: in this situation what role do theorems of this type actually play?

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1) One of the fundamental principles of the constructive interpretation of mathematical propositions is that every assertion about the existence of a constructive object satisfying a given condition is understood, in complete accord with the root property of constructive objects, as an assertion about the potential realizability of the construction of a constructive object satisfying the given condition.

A mathematician, having before him the indicated problem and at the same time some theorem provable by means of classical logic and asserting the existence of the required constructive objects for arbitrary admissible initially given objects, finds himself in essentially the same situation as when he did not have this theorem at his disposal. Desiring to actually use this theorem for the formulation of the required method, he is compelled to find a constructive proof, that is, he must prove (if this is possible) some proposition of constructive mathematics. If, not finding a constructive proof, he nevertheless regards the theorem as an encouraging stimulus for continuing his attempts to solve the problem, it can happen that he thus directs his mind along an absolutely hopeless path—the problem can turn out to be theoretically unsolvable, in spite of the existence of a suitable theorem of classical mathematics.

As an example of a problem for which just this situation occurs one can cite the problem of finding, for the concrete algorithmically given nondecreasing and bounded above sequence S of rational numbers constructed by E. Specker and described in §8.3.1, an algorithm (program for a computing machine with unbounded memory) which, for any natural number n, computes a subscript beginning with which the terms of the sequence are separated from each other by a distance less than 2^{-n}. Such an algorithm is impossible (see Theorem 8.3.1). At the same time, using a proof by "contradiction", we can deduce (for example, with the help of the corollary of §8.3.2) a theorem of classical mathematics (about constructive objects) which asserts that, for every n, there exists the required subscript.

One can exhibit a large number of examples of similar situations. A situation of a similar kind may arise, for example, in connection with theorems for the logical derivation of which one uses without any limitation proofs by "contradiction" or the law of the excluded middle. However, the cited logical methods of classical logic form only part of the "sources of nonconstructivity", only part of the obstacle on the way to the creation of a faithful picture telling us for which required constructive objects there are algorithms constructing (computing) them in terms of such-and-such initially given constructive objects, and for which there are no such algorithms.

IX. An obstacle of another kind, another "source of nonconstructivity" in mathematics, is connected with the widely practiced transition in classical mathematics from any assertion of the form

(*) "For any object α of type P, there exists an object β of type Q which is in the relation Γ to α"

to the assertion

(**) "There exists an operation ϕ associating with every object α of type P an object $\phi(\alpha)$ of type Q which is in the relation Γ to α".

Here Γ is a binary predicate; we shall assume that this predicate is invariant with respect to replacement of any pair of admissible objects by an equal pair (in the sense of equality of pairs induced by the relations of equality introduced for objects of type P and objects of type Q) and that, for any admissible value of the second argument, it is single-valued (up to equality of objects of type Q) in the first argument. Under these conditions the transition from (*) to (**) is often accompanied in classical mathematics by a transition from the predicate variant of the theory connected with the predicate to an operator variant in which there occurs an operation Φ replacing in a certain sense (together with the equality relation for objects of type Q) the predicate Γ. For the purposes of computational mathematics the operator variant is preferred, for obvious reasons. Mathematics offers many examples of the preference for such operator variants of theories.

The case where the proposition (*) cannot be proved by the methods of constructive mathematics does not require discussion— "nonconstructivity" is already present. Let us consider now the case where (*) is a proposition about constructive objects provable within constructive mathematics. In considering (*) and (**) as propositions of constructive mathematics it is necessary to think of the words "exists" and "operation" as being replaced by the words "is potentially realizable" and "algorithm", respectively.

In many cases the constructive reformulation of proposition (*) shows that it is an assertion about the realizability of an algorithm of a more complicated kind than that which is spoken about in proposition (**) (see [3, §8]). Hence the semantics of constructive logic and the corresponding apparatus of logical deduction does not always permit the transition to (**). In the case where the verification of the condition characterizing the concept "object of type P" includes within itself a non-algorithmizable search for a solution of some constructive task, proposition (**) can turn out to be refutable (see for example, Theorem 3.7.3 and the accompanying remark), and then the operator variant of the theory in the form suggested by (**) is impossible. However, in similar cases, one nevertheless succeeds in finding operator variants of theories of constructive mathematics, but they acquire a more complex form: as initially given objects for the algorithm Φ associated with the predicate Γ it is necessary to consider not individual objects of type P but rather systems of objects, each of which consists of some object H of type P (or a transcription of such an object) and additional objects carrying certain information about some solution of the constructive problem arising in the justification of the assertion "H is an object of type P".

The introduction into constructive mathematics of real FR-numbers, FR-constructs of various types, complete ciphers of uniformly continuous operators, metric spaces with a fixed algorithm for passage to the limit or with a fixed algorithm for constructing nets, etc. (see the main text of this paper), along with real F-numbers, F-constructs of appropriate kinds, uniformly continuous operators, metric spaces, etc., is dictated, in particular, by the tendency to have available, along with predicate variants of certain theories, also (and even preferably) operator variants, even at the expense of the introduction of objects of more complex kinds than in the corresponding theories of classical mathematics.

The transition, carried out in classical mathematics without any limitations, from predicate variants of the construction of theories to operator variants (on the basis of the transition from (*) to (**)) often produces mathematical theories for which the attempt to interpret the symbol for the operator Φ associated with the predicate Γ as a symbol for an algorithm applicable to all objects of type P and such that, for any object α of type P, the object $\Phi(\alpha)$ is in the relation Γ to α, immediately leads to confusion, in view of the impossibility of an algorithm with those properties. In similar cases in classical mathematics, in the transition to the operator variant, a change takes place in the type of the objects which can be considered as suitable initially given objects for the algorithm associated with the predicate Γ in the transition from the predicate to the operator variant of the theory admissible within constructive mathematics. Such a change leads to a certain disorientation of a mathematician dealing with the corresponding theory of classical mathematics, for example, in the light of the requirements of computational mathematics, the more so as, under the conditions of the multi-level character of contemporary mathematics, it is not always easy to discover the type of initially given objects for which a constructively justifiable operator variant of the theory is possible.

What has been said above is illustrated by the following example. In the classical theory of real numbers, one introduces a binary predicate "the real number q is the limit of the sequence of real numbers f" and one proves a theorem of the form (*), where the roles of α, β, P, Q, and Γ are played, respectively, by f, q, "fundamental sequence of real numbers", "real number", and "is the limit of". The transition to (**) is considered as the justification for the introduction of the operation lim of passage to the limit. Replacing in this theorem of the form (*) the concepts of classical mathematics by the corresponding concepts of constructive mathematics (with the concept "real number" we associate the concept "real F-number"), we obtain a proposition provable by the methods of constructive logic (see Theorem 10.5.3). However, the proposition ot the form (**) obtained by such a replacement is refutable and therefore the attempt to interpret the symbol lim (with the types of initially given objects borrowed from classical mathematics) as the symbol for an algorithm leads to obvious misunderstandings. An analysis of the proof of the constructive variant of the theorem of the form (*) just considered shows (see the proof of Theorem 10.4.5 and the remark after §10.4.6) that, as a basis for the operator construction of the theory of limits of sequences of real F-numbers, one can set an algorithm for which the initially given objects are pairs of the following form: the first term of a pair is a transcription of any fundamental sequence of real F-numbers, and the second term is the transcription of any algorithm transforming any natural number n into the transcription of a regulator of convergence in itself of the sequence of rational numbers which is based on the nth term of the sequence of F-numbers the transcription of which is the first term of the given pair (the terminology is explained in §3).

One can exhibit a large number of different examples of this kind from various branches of mathematics. We note that the kinds of "sources of nonconstructivity" in mathematics are not exhausted by the types considered above.

In constructive mathematics (in contradistinction to classical mathematics) the predicate variant and the operator variant of the construction of a theory are precisely delineated. Moreover, it is often necessary to carry out the transition from the first to the second variant by methods more complicated than in the corresponding theory of classical mathematics, but, in return, the possibility of changes in the types of relations between constructive objects is excluded. Thus, constructive mathematical analysis is a theory in which various mathematical phenomena are considered with more careful and more complete regard for the requirements of computational mathematics than in classical mathematical analysis.

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