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Special Functions and the Theory of Group Representations

N. Ja. Vilenkin

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Н. Я. ВИЛЕНКИН

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PREFACE

In this book we study the theory of special functions from the group-theoretic point of view. At first sight, the theory of special functions appears as a chaotic collection of formulas: apart from the fact that there exists an immense aggregate of the special functions themselves, for each of them there are, at present, all sorts of differential equations, integral representations, recurrence formulas, composition theorems and so on. To establish some kind of order in this chaos of formulas seems to be a completely hopeless task.

However, the development of the theory of group representations has now made it possible to comprehend the theory of the most important classes of special functions from a single point of view. We note that the appraisal of the importance of individual classes of special functions has changed greatly during the last hundred years. In the middle and the second half of the nineteenth century, elliptic and related functions were regarded as the most interesting. However, as F. Klein (see [77]) observed in one of his lectures, there exists another class of special functions, at least as important in view of its numerous applications to astronomy and mathematical physics, namely those associated with the hypergeometric function. The development of mathematics has now confirmed Klein's opinion; the hypergeometric function and its various special and degenerate cases, the functions of Bessel and Legendre, the orthogonal polynomials of Jacobi, Čebyšev, Laguerre and Hermite etc. play a big role in different branches of mathematics and its applications. This class of special functions is often called the special functions of mathematical physics. This is precisely the class of functions which lends itself to a group-theoretic treatment.

The connection between special functions and group representations was first discovered by É. Cartan (see [244]). (However, a connection between the

theory of special functions and the theory of invariants, which is one of the aspects of the theory of group representations, was established even earlier). The application of theory of representations to quantum mechanics played a significant part in the investigation part in the investigation of these connections.

Further work in this field was stimulated by the researches of I. M. Gel'fand and M. A. Naĭmark and their students and collaborators in the field of infinite-dimensional representations of groups. In the course of these researches, a connection was established between the theory of group representations and automorphic functions, the theory of special functions over finite fields was constructed, special functions were studied in homogeneous domains, and so on.

The aim of this book is a systematic account of the theory of special functions from the group-theoretic point of view. We have restricted ourselves here to the study of classical special functions and thus of the simplest groups.

The book consists of eleven chapters. In the first chapter basic concepts and facts of the theory of transformation groups and group representations are presented.

In the second chapter, two model examples are analyzed, namely the additive group of real numbers and the group of rotations of the circle. These groups lead, respectively, to the exponential and trigonometric functions. In this chapter certain facts of classical harmonic analysis, the theory of Fourier series and integrals in real and complex domains, are also presented.

The third chapter is devoted to representations of the group of rotations of the three-dimensional euclidean space and the locally isomorphic group $SU(2)$ of unitary unimodular matrices of the second order. The matrix elements of irreducible unitary representations of these groups are expressed by means of Legendre and Jacobi polynomials. The general properties of the matrix elements lead to various relations for these polynomials (orthogonality and completeness relations, recurrence formulas, composition theorem, etc.). At the end of the chapter we consider the discrete analogue of Jacobi polynomials, namely the Clebsch-Gordon coefficients.

In the fourth chapter representations of the group of motions of the euclidean plane are studied. The matrix elements of representations of this group are expressed by means of the Bessel functions, which allows one to derive a number of properties of Bessel functions from group-theoretic considerations. Further properties of Bessel functions and the closely related Hankel and Macdonald

functions are studied in Chapter V, devoted to representations of the group of motions of the pseudo-euclidean plane. In this chapter we derive a number of integrals involving the index of cylindrical functions. At the beginning of the chapter the group of linear transformations of the straight line and the theory of Γ-functions are considered.

The sixth and seventh chapters are devoted to representations of the group of real unimodular matrices of the second order. Representations of this group are connected with certain classes of special functions. In the sixth chapter the conical functions $P_{-\frac{1}{2}+i\lambda}(\operatorname{ch}\theta)$ are studied. In the seventh chapter we consider the realization of representations in the form of integral operators, with the hypergeometric function as the kernel. This leads to a number of properties of the hypergeometric function, some of them well known (the Mellin-Barnes integral representation), others new. In Chapter VIII we similarly study the confluent hypergeometric function and, in greater detail, the closely related Whittaker functions. It is shown that the theory of Whittaker functions is based on the study of representations of the group of third order triangular matrices. Starting from this, we derive various properties of Whittaker functions, e.g. integral representations, recurrence formulas, and continual composition theorems. In the same chapter Laguerre polynomials are considered, and a composition formula is derived for them.

All the groups considered in Chapters II–VIII have a very simple structure. More complicated groups are considered in Chapters IX–XI; they are the groups of motions of the n-dimensional sphere, of the n-dimensional Lobačevskiĭ space and of the n-dimensional euclidean space. Here, however, we do not consider all irreducible representations of these groups, but only the so-called representations of class 1, and we do not consider all matrix elements of representations, but only the matrix elements of "null column". This is found to be sufficient for constructing the theory of Gegenbauer polynomials and harmonic polynomials, and also for obtaining the decomposition of functions on the n-dimensional sphere and hyperboloid. In Chapter XI further properties of Bessel functions are derived.

Unfortunately the size of the book did not permit going into certain questions connected with the theory of representations and special functions. Thus the group of motions of the n-dimensional pseudo-euclidean space has not been considered. Hardly any light has been thrown upon asymptotic properties of special functions. However, the integral representations of these functions, presented in

the book, allow one to obtain their asymptotic expansions by the usual methods (see [6], [62]).

The author expresses his profound gratitude to I. M. Gel'fand, whose advice and instructions were made use of throughout the work on the book, and whose part in creating the book is difficult to overrate. The author also thanks M. A. Naimark, M. I. Graev, I. I. Pjateckiĭ-Šapiro, F. A. Berezin, F. I. Karpelevič, S. G. Gindikin, and D. P. Želobenko, with whom questions of the theory of group representations and the theory of special functions were often discussed. Numerous suggestions of M. I. Graev influenced the treatment of some questions and the final arrangement of the material.

The conversations with Ja. A. Smorodinskiĭ and his students, on the applications of the theory of group representations and integral transformations to physics, were highly useful for the author.

V. V. Cukerman took upon himself the difficult task of verifying the formulas. Great attention was paid to the manuscript, at all stages of its passage, by A. Z. Ryvkin. S. A. Vilenkina went through several variants of the manuscript. To them I express my gratitude for their help, which considerably facilitated preparation of the manuscript for printing.

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REMARKS AND BIBLIOGRAPHICAL NOTES

Chapter I

The theory of representations of linear transformation groups was originated by G. Frobenius [55]. Then Burnside and I. Schur found a substantially simpler approach by bringing the representation matrix itself into the foreground instead of its trace, namely the Frobenius character. The infinitesimal approach to representation theory was developed by É. Cartan [69]. The infinitesimal theory of Lie groups is expounded in the books of Pontrjagin [42], Helgason [56], Čebotarev [57], Chevalley [60], Eisenhart [61] and Kowalewski [78].

The general theory of continuous groups was, in the main, originated by L. S. Pontrjagin [42]. Invariant integration on Lie groups was introduced by Adolf Hurwitz in 1897 for proving theorems on invariants. In 1924 I. Schur [290] applied this process to representations of compact groups, in particular, the real orthogonal group. The existence of invariant measure on an arbitrary locally compact group was proved by Haar [252].

Induced representations of groups were utilized in a series of papers of I. M. Gel'fand and M. A. Naĭmark [17]. Concerning the general theory, see Mackey [277], Mautner [280] and Bruhat [240].

The theory of spherical functions on groups goes back to Cartan [244]; see Weyl [299] also. The subsequent development of the theory is given by Gel'fand, Naĭmark and Berezin [17], [105], [129], Vilenkin [116], Godement [251], Harish-Chandra [259], Mautner [281], Yoshizawa [303] and many other authors.

The results set forth here have, in the main, been obtained by I. Schur, F. Peter and H. Weyl. The fundamental result on completeness of the system of irreducible representations of a compact Lie group was proved by Peter and Weyl [284]. For infinite-dimensional representations, see Anna Hurevitsch [264]. These results were later carried over, in varying measure, to locally compact groups; see Rudin [90], Želobenko [178], Mackey [197,198], Naĭmark [204], Harish-Chandra [257–258] and Mautner [278]. Problems similar to those involved here were studied by M. G. Kreĭn [192,193].

Supplement to Chapter I. Concerning the topics touched upon here, see the books of I. M. Gel'fand, M. I. Graev and the author [14,15], and the paper by M. A. Naĭmark and S. V. Fomin [209].

Chapter II

The theory of trigonometrical series goes back to Bernoulli and Euler. These series were systematrically applied by Fourier and Cauchy in the 1820's for solving the problems of mathematical physics. Concerning the modern theory of trigonometrical series, see N. K. Bari, *Trigonometrical Series*, and A. Zygmund, *Trigonometrical Series*, volumes 1 and 2. The theory of Fourier integrals is expounded in the books of Bochner [5] and Titchmarsh [50]. Fourier transformation in the complex domain was first studied in detail by Wiener and Paley [12]. The proof of the inversion formula for Fourier transformation, given in the book, belongs to Gel'fand [132].

From the group-theoretic point of view, the theory of trigonometrical series and integrals is a special case of the theory of characters of commutative locally compact groups, constructed by L. S. Pontrjagin [42].

Chapter III

The functions $P_n(x)$ were introduced almost simultaneously by Legendre and Laplace in connection with the study of spheroidal attraction. They were the first example of orthogonal polynomials. The Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ are a generalization of the functions $P_n(x)$. Concerning the general theory of orthogonal polynomials, see Jackson [22] and Szegö [46]. For a bibliography on these polynomials, see [92].

The classical theory of Legendre functions is expounded in the books of Hobson [19], Whittaker and Watson [53], Lense [79], Robin [89] and Wangerin [97,98]. These functions are widely applied in classical and quantum physics. The study of the angular momentum operator in quantum mechanics led to a clarification of the connection between the polynomials of Legendre and Jacobi on the one hand, and representations of the group of rotations of three-dimensional euclidean space on the other. These matters are explained in the books of Bhagavantam and Venkatarayudu [2], Bauer [3], Beĭman [4], Van der Waerden [7], Wigner [11], Landau and Lifšic [30], Ljubarskiĭ [33], Heine [58], Hammermesch [72], Kahan [74], Racah [87] Weyl [99] and others.

A systematic construction of the theory of Legendre polynomials on the basis of the theory of group representations has been given by Gel'fand and Šapiro [155] see Gel'fand, Minlos and Šapiro [16] also. In these sources one can find an introduction to, and a detailed study of, the functions $P^{l}_{mn}(x)$, which are similar to Jacobi polynomials, yet somewhat different from them. The results of §§5.1 and 5.2 are due to the author [115,117].

The Clebsch-Gordan coefficients are employed for solving problems in spectroscopy; see Jucys, Levinsonas and Vanagas [63] and the books mentioned above. The derivation of properties of these coefficients, given here, is due basically to the author.

Chapter IV

The function $J_0(x)$ was considered in 1738 by Daniel Bernoulli, while functions $J_n(x)$ with integral values of n were considered in 1764 by Euler, and studied in detail by Bessel in 1824. Concerning the classical theory of Bessel functions, see Watson [8], Gray and Matthews [21], the handbook *Higher Transcendental Functions* [73] and Nielsen [83]. Applications to problems in mathematical physics are given in the books of Korenev [27], Lebedev [31], Morse and Feshbach [36] and Ufljand [54].

The author [113] has constructed certain sections of the theory of Bessel functions on the basis of the theory of representations of the group of euclidean plane motions. See Miller [81], Konstantinova and Sokolik [190] and Inonu and Wigner [265]. Representations of the universal covering group for $M(2)$ have been studied by Thoma [297]. For a generalization to the $\mathfrak{P}$-adic case, see Saito [289].

Chapter V

Representations of the group of linear transformations of the straight line have been studied by Gel'fand and Naĭmark [145]; see Kohari [267] also. The functions $\Gamma(x)$ and $B(x, y)$ were introduced by Euler. The account of the theory of these functions, given here, belongs to the author. Various generalizations of the Γ-function have been considered by Godement [160], Gel'fand and Graev [143] and Gindikin [158].

The classical theory of the Macdonald and Hankel functions is presented in the treatise of Watson [8], already mentioned. The author has studied these functions on the basis of the theory of representations of the group of pseudo-euclidean

plane motions. Some of the results (the composition theorem, for example) have been obtained independently by V. S. Ryko. It is possible that some of the formulas given here are new.

Chapter VI

The group of real unitary matrices of the second order (i.e. the group of quasi-unitary matrices of the second order) was the first noncompact, noncommutative group the theory of whose representations was studied in detail (Bargmann [236]). Interest in the representations of this group and the closely related group of unitary complex matrices of the second order was evoked by the role of these groups in physics (they are isomorphic to the three-dimensional and four-dimensional Lorentz groups respectively). Representations of the group of complex second order matrices were first studied by Gel'fand and Naĭmark [144]. The results obtained here were used for a fundamental study of representations of semisimple groups (Gel'fand and Naĭmark [17,147], Gel'fand and Graev [135,136,137,138], Berezin [103,104], Želobenko [176,177], Ehrenpreis and Mautner [249], Harish-Chandra [253–259] and others). Many of the results of harmonic analysis on groups were first obtained on groups of the second order; moreover, all of these have not as yet been carried over to a more general case (Bogaevskiĭ [109], Gel'fand and Fomin [152], Želobenko [172–174], Naĭmark [201,203,207], Romm [220, 221], Kunze and Stein [269,270] and Pukánszky [285,287].

The exposition in this chapter basically follows Bargmann. Many of the topics are presented in the book of Gel'fand, Graev and the author [15], Chapter 7. Concerning other derivations of the Plancherel formula on the group $SL(2, R)$, see Gel'fand and Graev [136] and Pukánszky [287].

Chapter VII

The hypergeometric function was introduced by Gauss. Concerning the classical theory of this function, see Kratzer and Franz [28], *Higher Transcendental Functions*, volume 1 [73], and Klein [77]. The account of the theory given here belongs to the author [122,123]. Several formulas of this chapter are new.

Chapter VIII

Concerning the theory of confluent hypergeometric functions and the closely related Whittaker functions, see Kratzer and Franz [28], Whittaker and Watson [53], Bailey [66], Buchholz [68], *Higher Transcendental Functions*, volume 1 [73],

and Tricomi [94]. The theory of representations of the group of triangular third order matrices is a special case of the general theory of representations of solvable and nilpotent Lie groups (see Kirillov [188], Dixmier [167,248]).

The connection between the theory of Whittaker functions and the theory of representations of the group of triangular matrices was established by the author [125]; see Miller [81] also. Several formulas of this chapter are new.

Chapter IX

Schur [290], Cartan [242,243], Weyl [300] and Brauer [238] have given the classification of representations of the orthogonal group and calculated the characters of these representations. Concerning these results, see Murnaghan [37,82] and Weyl [10] also. Gel'fand and Cetlin [154] have calculated the matrix elements of irreducible infinitesimal representations of this group. The expression for the matrix elements of representations in terms of Euler parameters has been obtained by the author [114,121] and Lambina [195].

The theory of spherical functions in the n-dimensional case is expounded in the books of Appell and Kampé de Fériet [65] and Nielsen [85]. In *Higher Transcendental Functions*, volume 2 [73], this theory is presented essentially on the basis of representation theory. The realization of representations given in our book belongs to the author, as does the theory of polyspherical functions. See Kirillov [186] also.

The connection between the theory of group representations and Gegenbauer polynomials was established by Cartan [244].

Chapter X

Representations of the group of hyperbolic rotations of n-dimensional space and the related special functions were obtained, independently, by the author [114,119,121] and Takahashi [292]; see Hirai [262] also.

In the four-dimensional case these representations have been completely described by Gel'fand and Naĭmark [144,16,39,201]. These representations, their matrix elements and related integral transformations have been studied in a number of papers (see, for example, Golodec [162], Dolginov, Toptygin and Moskalev [169-171], Želobenko [173,174], Levinsonas and Jucys [196], Popov [218], Romm [220,221], Šapiro [226,226a], Èskin [228]. Takahashi [291] and Tamm [294].

The horisphere method for decomposing representations into irreducible components has been developed by Gel'fand and Graev [141,142]. The derivation of

the Fock-Mehler integral transformation on the basis of the horisphere method has been given by the author [119, 121]. It can be shown that other integral transformations (of Kontorovich-Lebedev, Olevskiĭ [214] and others) are also obtained by this method. For the four-dimensional case this has been done by the author and Smorodinskiĭ [127]. Polyspherical and horispherical functions on the hyperboloid have been introduced by the author [124]. In the three-dimensional case various orthogonal coordinate systems, in which the Laplace operator admits separation of variables, have been studied by Olevskiĭ [215].

Chapter XI

The study of properties of Bessel functions in connection with the theory of representations of the group of n-dimensional euclidean space motions has been carried out by the author [114, 121] and independently by Orihara [283]. Ito [266] has described all irreducible unitary representations of this group. Bessel functions of matrix argument have been studied by Herz [261]. Concerning the theory of Hermite polynomials, see Szegö [46].

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Latin alphabet

(A) – matrix of the operator A, 9

A^* – operator Hermitian-adjoint to A, 13

$A * B$ – (semidirect) cross-product of groups, 197

$A \otimes B$ – Kronecker product of operators, 63

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$W_{\lambda,\mu}(z)$, $M_{\lambda,\mu}(z)$ – Whittaker functions, 400

$[\mathbf{x}, \mathbf{y}]$ – bilinear form in pseudo-euclidean space, 503

$Y_{lk}(\phi, \theta)$ – associated spherical functions, 129

$Z_\nu(z)$ – cylindrical function, 129

German alphabet

$\mathfrak{B}^{n\sigma}$ – 509

$\mathfrak{D}$ – space of infinitely differentiable functions on the circle, 200

$\mathfrak{D}_x$ – space of the function $\phi(z)$, 295, 296

$\mathfrak{H}_l$ – invariant subspaces, in which the representations $T_l(u)$ are realized, 175

$\mathfrak{H}_l$ – space of homogeneous polynomials of degree $2l$, 108

$\mathfrak{H}^{nl}$ – space of homogeneous harmonic polynomials, 442

$\mathfrak{L}'$ – space adjoint to $\mathfrak{L}$, 12, 59

$\mathfrak{L}^2(S^{n-1})$ – space of square-summable functions on the sphere, 440

$\mathfrak{L}_1 + \mathfrak{L}_2$ – direct sum of spaces, 15

$\mathfrak{L}_1 \otimes \mathfrak{L}_2$ – Kronecker (tensor) product of spaces, 60

$\mathfrak{L}_n^2$, ${}_m\mathfrak{L}^2$, ${}_m\mathfrak{L}_n^2$ – subspaces of functions on $SU(2)$, 164–167

$\mathfrak{M} = G/H$ – homogeneous space, 25

$\mathfrak{P}_l(\operatorname{ch}\tau)$ – Legendre functions, 314

$\mathfrak{P}_l^m(z)$ – associated Legendre functions, 315, 316

$\mathfrak{P}_{mn}^l(\operatorname{ch}\tau)$, $\mathfrak{P}_{mn}^l(z)$ – Jacobi functions, 309, 310

$\mathfrak{R}^{nl}$ – space of homogeneous polynomials, 441

$\mathfrak{S}$ – 82, 225

Greek alphabet

$\mathrm{B}(x, y)$ – beta function, 243

$\Gamma(z)$ – gamma function, 237

Δ, Δ_0 – Laplace operators, 143–145

Δ_{lm} – restriction of the Laplace operator, 142

δ_{ij} – Kronecker symbol, 10

$\Lambda^{n-1}(R)$ – $(n-1)$-dimensional Lobačevskiĭ space, 506

$\Xi_K^l(x)$ – canonical basis in $\mathfrak{H}^{nl}$, 466, 468

$\Phi(\alpha; \gamma; z)$ – confluent hypergeometric function, 401

ϕ, θ, ψ – Euler angles, 98

ϕ, τ, ψ – Euler angles, 295

$\chi_l(u)$, $\chi_T(g)$ – characters, 19, 170

Ω – subgroup of hyperbolic rotations, 250

Special symbols

$\otimes$ – Kronecker (tensor) product, 18, 60, 63, 176

$\Sigma \oplus T_k(g)$ or $\Sigma\, T_k(g)$ – direct orthogonal sum of representations, 16

$\Box$ – the wave operator, 541

$\Box_0$ – Laplace operator on a hyperboloid, 542

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A standard scheme for a relation between special functions and group representation theory is the following: certain classes of special functions are interpreted as matrix elements of irreducible representations of a certain Lie group, and then properties of special functions are related to (and derived from) simple well-known facts of representation theory. The book combines the majority of known results in this direction. In particular, the author describes connections between the exponential functions and the additive group of real numbers (Fourier analysis), Legendre and Jacobi polynomials and representations of the group $SU(2)$, and the hypergeometric function and representations of the group $SL(2,R)$, as well as many other classes of special functions.

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