The Functional Method and Its Applications

by

E. V. Voronovskaja

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PREFACE

This book can be used as a textbook for graduate students of mathematics and those preparing for examinations in the theory of functions, and in particular in the theory of best uniform approximation.

The material presented here may also be useful for engineers and students of radio and electrical engineering who are concerned with problems of the theory of filters, amplifiers, pulse technology, etc.

To make it easier for engineers to read the book, it contains a brief introductory section on the theory of moment sequences and the simplest theorems of functional analysis.

The first part of the book contains an exposition of the new theory which was announced in a series of papers by the author between 1934 and 1953; a detailed exposition has not been published previously. The second part illustrates the theory by solving a number of problems of Čebyšev type which could not be handled by classical methods.

The author hopes that the availability of this new method and its effectiveness in solving problems of various kinds will help graduate students in finding thesis topics, and engineers in the theoretical analysis of their experimental problems.

The author wishes to thank her student M. Ja. Zinger, and E. L. Rabkin, assistant in the department of mathematics of the Leningrad Electrotechnical Institute of Communication, for their help in preparing this book for publication.

Leningrad, 8 July 1963

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APPENDIX

DERIVATIVE FUNCTIONALS OF AN ALGEBRAIC POLYNOMIAL AND V. A. MARKOV'S THEOREM[1)]

V. A. GUSEV

The functional method is applied to investigate the exact upper bounds of the derivatives of order k of an algebraic polynomial at each point of the interval $[0,1]$. A new proof of V. A. Markov's theorem is given and some new results are obtained.

V. A. Markov's theorem [1][2)] gives a bound for the derivative of order $k \leqq n$ of an algebraic polynomial $P_n(x)$: if

$$\max_{[-1,1]} |P_n(x)| = M, \tag{1}$$

then

$$\max_{[-1,1]} |P_n^{(k)}(x)| \leqq \frac{n^2(n^2-1^2)\cdots(n^2-\overline{k-1}^2)}{1\cdot 3\cdots(2k-1)}\cdot M \quad (k=1,2,\cdots,n). \tag{2}$$

New and simpler proofs of this theorem were given by S. N. Bernšteĭn [2] and Duffin and Schaeffer [3].

The functional method developed by E. V. Voronovskaja [4]—[6] makes it possible to study the problem considerably more completely than in [2] and [3], and in a considerably shorter way than in [1].

The case $k = 1$ was studied by E. V. Voronovskaja [7]. In the present paper the results obtained in [7] for the first derivative are extended to derivatives of higher order; the terminology of [7] is retained here.

We consider the interval $[0,1]$ (this is the most convenient for the functional method). Then (2) takes the form

$$\max_{[0,1]} |P_n^{(k)}(x)| \leqq \frac{2^k n^2(n^2-1^2)\cdots(n^2-\overline{k-1}^2)}{1\cdot 3\cdots(2k-1)}\cdot M_{[0,1]} = M_{[0,1]}T_n^{(k)}(1), \tag{2'}$$

where

$$M_{[0,1]} = \max_{[0,1]}|P_n(x)|, \quad T_n(x) = \cos n \arccos(2x-1).$$

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1) Translation of Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 367-384.

2) Numbers in brackets refer to the Bibliography at the end of this Appendix.

We shall treat the kth derivative of $P_n(x)$ at a point ξ as the linear functional $F_\xi^{(k)}$, defined by the finite sequence (or segment)

$$(\mu_i^{(k)})_{i=0}^n = 0_0, 0_1, \cdots, 0_{k-1}, k!, (k+1)!\xi, \cdots, \frac{n!}{(n-k)!}\xi^{n-k} \tag{3}$$
$$(k = 1, 2, \cdots, n; \ -\infty < \xi < \infty),$$

on the set of polynomials $\{P_n(x)\}$ of degree at most n, so that

$$F_\xi^{(k)}(x^i) = 0 \qquad \text{for } i = 0, 1, \cdots, k-1,$$
$$F_\xi^{(k)}(x^i) = \frac{i!}{(i-k)!}\xi^{i-k} \text{ for } i = k, k+1, \cdots, n$$

and

$$F_\xi^{(k)}[P_n(x)] = P_n^{(k)}(\xi).$$

Since the space $\{P_n(x)\}$ is finite-dimensional, for each ξ there is a polynomial $Q_n(x, \xi)$, called extremal or serving, such that when

$$\max_{[0,1]} |Q_n(x, \xi)| = 1$$

(reduced polynomial) we have the equation

$$N_k(\xi) = [Q_n^{(k)}(x, \xi)]_{x=\xi},$$

where $N_k(\xi)$ is the norm of the functional (3). We have

$$\max_{[0,1]} |P_n^{(k)}(\xi)| \leq M_{[0,1]} \cdot N_k(\xi) = M_{[0,1]} \cdot [Q_n^{(k)}(x, \xi)]_{x=\xi} \quad (k = 1, 2, \cdots, n).$$

This is a sharpening of V. A. Markov's inequality (2′). Thus the problem is to determine the extremal polynomial and the norm of the functional (3).

REMARK 1. Putting $\mu_i^{(k)} = \mu_{i,0}^{(k)}$, we form the table of differences $(\mu_{i,p}^{(k)})$ of the segment (3), where

$$\mu_{l,p}^{(k)} = \mu_{l,p-1}^{(k)} - \mu_{l+1,p-1}^{(k)};$$

then we obtain

$$(\mu_{0j}^{(k)}) = 0_0, 0_1, \cdots, 0_{k-1}, (-1)^k k!, (-1)^k (k+1)!(1-\xi), \cdots$$
$$\cdots, \frac{(-1)^k n!}{(n-k)!}(1-\xi)^{n-k}.$$

This functional has the same norm (cf. [**4**]) as (3). Consequently

$$N_k(\xi) = N_k(1-\xi) \quad (k = 1, 2, \cdots, n; \ -\infty < \xi < \infty). \tag{4}$$

REMARK 2. For $\xi < 0$ the segment (3) has alternating signs, and consequently

$$Q_n(x, \xi) = + T_n(x),$$

if $k \equiv n \pmod 2$, and

$$Q_n(x, \xi) = - T_n(x),$$

if $k + 1 \equiv n \pmod 2$. In view of (4), for each ξ outside $[0, 1]$ we have

$$Q_n(x, \xi) = \pm T_n(x), \quad N_k(\xi) = | T_n^{(k)}(\xi) | \quad (k = 1, 2, \cdots, n).$$

We recall a necessary and sufficient condition that a given reduced polynomial $L_n(x)$ $(\max_{[0,1]} | L_n(x) | = 1)$ with distribution $(\overset{\pm}{\sigma}_i)_1^s$ $(0 \leqq \sigma_1 < \sigma_2 < \cdots < \sigma_s \leqq 1;\ L_n(\overset{+}{\sigma}_i) = +1, L_n(\overset{-}{\sigma}_i) = -1)$ is extremal for a given segment $(\mu_i)_0^n$. The test for extremal character is as follows (cf. [7]): the system of $n + 1$ equations in the s unknowns δ_i

$$\text{(V)} \qquad \sum_{i=1}^{s} \delta_i \sigma_i^l = \mu_l \quad (l = 0, 1, \cdots, n)$$

has the properties 1) it is consistent, and 2) it has a solution such that either $\operatorname{sgn} \delta_i = L_n(\sigma_i)$ or $\delta_i = 0$ (but not all are zero).

THEOREM 1. *On the interval $[0, 1]$ there are $n - k + 1$ Čebyšev intervals $[\alpha_i^{(k)}, \beta_i^{(k)}]$ $(k = 1, 2, \cdots, n;\ i = 1, 2, \cdots, n - k + 1)$ at whose points the functional (3) is served by one of the polynomials $\pm T_n(x)$. The endpoints of the intervals are the roots $\alpha_2^{(k)}, \alpha_3^{(k)}, \cdots, \alpha_{n-k+1}^{(k)}$ of the equation*

$$\frac{d^k}{dx^k}\left(\frac{R_{n+1}(x)}{x}\right) = 0,$$

and $\alpha_1^{(k)} = 0$, and the roots $\beta_1^{(k)}, \beta_2^{(k)}, \cdots, \beta_{n-k}^{(k)}$ of the equation

$$\frac{d^k}{dx^k}\left(\frac{R_{n+1}(x)}{x-1}\right) = 0,$$

and $\beta_{n-k+1}^{(k)} = 1$; here $R_{n+1}(x) = \prod_{i=0}^{n} (x - \tau_i)$ is the resolvent of $T_n(x)$, and $(\tau_i)_0^n$ are its nodes, i.e. its points of maximum deviation from zero.

PROOF. Let us apply the test for extremal character to $L_n(x) = \pm T_n(x)$. In this case $s = n + 1$ and condition 1) drops out, while condition 2) is that $\delta_0, \delta_1, \cdots, \delta_n$ have alternating signs. The system (V) for the segment (3) takes the form

$$\sum_{i=0}^{n} \delta_i \tau_i^l = 0 \qquad (l = 0, 1, \cdots, k - 1),$$

$$\sum_{i=0}^{n} \delta_i \tau_i^l = \frac{l!}{(l-k)!} \xi^{l-k} \quad (l = k, k + 1, \cdots, n),$$

and its solution is given by

$$\delta_j = \frac{(-1)^{n-j}}{\prod\limits_{i \neq j} |\tau_j - \tau_i|} \cdot \left[\frac{d^k}{dx^k} \left(\frac{R_{n+1}(x)}{x - \tau_j} \right) \right]_{x=\xi} \quad (j = 0, 1, \cdots, n). \tag{5}$$

We shall need the following three lemmas of V. A. Markov [1].

Lemma 1. *If the equation* $G_s(x) = x^s + a_1 x^{s-1} + \cdots + a_{s-1}x + a_s = 0$ *has no complex roots, then*

$$[G_s^{(k)}(x)]^2 - G_s^{(k-1)}(x) \cdot G_s^{(k+1)}(x) > 0$$

for all $k \leqq s$ *and all real* x *except for roots of* $G_s(x) = 0$ *of multiplicity greater than* k.

Lemma 2. *Let* $G(x) = \prod_{i=1}^{s} (x - x_i)$ $(x_k \neq x_j)$, *let* z *be a root of* $G^{(k)}(x) = 0$ *and let*

$$G_l(x) = \frac{G(x)}{x - x_l} \quad (l = 1, 2, \cdots, s);$$

then the numbers $G_1^{(k)}(z), G_2^{(k)}(z), \cdots, G_s^{(k)}(z), G^{(k+1)}(z)$ *all have the same sign.*

Lemma 3. *Let*

$$G(x) = A \cdot \prod_{i=1}^{s} (x - a_i), \quad H(x) = B \cdot \prod_{i=1}^{s} (x - b_i) \quad (A > 0, B > 0)$$

and $b_1 < a_1 < b_2 < a_2 < \cdots < b_s < a_s$. *If* $G^{(k)}(z) = 0$ *then*

$$\frac{H^{(k)}(z)}{G^{(k+1)}(z)} > 0.$$

Corollary. *It follows immediately from Lemma 3 that the roots of* $G^{(k)}(x) = 0$ *and of* $H^{(k)}(x) = 0$ *are interlaced. This is also true when the degrees of* $G(x)$ *and* $H(x)$ *differ by unity (the roots of* $G(x)$ *and* $H(x)$ *separate each other).*

Let $\Phi_j(x)$ be the polynomial

$$\frac{R_{n+1}(x)}{x - \tau_j} \quad (j = 0, 1, \cdots, n)$$

and let $\xi_{j,k}^{(1)}, \xi_{j,k}^{(2)}, \cdots, \xi_{j,k}^{(n-k)}$ be the roots of $\Phi_j^{(k)}(x)$; then $\xi_{0,k}^{(i)} = \alpha_{i+1}^{(k)}$ and $\xi_{n,k}^{(i)} = \beta_i^{(k)}$; let $\theta_1^{(k)}, \theta_2^{(k)}, \cdots, \theta_{n-k+1}^{(k)}$ be the roots of $R_{n+1}^{(k)}(x)$. It follows from the results of Voronovskaja for $k = 1$ (cf. [7]) that the following inequalities hold for the roots of $\Phi_0'(x), \Phi_1'(x), \cdots, \Phi_n'(x)$ and $R_{n+1}'(x)$:

$$0 < \theta_1^{(1)} < \beta_1^{(1)} < \xi_{n-1,1}^{(1)} < \xi_{n-2,1}^{(1)} < \cdots < \xi_{1,1}^{(1)} < \alpha_2^{(1)} < \theta_2^{(1)} < \beta_2^{(1)} < \xi_{n-1,1}^{(2)} < \cdots$$
$$\cdots < \xi_{1,1}^{(2)} < \alpha_3^{(1)} < \theta_3^{(1)} < \beta_3^{(1)} < \cdots < \alpha_{n-1}^{(1)} < \theta_{n-1}^{(1)} < \beta_{n-1}^{(1)} < \xi_{n-1,1}^{(n-1)} < \cdots$$
$$\cdots < \xi_{1,1}^{(n-1)} < \alpha_n^{(1)} < \theta_n^{(1)} < 1.$$

Applying the corollary of Lemma 3 to each pair of the polynomials $\Phi_0'(x), \Phi_1'(x), \cdots, \Phi_n'(x)$, $R_{n+1}'(x)$, we see that there are similar inequalities for the roots of the kth derivatives $\Phi_0^{(k)}(x), \Phi_1^{(k)}(x), \cdots, \Phi_n^{(k)}(x)$, $R_{n+1}^{(k)}(x)$:

$$0 < \theta_1^{(k)} < \beta_1^{(k)} < \xi_{n-1,k}^{(1)} < \xi_{n-2,k}^{(1)} < \cdots < \xi_{1,k}^{(1)} < \alpha_2^{(k)} < \theta_2^{(k)} < \beta_2^{(k)} < \cdots$$
$$\cdots < \alpha_i^{(k)} < \theta_i^{(k)} < \beta_i^{(k)} < \xi_{n-1,k}^{(i)} < \xi_{n-2,k}^{(i)} < \cdots$$
$$\cdots < \xi_{1,k}^{(i)} < \alpha_{i+1}^{(k)} < \theta_{i+1}^{(k)} < \beta_{i+1}^{(k)} < \cdots$$
$$\cdots < \alpha_{n-k}^{(k)} < \theta_{n-k}^{(k)} < \beta_{n-k}^{(k)} < \xi_{n-1,k}^{(n-k)} < \xi_{n-2,k}^{(n-k)} < \cdots$$
$$\cdots < \xi_{1,k}^{(n-k)} < \alpha_{n-k+1}^{(k)} < \theta_{n-k+1}^{(k)} < 1.$$

Therefore inside each interval of the form

$$[\alpha_i^{(k)}, \beta_i^{(k)}] \quad (k = 1, 2, \cdots, n-1;\ i = 1, 2, \cdots, n-k+1)$$

there is a single root of $R_{n+1}^{(k)}(x)$, and each of the derivatives $\Phi_0^{(k)}(x)$, $\Phi_1^{(k)}(x), \cdots, \Phi_n^{(k)}(x)$ preserves its sign. Applying Lemma 2 to the polynomials $R_{n+1}(x)$, $\Phi_0(x), \cdots, \Phi_n(x)$, we reach the conclusion that all the numbers $\Phi_0^{(k)}(\theta_i^{(k)}), \Phi_1^{(k)}(\theta_i^{(k)}), \cdots, \Phi_n^{(k)}(\theta_i^{(k)})$ have the same sign (plus for $i = n-k+1, n-k-1, n-k-3, \cdots$, and minus for $i = n-k$, $n-k-2, n-k-4, \cdots$). It follows that if $\xi \in [\alpha_i^{(k)}, \beta_i^{(k)}]$, all the numbers $\Phi_0^{(k)}(\xi), \Phi_1^{(k)}(\xi), \cdots, \Phi_n^{(k)}(\xi)$ have the same sign, and in addition only $\Phi_0^{(k)}(\xi) = 0$ at $\xi = \alpha_i^{(k)}$ $(i = 1, 2, \cdots, n-k+1)$, while only $\Phi_n^{(k)}(\xi) = 0$ at $\xi = \beta_i^{(k)}$ $(i = 1, 2, \cdots, n-k)$. Turning to formula (5), we see that $\delta_0, \delta_1, \cdots, \delta_n$ have alternating signs, and condition 2) of the test is satisfied if $\xi \in [\alpha_i^{(k)}, \beta_i^{(k)}]$. At the left-hand endpoints $(\alpha_i^{(k)})_{i=2}^{n-k+1}$ loses the weight δ_0 at the node $\tau_0 = 0$, i.e. $\delta_0 = 0$, and at the right-hand endpoints $(\beta_i^{(k)})_{i=1}^{n-k}$ loses the weight δ_n at the node $\tau_n = 1$, i.e. $\delta_n = 0$. Consequently service of the functional $F_\xi^{(k)}$ by the polynomials $\pm T_n(x)$ ends (or begins) at these points (cf. [**4**], [**5**]).

Thus in the intervals $[\alpha_{n-k+1}^{(k)}, 1], [\alpha_{n-k-1}^{(k)}, \beta_{n-k-1}^{(k)}], \cdots$ the functional $F_\xi^{(k)}$ is served by $+T_n(x)$; and in the intervals

$$[\alpha_{n-k}^{(k)}, \beta_{n-k}^{(k)}], [\alpha_{n-k-2}^{(k)}, \beta_{n-k-2}^{(k)}], \cdots,$$

by $-T_n(x)$. When $k = n$ formula (5) takes the form

$$\delta_j = \frac{(-1)^{n-j} n!}{\prod_{i \neq j} |\tau_j - \tau_i|} \quad (j = 0, 1, \cdots, n),$$

and $F_\xi^{(n)}$ is served by $+T_n(x)$ for $\xi \in [0,1]$. This completes the proof of Theorem 1.

THEOREM 2. *Between the Čebyšev intervals there are open Zolotarev intervals* $(\beta_i^{(k)}, \alpha_{i+1}^{(k)})$ $(k = 1, 2, \cdots, n-1;\ i = 1, 2, \cdots, n-k)$ *at whose points the functional* (3) *is served by all the polynomials of passport* $[n, n, 0]$[3] *(denoted by* $Q_n(x, \vartheta)$*) and only by these, and indeed by each one at that point of each interval where*

$$\left[\frac{\partial^k R_n(x, \vartheta)}{\partial x^k}\right]_{\substack{x=\xi \\ \vartheta=\vartheta_\xi}} = 0.$$

Here $R_n(x, \vartheta) = \prod_{i=1}^n (x - \sigma_i)$ *is the resolvent of* $Q_n(x, \vartheta)$, $(\bar{\bar{\sigma}}_i)_1^n$ *is its distribution, and* ϑ *is the variable leading coefficient of* $Q_n(x, \vartheta)$ $(-2^{2n-1} < \vartheta < 2^{2n-1})$.

PROOF. Let $L_n(x)$ be any polynomial of passport $[n, n, 0]$, let $(\bar{\bar{\sigma}}_i)_1^n$ be its distribution, let

$$R_n(x) = \prod_{i=1}^n (x - \sigma_i) = \sum_{l=0}^n r_l x^l$$

be its resolvent, and let ξ_0 be a root of $R_n^{(k)}(x) = 0$. We shall show that $L_n(x)$ serves the functional $F_{\xi_0}^{(k)}$. The first n equations of the system (V) take the form

$$\sum_{i=1}^n \delta_i \sigma_i^l = 0 \qquad (l = 0, 1, \cdots, k-1),$$

$$\sum_{i=1}^n \delta_i \sigma_i^l = \frac{l!}{(l-k)!} \xi_0^{l-k} \quad (l = k, k+1, \cdots, n-1),$$

and the solution is given by

$$\delta_j = \frac{(-1)^{n-j-1}}{\prod_{i \neq j} |\sigma_j - \sigma_i|} \left[\frac{d^k}{dx^k}\left(\frac{R_n(x)}{x - \sigma_j}\right)\right]_{x=\xi_0} \quad (j = 1, 2, \cdots, n).$$

Condition 2) of the test for extremal character is satisfied since, by Lemma 2, the numbers

[3] See [**6**].

$$\left[\frac{d^k}{dx^k}\left(\frac{R_n(x)}{x-\sigma_1}\right)\right]_{x=\xi_0}, \left[\frac{d^k}{dx^k}\left(\frac{R_n(x)}{x-\sigma_2}\right)\right]_{x=\xi_0}, \cdots, \left[\frac{d^k}{dx^k}\left(\frac{R_n(x)}{x-\sigma_n}\right)\right]_{x=\xi_0}$$

have the same sign. Condition 1), that

$$\frac{n!}{(n-k)!}\xi_0^{n-k} = \sum_{i=1}^{n} \delta_i \sigma_i^n,$$

i.e. that the entire system (V) is consistent, yields the equation

$$F_\xi^{(k)}[R_n(x)] = R_n^{(k)}(\xi) = \sum_{l=0}^{n} r_l \sum_{i=1}^{n} \delta_i \sigma_i^l = \sum_{i=1}^{n} \delta_i R_n(\sigma_i) = 0,$$

i.e. $R_n^{(k)}(\xi) = 0$, which is satisfied at ξ_0 and nowhere else.

Thus each polynomial $L_n(x)$ serves $F_\xi^{(k)}$ at $n-k$ points, the roots of $R_n^{(k)}(x) = 0$, lying one in each of the intervals $(\beta_i^{(k)}, \alpha_{i+1}^{(k)})$ $(i = 1, 2, \cdots, n-k)$.

If we delete from the set $\{Q_n(x, \vartheta)\}$ the Čebyšev transformations $\pm T_n(\nu x)$ and $\pm T_n(\nu \cdot \overline{1-x})$ that the set contains, the remaining polynomials are, up to constant factors, exactly the polynomials of E. I. Zolotarev [8], which he expressed in terms of elliptic functions; we denote them by $Z_n(x, \vartheta)$.

At the endpoints $(\beta_i^{(k)})_{i=1}^{n-k}$ and $(\alpha_i^{(k)})_{i=2}^{n-k+1}$ the functional $F_\xi^{(k)}$ loses its weight at, respectively, the nodes $\tau_n = 1$ $(\delta_n = 0)$ or $\tau_0 = 0$ $(\delta_0 = 0)$; hence in $(\beta_i^{(k)}, \alpha_{i+1}^{(k)})$, as ξ moves from $\beta_i^{(k)}$ to $\alpha_{i+1}^{(k)}$ (assuming, for definiteness, that $F_\xi^{(k)}$ is served in $[\alpha_i^{(k)}, \beta_i^{(k)}]$ by $+T_n(x)$), service changes in the following order:

$$+T_n(\nu x)\ \left(\cos^2\frac{\pi}{2n} \leqq \nu < 1\right), \quad +Z_n(x,\vartheta)\ \left(0 < \vartheta < 2^{2n-1}\cos^{2n}\frac{\pi}{2n}\right),$$
$$-T_{n-1}(x), \quad (-1)^{n-1}Z_n(1-x,\vartheta), \quad (-1)^{n-1}T_n(\nu\cdot\overline{1-x})$$

without a break, according to the theorem on continuous deformation (cf. [4], [5]), and without repetition until the extremal polynomial becomes $-T_n(x)$ at $\alpha_{i+1}^{(k)}$.

This completes the proof of Theorem 2.

Therefore if E_T is the set of points of Čebyšev intervals and E_z is its complement with respect to $[0,1]$, we have the following inequalities for reduced polynomials:

$$|P_n^{(k)}(\xi)| \leqq \begin{cases} |T_n^{(k)}(\xi)| = N_k(\xi) & \text{for } \xi \in E_T, \\ |Q_n^{(k)}(\xi, \vartheta_\xi)| = N_k(\xi) & \text{for } \xi \in E_z, \end{cases}$$

where ϑ_ξ and ξ are connected, by Theorem 2, by the equation

(6) $$\frac{\partial^k R_n(\xi, \vartheta)}{\partial \xi^k} = 0.$$

The norm $N_k(\xi)$ is a continuous function of ξ.

THEOREM 3. *In each interior Čebyšev interval the norm* $N_k(\xi)$ *takes its maximum value*

$$\max_{[\alpha_i^{(k)}, \beta_i^{(k)}]} N_k(\xi) = N_k(\gamma_i^{(k)})$$

just once, where $(\gamma_i^{(k)})_{i=2}^{n-k}$ *are the roots of* $T_n^{(k+1)}(x) = 0$; *in the two boundary Čebyšev intervals the norm* $N_k(\xi)$ *decreases monotonically from the endpoints of* $[0,1]$ *inward.*

PROOF. Theorem 3 of Voronovskaja [7] shows that the roots $(\gamma_i^{(1)})_{i=2}^{n-1}$ of $T_n''(x)$ lie one each in the interior intervals $[\alpha_i^{(1)}, \beta_i^{(1)}]$ $(i = 2, 3, \cdots, n-1)$. Applying the corollary of V. A. Markov's Lemma 3 to each pair of the polynomials $\Phi_0'(x)$, $T_n''(x)$, $\Phi_n'(x)$, we see that for each $k \leqq n-2$ the roots $(\gamma_i^{(k)})_{i=2}^{n-k}$ of $T_n^{(k+1)}(x)$ lie one each in the interior intervals $[\alpha_i^{(k)}, \beta_i^{(k)}]$ $(i = 2, 3, \cdots, n-k)$. It follows that the norm $N_k(\xi)$ attains its maximum once in each interior Čebyšev interval, and decreases monotonically, from the endpoints of $[0,1]$ inward, in the boundary Čebyšev intervals, q.e.d.

REMARK. If $k \equiv n \pmod 2$, there is a Čebyšev interval

$$[\alpha_{(n-k+2)/2}^{(k)}, 1 - \alpha_{(n-k+2)/2}^{(k)}],$$

containing the point $\xi = \gamma_{(n-k+2)/2}^{(k)} = \frac{1}{2}$, and

$$N_k\left(\frac{1}{2}\right) = \left| T_n^{(k)}\left(\frac{1}{2}\right) \right| = \begin{cases} 2^k n(n^2 - 1^2) \cdots (n^2 - \overline{k-2}^2) & \text{for } n \text{ odd}, \\ 2^k n^2(n^2 - 2^2) \cdots (n^2 - \overline{k-2}^2) & \text{for } n \text{ even}. \end{cases}$$

We turn to the norm $N_k(\xi)$ on the Zolotarev intervals. In each interval $(\beta_i^{(k)}, \alpha_{i+1}^{(k)})$ there is a unique point $\xi_{i,k}^*$ $(i = 1, 2, \cdots, n-k)$ at which the functional $F_\xi^{(k)}$ is served by one of the polynomials $\pm T_{n-1}(x)$. The points $(\xi_{i,k}^*)_{i=1}^{n-k}$ can be found as the roots of $R_n^{(k)}(x) = 0$, where $R_n(x)$ is the resolvent of $T_{n-1}(x)$. We call $(\beta_i^{(k)}, \xi_{i,k}^*)$ and $(\xi_{i,k}^*, \alpha_{i+1}^{(k)})$ the left-hand and right-hand parts of the interval $(\beta_i^{(k)}, \alpha_{i+1}^{(k)})$.

THEOREM 4. *In the interval* $(\beta_i^{(k)}, \alpha_{i+1}^{(k)})$ $(i = 1, 2, \cdots, n-k)$ *the norm* $N_k(\xi)$ *varies monotonically at each point* ξ *at which the* $(k+1)$*th derivative of the extremal polynomial is not zero.*

PROOF. Suppose for definiteness that the extremal polynomial at $\xi = \beta_i^{(k)}$ is $+T_n(x)$. Then in some interval $(\beta_i^{(k)}, A_i^{(k)})$, $\beta_i^{(k)} < A_i^{(k)} < \xi_{i,k}^*$, the Čebyšev transformations $T_n(\nu x)$ will be extremal (cf. Theorem 2). We have

$$N_k(\xi) = T_n^{(k)}(\nu\xi) \cdot \nu^k$$

and

$$N_k'(\xi) = -kT_n^{(k)}(\nu\xi) \cdot \frac{[\beta_i^{(k)}]^k}{\xi^{k+1}};$$

in fact, the quantity $T_n^{(k)}(\nu\xi) \cdot \nu^k$ must be maximized over ν, i.e.

$$T_n^{(k+1)}(\nu\xi) \cdot \xi\nu^k + T_n^{(k)}(\nu\xi)k\nu^{k-1} = 0,$$

or

$$T_n^{(k+1)}(\nu\xi) \cdot \nu\xi + kT_n^{(k)}(\nu\xi) = 0,$$

whence it follows that $\nu\xi = \text{const}$. Therefore $\nu = \beta_i^{(k)}/\xi$, $A_i^{(k)} = \beta_i^{(k)}/\cos^2(\pi/2n)$ ($\xi = \beta_i^{(k)}$ for $\nu = 1$; $\xi = A_i^{(k)}$ for $\nu = \cos^2(\pi/2n)$); $T_n^{(k)}(\nu\xi) > 0$, $N_k'(\xi) < 0$, and the norm $N_k(\xi)$ decreases in $(\beta_i^{(k)}, A_i^{(k)})$. The monotone character of the norm is proved similarly for the other Čebyšev transformations.

Thus it remains to prove the theorem for the part of $(\beta_i^{(k)}, \alpha_{i+1}^{(k)})$ in which the polynomials $Z_n(x, \vartheta)$ are extremal. Let $A_i^{(k)} < \xi_1 < \xi_{i,k}^*$; then $Z_n(x, \vartheta_{\xi_1})$ serves $F_{\xi_1}^{(k)}$, and

$$N_k(\xi_1) = [Z_n^{(k)}(x, \vartheta_{\xi_1})]_{x=\xi_1},$$
$$[R_n^{(k)}(x, \vartheta_{\xi_1})]_{x=\xi_1} = 0,$$

where $R_n(x, \vartheta_{\xi_1})$ is the resolvent of $Z_n(x, \vartheta_{\xi_1})$ (cf. Theorem 2). We have

$$N_k'(\xi) = \frac{\partial^{k+1} Z_n(\xi, \vartheta)}{\partial \xi^{k+1}} + \frac{\partial^{k+1} Z_n(\xi, \vartheta)}{\partial \xi^k \partial \vartheta} \frac{d\vartheta}{d\xi}.$$

Using the fundamental relation connecting a Zolotarev polynomial and its resolvent,

$$\frac{\partial Z_n(\xi, \vartheta)}{\partial \vartheta} = R_n(\xi, \vartheta)$$

(cf. [**4**], [**6**]), we obtain

$$\frac{\partial^{k+1} Z_n(\xi, \vartheta)}{\partial \xi^k \partial \vartheta} = \frac{\partial^k R_n(\xi, \vartheta)}{\partial \xi^k}.$$

consequently

$$N_k'(\xi_1) = \left(\frac{\partial^{k+1} Z_n(\xi, \vartheta)}{\partial \xi^{k+1}}\right)_{\substack{\xi=\xi_1 \\ \vartheta=\vartheta_{\xi_1}=\vartheta_1}} + \left(\frac{\partial^k R_n(\xi, \vartheta)}{\partial \xi^k} \cdot \frac{d\vartheta}{d\xi}\right)_{\substack{\xi=\xi_1 \\ \vartheta=\vartheta_{\xi_1}=\vartheta_1}} = Z_n^{(k+1)}(\xi_1\vartheta_1).$$

Remark. The norm $N_k(\xi)$ has a continuous derivative $N_k'(\xi)$. In fact,

$$\lim_{\xi\to\beta_i^{(k)}-0} N_k'(\xi) = N_k'(\beta_i^{(k)}-0) = T_n^{(k+1)}(\beta_i^{(k)})$$

and

$$\lim_{\xi\to\beta_i^{(k)}+0} N_k'(\xi) = N_k'(\beta_i^{(k)}+0) = -\frac{k}{\beta_i^{(k)}} T_n^{(k)}(\beta_i^{(k)}).$$

Since

$$A_i^{(k)} = \frac{\beta_i^{(k)}}{\cos^2(\pi/2n)}$$

and

$$Z_n\left(x, 2^{2n-1}\cos^{2n}\frac{\pi}{2n}\right) = T_n\left(x\cos^2\frac{\pi}{2n}\right),$$

then

$$\lim_{\xi\to A_i^{(k)}+0} N_k'(\xi) = N_k'(A_i^{(k)}+0) = \cos^{2(k+1)}\frac{\pi}{2n}\cdot T_n^{(k+1)}(\beta_i^{(k)})$$

and

$$\lim_{\xi\to A_i^{(k)}-0} N_k'(\xi) = N_k'(A_i^{(k)}-0) = -kT_n^{(k)}(\beta_i^{(k)})\frac{\cos^{2(k+1)}(\pi/2n)}{\beta_i^{(k)}}.$$

Using the relation

$$T_n^{(k+1)}(\beta_i^{(k)})\beta_i^{(k)} + kT_n^{(k)}(\beta_i^{(k)}) = 0$$

(cf. Theorem 4), we see that

$$N_k'(\beta_i^{(k)}-0) = N_k'(\beta_i^{(k)}+0)$$
$$N_k'(A_i^{(k)}-0) = N_k'(A_i^{(k)}+0).$$

By the symmetry of the norm (cf. formula (4)) we infer the continuity of $N_k'(\xi)$ at the points $(\alpha_i^{(k)})_{i=2}^{n-k+1}$ and $(1-A_i^{(k)})_{i=1}^{n-k}$. At the remaining points $N_k'(\xi)$ is obviously continuous.

We now find an expression for the second derivative $N_k''(\xi)$ in the interval $(A_i^{(k)}, \xi_{i,k}^*)$. Since

$$N_k'(\xi) = Z_n^{(k+1)}(\xi, \vartheta_\xi),$$

we have

$$N_k''(\xi) = \frac{\partial Z_n^{(k+1)}(\xi,\vartheta_\xi)}{\partial\xi} + \frac{\partial Z_n^{(k+1)}(\xi,\vartheta_\xi)}{\partial\vartheta}\cdot\frac{d\vartheta}{d\xi}$$
$$= Z_n^{(k+2)}(\xi,\vartheta_\xi) + R_n^{(k+1)}(\xi,\vartheta_\xi)\cdot\frac{d\vartheta}{d\xi}.$$

Differentiating (6) with respect to ξ, we obtain

$$R_n^{(k+1)}(\xi,\vartheta)+\frac{\partial R_n^{(k)}(\xi,\vartheta)}{\partial\vartheta}\cdot\frac{d\vartheta}{d\xi}=0,$$

whence we find

$$\frac{d\vartheta}{d\xi}=-\frac{R_n^{(k+1)}(\xi,\vartheta)}{\partial R_n^{(k)}(\xi,\vartheta)/\partial\vartheta}.$$

We now compute $\partial R_n(x,\vartheta)/\partial\vartheta$. Since

$$R_n(x,\vartheta)=\prod_{i=1}^{n}[x-\sigma_i(\vartheta)],$$

we have

$$\frac{\partial R_n(x,\vartheta)}{\partial\vartheta}=-\sum_{i=1}^{n}\frac{d\sigma_i}{d\vartheta}\cdot R_i(x,\vartheta),$$

where

$$R_i(x,\vartheta)=\frac{R_n(x,\vartheta)}{x-\sigma_i}.$$

In addition, the following relation holds between $Z_n(x,\vartheta)$ and its resolvent (cf. [**6**]):

$$n\vartheta(x-\lambda)R_n(x,\vartheta)=x(x-1)Z_n'(x,\vartheta), \tag{7}$$

where λ is the zero of $Z_n'(x,\vartheta)$ outside $[0,1]$.

Differentiating (7) with respect to ϑ, we obtain

$$n(x-\lambda)R_n(x,\vartheta)-n\vartheta R_n(x,\vartheta)\frac{d\lambda}{d\vartheta}+n\vartheta(x-\lambda)\frac{\partial R_n(x,\vartheta)}{\partial\vartheta}$$
$$=x(x-1)\frac{\partial Z_n'(x,\vartheta)}{\partial\vartheta}=x(x-1)R_n'(x,\vartheta)=x(x-1)\sum_{i=1}^{n}R_i(x,\vartheta).$$

Putting $x=\sigma_i$ in the extreme parts of the preceding equation, we find

$$n\vartheta(\sigma_i-\lambda)\frac{\partial R_n(\sigma_i,\vartheta)}{\partial\vartheta}=\sigma_i(\sigma_i-1)R_i(\sigma_i,\vartheta).$$

But

$$\frac{\partial R_n(\sigma_i,\vartheta)}{\partial\vartheta}=-\frac{d\sigma_i}{d\vartheta}\cdot R_i(\sigma_i,\vartheta),$$

hence

This completes the proof of Theorem 4.

$$\frac{d\sigma_i}{d\vartheta} = \frac{\sigma_i(1-\sigma_i)}{n\vartheta(\sigma_i - \lambda)}$$

and

$$\frac{\partial R_n(x,\vartheta)}{\partial\vartheta} = \sum_{i=1}^{n} \frac{\sigma_i(\sigma_i - 1)}{n\vartheta(\sigma_i - \lambda)} R_i(x,\vartheta).$$

Thus

$$\frac{d\vartheta}{d\xi} = \frac{n\vartheta R_n^{(k+1)}(\xi,\vartheta)}{\sum_{i=1}^{n} \frac{\sigma_i(1-\sigma_i)}{\sigma_i - \lambda} R_i^{(k)}(\xi,\vartheta)}.$$

We introduce the notation

$$\phi(x,\vartheta) = \sum_{i=1}^{n} \frac{\sigma_i(\sigma_i - 1)}{\sigma_i - \lambda} R_i(x,\vartheta)$$

and

$$\psi(x,\vartheta) = \sum_{i=1}^{n} \frac{R_i(x,\vartheta)}{\sigma_i - \lambda};$$

then we have

$$\begin{aligned}\phi(x,\vartheta) &= \sum_{i=1}^{n} (\sigma_i - 1 + \lambda) R_i(x,\vartheta) + \lambda(\lambda - 1)\psi(x,\vartheta) \\ &= -\sum_{i=1}^{n} (x - \sigma_i) R_i(x,\vartheta) + (x - 1 + \lambda)\sum_{i=1}^{n} R_i(x,\vartheta) + \lambda(\lambda-1)\psi(x,\vartheta) \qquad (8) \\ &= -nR_n(x,\vartheta) + (x - 1 + \lambda) R_n'(x,\vartheta) + \lambda(\lambda-1)\psi(x,\vartheta).\end{aligned}$$

Put

$$\chi(x,\vartheta) = (x-\lambda)\psi(x,\vartheta);$$

then

$$\chi(\sigma_i,\vartheta) = (\sigma_i - \lambda)\psi(\sigma_i,\vartheta) = R_i(\sigma_i,\vartheta), \quad \chi(\lambda,\vartheta) = 0,$$

whence it is clear that

$$\chi(x,\vartheta) = (x-\lambda)\psi(x,\vartheta) = R_n'(x,\vartheta) - \frac{R_n'(\lambda,\vartheta)}{R_n(\lambda,\vartheta)} R_n(x,\vartheta). \qquad (9)$$

THEOREM 5. *In each Zolotarev interval the norm* $N_k(\xi)$ *has just one minimum, at the point* $\xi = \xi_{0,i}^{(k)}$ *at which* $Z_n^{(k+1)}(\xi,\vartheta_\xi) = 0$, *and*

$$N_k(\xi_{0,i}^{(k)}) = \min_{(\beta_i^{(k)},\alpha_{i+1}^{(k)})} N_k(\xi) \leqq |T_{n-1}^{(k)}(\xi_{i,k}^*)|$$

$$(i = 1, 2, \cdots, n-k;\ k = 1, 2, \cdots, n-1).$$

If $\beta_i^{(k)} > \frac{1}{2}$ *we have* $\beta_i^{(k)} < \xi_{0,i}^{(k)} < \xi_{i,k}^*$; *if* $\alpha_{i+1}^{(k)} < \frac{1}{2}$ *we have* $\xi_{i,k}^* < \xi_{0,i}^{(k)} < \alpha_{i+1}^{(k)}$.

PROOF. Let $Z_n^{(k+1)}(\xi_0, \vartheta_{\xi_0}) = 0$, where $\beta_i^{(k)} < \xi_0 < \xi_{i,k}^*$. We differentiate (7) $k+1$ times with respect to x, and then put $x = \xi_0$ and $\vartheta = \vartheta_{\xi_0} = \vartheta_0$. Taking into account that $R_n^{(k)}(\xi_0, \vartheta_0) = 0$ and $Z_n^{(k+1)}(\xi_0, \vartheta_0) = 0$, we obtain

$$(10) \quad n\vartheta_0(\xi_0 - \lambda) R_n^{(k+1)}(\xi_0, \vartheta_0) = \xi_0(\xi_0 - 1) Z_n^{(k)}(\xi_0, \vartheta_0) + (k+1) k Z_n^{(k)}(\xi_0, \vartheta_0).$$

Now differentiate (8) and (9) k times with respect to x, and then put $x = \xi_0$ and $\vartheta = \vartheta_0$; we obtain

$$\chi^{(k)}(\xi_0, \vartheta_0) = (\xi_0 - \lambda)\psi^{(k)}(\xi_0, \vartheta_0) + k\psi^{(k-1)}(\xi_0, \vartheta_0) = R_n^{(k-1)}(\xi_0, \vartheta_0),$$

whence

$$\psi^{(k)}(\xi_0, \vartheta_0) = \frac{R_n^{(k+1)}(\xi_0, \vartheta_0) - k\psi^{(k-1)}(\xi_0, \vartheta_0)}{\xi_0 - \lambda}$$

and therefore

$$(11) \quad \begin{aligned} \phi^{(k)}(\xi_0, \vartheta_0) &= (\xi_0 - 1 + \lambda) R_n^{(k+1)}(\xi_0, \vartheta_0) \\ &\quad + \lambda(\lambda - 1) \frac{R_n^{(k+1)}(\xi_0, \vartheta_0) - k\psi^{(k-1)}(\xi_0, \vartheta_0)}{\xi_0 - \lambda} \\ &= \frac{\xi_0(\xi_0 - 1)}{\xi_0 - \lambda} R_n^{(k+1)}(\xi_0, \vartheta_0) - \frac{k\lambda(\lambda - 1)}{\xi_0 - \lambda} \psi^{(k-1)}(\xi_0, \vartheta_0). \end{aligned}$$

Since

$$\begin{aligned} N_k''(\xi_0) &= Z_n^{(k+2)}(\xi_0, \vartheta_0) + R_n^{(k+1)}(\xi_0, \vartheta_0) \left(\frac{d\vartheta}{d\xi}\right)_{\xi = \xi_0} \\ &= Z_n^{(k+2)}(\xi_0, \vartheta_0) - \frac{n\vartheta_0 R_n^{(k+1)}(\xi_0, \vartheta_0)}{\dfrac{\phi^{(k)}(\xi_0, \vartheta_0)}{R_n^{(k+1)}(\xi_0, \vartheta_0)}}, \end{aligned}$$

we find, by using (10) and (11), that

$$(12) \quad N_k''(\xi_0) = \frac{kZ_n^{(k)}(\xi_0, \vartheta_0)}{\lambda - \xi_0} \, \frac{k + 1 + \lambda(\lambda - 1) \dfrac{\psi^{(k-1)}(\xi_0, \vartheta_0)}{R_n^{(k+1)}(\xi_0, \vartheta_0)} \cdot \dfrac{Z_n^{(k+2)}(\xi_0, \vartheta_0)}{Z_n^{(k)}(\xi_0, \vartheta_0)}}{\dfrac{\xi_0(\xi_0 - 1)}{\xi_0 - \lambda} - \dfrac{k\lambda(\lambda - 1)}{\xi_0 - \lambda} \cdot \dfrac{\psi^{(k-1)}(\xi_0, \vartheta_0)}{R_n^{(k+1)}(\xi_0, \vartheta_0)}}.$$

This is similar to V. A. Markov's formula (118) [1]. Now we have

$$\frac{\xi_0(\xi_0 - 1)}{\xi_0 - \lambda} - \frac{k\lambda(\lambda - 1)}{\xi_0 - \lambda} \cdot \frac{\psi^{(k-1)}(\xi_0, \vartheta_0)}{R_n^{(k+1)}(\xi_0, \vartheta_0)} = \frac{\phi^{(k)}(\xi_0, \vartheta_0)}{R_n^{(k+1)}(\xi_0, \vartheta_0)} = -\frac{n\vartheta_0}{(d\vartheta/d\xi)_{\xi = \xi_0}} > 0,$$

since as ξ increases from $A_i^{(k)}$ to $\xi_{i,k}^*$ the parameter ϑ decreases monotonically and $d\vartheta/d\xi < 0$; in addition,

$$\frac{Z_n^{(k+2)}(\xi_0, \vartheta_0)}{Z_n^{(k)}(\xi_0, \vartheta_0)} < 0,$$

by Markov's Lemma 1, and

$$\frac{\psi^{(k-1)}(\xi_0, \vartheta_0)}{R_n^{(k+1)}(\xi_0, \vartheta_0)} < 0,$$

by Markov's Lemma 3. Thus $N_k''(\xi_0)$ and $Z_n^{(k)}(\xi_0, \vartheta_0) = N_k(\xi_0)$ have the same sign. Consequently if we have

$$Z_n^{(k+1)}(\xi_0, \vartheta_0) = 0,$$

then

$$N_k(\xi_0) = \min_{(\beta_i^{(k)}, \alpha_{i+1}^{(k)})} N_k(\xi),$$

and there can be no more than one such point ξ_0 in each Zolotarev interval. We now show that there exists one such point in each $(\beta_i^{(k)}, \alpha_{i+1}^{(k)})$. We have

$$(n-1)2^{2n-3}R_n(x) = x(x-1)T_{n-1}'(x),$$

where $R_n(x)$ is the resultant of $T_{n-1}(x)$. Differentiating the preceding equation k times, we obtain

$$(n-1)2^{2n-3}R_n^{(k)}(x) = x(x-1)T_{n-1}^{(k+1)}(x) + k(2x-1)T_{n-1}^{(k)}(x) \\ + k(k+1 T_{n-1}^{(k-1)}(x).$$

In addition,

$$x(1-x)T_{n-1}^{(k+1)}(x) - (2k-1)(x-\tfrac{1}{2})T_{n-1}^{(k)}(x) \\ + (\overline{n-1}^2 - \overline{k-1}^2)T_{n-1}^{(k-1)}(x) = 0.$$

In the last two equations put $x = \xi_{i,k}^*$, remember that $R_n^{(k)}(\xi_{i,k}^*) = 0$, and then eliminate $T_{n-1}^{(k-1)}(\xi_{i,k}^*)$ to obtain

$$\xi_{i,k}^*(1-\xi_{i,k}^*)[\overline{n-1}^2 - \overline{k-1}^2 + k(k+1)]T_{n-1}^{(k+1)}(\xi_{i,k}^*) \\ = k(2\xi_{i,k}^* - 1)[\overline{n-1}^2 - \overline{k-1}^2 + (k-\tfrac{1}{2})(k+1)]T_{n-1}^{(k)}(\xi_{i,k}^*).$$

If $\xi_{i,k}^* > \frac{1}{2}$ in this equation, i.e. $\beta_i^{(k)} > \frac{1}{2}$, we have

$$\operatorname{sgn} T_{n-1}^{(k+1)}(\xi_{i,k}^*) = \operatorname{sgn} T_{n-1}^{(k)}(\xi_{i,k}^*)$$

and $N_k'(\xi_{i,k}^*) > 0$. On the other hand,

$$N_k'(A_i^{(k)}) = -k|T_n^{(k)}(\beta_i^{(k)})|\frac{\cos^{2(k+1)}(\pi/2n)}{\beta_i^{(k)}} < 0.$$

Consequently there is a unique point $\xi_{0,i}^{(k)}$ in the interval $(A_i^{(k)}, \xi_{i,k}^*)$ such that

$$N_k'(\xi_{0,i}^{(k)}) = 0.$$

By (4), this point $\xi_{0,i}^{(k)}$ lies in the interval $(\xi_{i,k}^*, \alpha_{i+1}^{(k)})$ if $\alpha_{i+1}^{(k)} < \frac{1}{2}$. This completes the proof of Theorem 5.

REMARK 1. If $k + 1 \equiv n \pmod 2$ there is a Zolotarev interval

$$(\beta_{(n-k+1)/2}^{(k)}, 1 - \beta_{(n-k+1)/2}^{(k)})$$

containing the point

$$\xi = \xi_{0,(n-k+1)/2}^{(k)} = \xi_{(n-k+1)/2,k}^* = \tfrac{1}{2},$$

and

$$N_k(\tfrac{1}{2}) = |\, T_{n-1}^{(k)}(\tfrac{1}{2})\,|$$
$$= \begin{cases} 2^k(n-1)(\overline{n-1^2} - 1^2) \cdots (\overline{n-1^2} - \overline{k-2^2}) & \text{for } n \text{ even}, \\ 2^k(n-1)^2(\overline{n-1^2} - 2^2) \cdots (\overline{n-1^2} - \overline{k-2^2} & \text{for } n \text{ odd}. \end{cases}$$

REMARK 2. Since $T_n(x) = \cos n\theta$, where $\theta = \arccos(2x-1)$, we have

$$T_n'(x) = -n \sin n\theta \frac{d\theta}{dx} = \frac{2n \sin n\theta}{\sin\theta} = 4n[\cos(n-1)\theta + \cos(n-3)\theta + \cdots]$$

and

$$T_n^{(k)}(x) = \sum_{i=0}^{n-k} a_{k,i} \cos i\theta,$$

with $a_{k,i} \geqq 0$ (cf. [2]). It follows that

$$\max_{[0,1]} N_k(\xi) = T_n^{(k)}(1),$$

and we obtain V. A. Markov's inequality (2′).

THEOREM 6. *The discontinuities of the second derivative $N_k''(\xi)$ of the norm are*

$$(\alpha_i^{(k)})_{i=2}^{n-k+1}, \quad (\beta_i^{(k)})_{i=1}^{n-k}, \quad (A_i^{(k)})_{i=1}^{n-k} \quad \textit{and} \quad (1 - A_i^{(k)})_{i=1}^{n-k}$$

PROOF. We have

$$N_k''(\beta_i^{(k)} - 0) = T_n^{(k+2)}(\beta_i^{(k)}),$$
$$N_k''(\beta_i^{(k)} + 0) = \frac{k(k+1)}{[\beta_i^{(k)}]^2} T_n^{(k)}(\beta_i^{(k)}) = -\frac{k+1}{\beta_i^{(k)}} T_n^{(k+1)}(\beta_i^{(k)})$$

and consequently

$$N_k''(\beta_i^{(k)}-0)-N_k''(\beta_i^{(k)}+0)=T_n^{(k+2)}(\beta_i^{(k)})+\frac{k+1}{\beta_i^{(k)}}\,T_n^{(k+1)}(\beta_i^{(k)})$$
$$=\frac{n2^{2n-1}}{\beta_i^{(k)}}\cdot\Phi_n^{(k+1)}(\beta_i^{(k)})\neq 0,$$
$$(n2^{2n-1}\Phi_n(x)=xT_n'(x)).$$

We have $\Phi_n^{(k+1)}(\beta_i^{(k)})>0$ for $i=n-k,\ n-k-2,\ n-k-4,\cdots$, and $\Phi_n^{(k+1)}(\beta_i^{(k)})<0$ for $i=n-k-1, n-k-3,\cdots$. Hence if we use the fact that the extremal polynomial is $+T_n(x)$ in the intervals $[\alpha_{n-k+1}^{(k)},1]$, $[\alpha_{n-k-1}^{(k)},\beta_{n-k-1}^{(k)}],\cdots$, and $-T_n(x)$ in $[\alpha_{n-k}^{(k)},\beta_{n-k}^{(k)}],[\alpha_{n-k-2}^{(k)},\beta_{n-k-2}^{(k)}],\cdots$ (cf. Theorem 1), we have

$$N_k''(\beta_i^{(k)}+0)>N_k''(\beta_i^{(k)}-0).$$

Using formula (4), we find that at the points $(\alpha_i^{(k)})_{i=2}^{n-k+1}$

$$N_k''(\alpha_i^{(k)}+0)<N_k''(\alpha_i^{(k)}-0).$$

In addition,

$$N_k''(A_i^{(k)}-0)=\frac{k(k+1)}{[\beta_i^{(k)}]^2}\,T_n^{(k)}(\beta_i^{(k)})\cos^{2(k+2)}\frac{\pi}{2n}$$
$$=-\frac{k+1}{\beta_i^{(k)}}\,T_n^{(k+1)}(\beta_i^{(k)})\cos^{2(k+2)}\frac{\pi}{2n}$$

and

$$N_k''(A_i^{(k)}+0)=\left[Z_n^{(k+2)}(\xi,\vartheta)+R_n^{(k+1)}(\xi,\vartheta)\cdot\frac{d\vartheta}{d\xi}\right]_{\substack{\xi=A_i^{(k)}\\ \vartheta=2^{2n-1}\cos^{2n}(\pi/2n)}}$$
$$=\cos^{2(k+2)}\frac{\pi}{2n}\cdot T_n^{(k+2)}(\beta_i^{(k)})+\frac{\Phi_n^{(k+1)}(\beta_i^{(k)})}{\cos^{2(n-k-1)}(\pi/2n)}\cdot\left(\frac{d\vartheta}{d\xi}\right)_{\xi=A_i^{(k)}},$$

since $A_i^{(k)}=\beta_i^{(k)}/\cos^2(\pi/2n)$ and

$$R_n\left(x,2^{2n-1}\cos^{2n}\frac{\pi}{2n}\right)=\prod_{i=1}^{n}\left(x-\frac{\tau_{i-1}}{\cos^2(\pi/2n)}\right)=\frac{\Phi_n(x\cos^2(\pi/2n))}{\cos^{2n}(\pi/2n)}$$

is the resolvent of the polynomial

$$Z_n\left(x,2^{2n-1}\cos^{2n}\frac{\pi}{2n}\right)=T_n\left(x\cos^2\frac{\pi}{2n}\right).$$

Consequently

$$N_k''(A_i^{(k)}+0)-N_k''(A_i^{(k)}-0)$$
$$=\cos^{2(k+2)}\frac{\pi}{2n}\cdot\Phi_n^{(k+1)}(\beta_i^{(k)})\left[\frac{n2^{2n-1}}{\beta_i^{(k)}}+\frac{(d\vartheta/d\xi)_{\xi=A_i^{(k)}}}{\cos^{2(n+1)}(\pi/2n)}\right],\tag{13}$$

where

$$\left(\frac{d\vartheta}{d\xi}\right)_{\xi=A_i^{(k)}} = -n\left[\frac{\vartheta R_n^{(k+1)}(\xi,\vartheta)}{\phi^{(k)}(\xi,\vartheta)}\right]_{\substack{\xi=A_i^{(k)} \\ \vartheta=2^{2n-1}\cos^{2n}(\pi/2n)}}$$
$$= -\frac{n2^{2n-1}\cos^{2(k+1)}(\pi/2n)\cdot\Phi_n^{(k+1)}(\beta_i^{(k)})}{\phi^{(k)}(A_i^{(k)},2^{2n-1}\cos^{2n}(\pi/2n))}.$$

Let us calculate $\phi^{(k)}(A_i^{(k)},2^{2n-1}\cos^{2n}(\pi/2n))$. Since

$$\phi\left(x,2^{2n-1}\cos^{2n}\frac{\pi}{2n}\right)$$
$$= -\frac{n\Phi_n\left(x\cos^2\frac{\pi}{2n}\right)}{\cos^{2n}\frac{\pi}{2n}} + \frac{x\Phi_n'\left(x\cos^2\frac{\pi}{2n}\right)}{\cos^{2(n-1)}\frac{\pi}{2n}} - \frac{R_n\left(x,2^{2n-1}\cos^{2n}\frac{\pi}{2n}\right)}{x-1}$$

(cf. formula (8)), if we differentiate k times and put $x = A_i^{(k)}$ we obtain

$$\phi^{(k)}\left(A_i^{(k)},2^{2n-1}\cos^{2n}\frac{\pi}{2n}\right) = \frac{\beta_i^{(k)}\Phi_n^{(k+1)}(\beta_i^{(k)})}{\cos^{2(n-k-1)}(\pi/2n)} - \Delta_n^{(k)}(A_i^{(k)}), \tag{14}$$

where

$$\Delta_n^{(k)}(A_i^{(k)}) = \left\{\frac{d}{dx^k}\left[\frac{R_n(x,2^{2n-1}\cos^{2n}(\pi/2n))}{x-1}\right]\right\}_{x=A_i^{(k)}}.$$

We now substitute the formula for $(d\vartheta/d\xi)_{\xi=A_i^{(k)}}$ into (13); then, using (14), we find after some calculation

$$N_k''(A_i^{(k)}+0) - N_k''(A_i^{(k)}-0)$$
$$= \left[\frac{\sin^2\frac{\pi}{2n}}{\cos^{2(n-k-1)}\frac{\pi}{2n}}\cdot\Phi_n^{(k+1)}(\beta_i^{(k)}) + \frac{\cos^2\frac{\pi}{2n}}{\beta_i^{(k)}}\cdot\Delta_n^{(k)}(A_i^{(k)})\right]\left(\frac{d\vartheta}{d\xi}\right)_{\xi=A_i^{(k)}} \neq 0,$$

since $\Phi_n^{(k+1)}(\beta_i^{(k)})$ and $\Delta_n^{(k)}(A_i^{(k)})$ have the same sign by Markov's Lemma 2.
Just as for the points $(\beta_i^{(k)})_{i=1}^{n-k}$ and $(\alpha_i^{(k)})_{i=2}^{n-k+1}$, we see that

$$N_k''(A_i^{(k)}+0) > N_k''(A_i^{(k)}-0)$$

and

$$N_k''(1-A_i^{(k)}+0) < N_k''(1-A_i^{(k)}-0).$$

At the remaining points $N_k''(\xi)$ is clearly continuous.

Theorem 7. *The sum of the lengths of the Čebyšev intervals, or the measure of the set E_T, is k/n.*

PROOF. We have

$$\operatorname{mes} E_T = \sum_{i=1}^{n-k+1} (\beta_i^{(k)} - \alpha_i^{(k)}) = 1 + \sum_{i=1}^{n-k} \beta_i^{(k)} - \sum_{i=2}^{n-k+1} \alpha_i^{(k)}.$$

In addition,

$$(15)\qquad \begin{aligned} \Phi_0^{(k)}(x) &= A_k\left(x^{n-k} - \sum_{i=2}^{n-k+1} \alpha_i^{(k)} \cdot x^{n-k-1} + \cdots\right), \\ \Phi_n^{(k)}(x) &= B_k\left(x^{n-k} - \sum_{i=1}^{n-k} \beta_i^{(k)} \cdot x^{n-k-1} + \cdots\right). \end{aligned}$$

On the other hand, since

$$\Phi_j(x) = x^n - \left(\frac{n+1}{2} - \tau_j\right) x^{n-1} + \cdots,$$

we have

$$(16)\qquad \begin{aligned} \Phi_0^{(k)}(x) &= \frac{n!}{(n-k)!} x^{n-k} - \frac{n+1}{2} \frac{(n-1)!}{(n-k-1)!} x^{n-k-1} + \cdots, \\ \Phi_n^{(k)}(x) &= \frac{n!}{(n-k)!} x^{n-k} - \frac{n-1}{2} \frac{(n-1)!}{(n-k-1)!} x^{n-k-1} + \cdots. \end{aligned}$$

Comparing the expansions (15) and (16), we find that

$$\sum_{i=2}^{n-k+1} \alpha_i^{(k)} = \frac{n+1}{2}\left(1 - \frac{k}{n}\right), \quad \sum_{i=1}^{n-k} \beta_i^{(k)} = \frac{n-1}{2}\left(1 - \frac{k}{n}\right),$$

whence we obtain

$$\operatorname{mes} E_T = k/n.$$

It follows from Theorem 7 that as $n \to \infty$ with k fixed, the Čebyšev intervals contract to points, i.e. the Zolotarev polynomials displace the polynomials $T_n(x)$ on $[0, 1]$. Conversely, if $n - k$ is fixed and $n \to \infty$, the polynomials $T_n(x)$ displace the Zolotarev polynomials.

In conclusion we compare the exact majorant $N_k(\xi)$ with Bernšteĭn's majorant [**9**]:

$$(17)\qquad |P_n^{(k)}(\xi)| < \left[\frac{k}{\xi(1-\xi)}\right]^{k/2} \cdot n(n-1)\cdots(n-k+1) = B_k(\xi)$$

and Duffin and Schaeffer's majorant [**3**]:

$$(18)\qquad |P_n^{(k)}(\xi)| \leqq |T_n^{(k)}(\xi) + iS_n^{(k)}(\xi)| \quad (S_n(x) = \sin n \arccos(2x-1))$$

(we have written both inequalities for reduced polynomials on $[0, 1]$; for $k = 1$ they are the same).

For $k = 2, 3, \cdots, n$ the majorant (17) never equals the exact majorant $N_k(\xi)$ (cf. [9], p. 27).

For large n, (17) was sharpened by Bernšteĭn [10], who obtained

$$N_k(\xi) \sim \frac{n^k}{[\xi(1-\xi)]^{k/2}} \text{ for } \xi \in [0,1];$$

it follows from this that

$$\lim_{n\to\infty} \frac{B_k(\xi)}{N_k(\xi)} = k^{k/2} \text{ for } \xi \in [0,1].$$

The best approximation to $N_k(\xi)$ is given by (18), which is tangent to $N_k(\xi)$ in each Čebyšev interval at $\xi = \vartheta_i^{(k)}$ $(k = 1, 2, \cdots, n;\ i = 1, 2, \cdots, n-k+1)$, where $(\vartheta_i^{(k)})_{i=1}^{n-k+1}$ are the roots of $S_n^{(k)}(x) = 0$.

For example, if $k+1 \equiv n \pmod 2$, we have for fixed k and large n

$$|T_n^{(k)}(\tfrac{1}{2}) + iS_n^{(k)}(\tfrac{1}{2})| - N_k(\tfrac{1}{2}) \sim k2^k n^{k-1}$$

and

$$\frac{|T_n^{(k)}(\tfrac{1}{2}) + iS_n^{(k)}(\tfrac{1}{2})|}{N_k(\tfrac{1}{2})} \sim 1.$$

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