Differential Equations of the Second Order with Retarded Argument

by

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DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH RETARDED ARGUMENT. SOME PROBLEMS OF THE THEORY OF VIBRATIONS OF SYSTEMS WITH RETARDATION

by
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ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ ВТОРОГО ПОРЯДКА С ЗАПАЗДЫВАЮЩИМ АРГУМЕНТОМ

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PREFACE

The theory of differential equations with deviating arguments is a relatively new and rapidly developing branch of the theory of ordinary differential equations. Numerous research papers have been devoted to this theory. There are also a few monographs which in part or entirely are concerned with various aspects of differential equations with deviating arguments; these monographs are by A. D. Myškis [4], L. È. Èl′sgol′c [3], [5], N. N. Krasovskiĭ [1], E. Pinney [1], and R. Bellman and K. Cooke [1].

Among these monographs the one most closely related to the present treatise is that of Myškis [4].

However, as opposed to Myškis' book, where the entire exposition is devoted to the very general case of a "distributed" delay, we in the present book are only concerned with the simpler case of equations with a "concentrated" delay. The results presented here may most of the time be generalized to equations depending on a distributed delay; nevertheless, in order to facilitate reading of the book, we have found it advisable to limit ourselves to the case of a concentrated delay.

For ease of reading, proofs are given nearly everywhere. If a certain result already has appeared in a monograph, a reference is given. For the reader of this book a knowledge of mathematical analysis and elementary ordinary differential equations is required. Only in special places do we use elementary facts of the theory of functions of a real variable and functional analysis.

The author hopes that the present book will be of interest both for mathematicians working in the theory of ordinary differential equations and also for a significantly wider circle of readers, physicists and research engineers dealing with systems with retardations.

The author takes this opportunity to express his deep gratitude to A. D. Myškis and L. È. Èl′sgol′c for their interest and their many valuable comments about the author's work on which this book is based.

In this book we present and discuss several results presented in the seminar on differential equations with deviating arguments conducted by L. È. Èl′sgol′c. The author wishes to thank his many colleagues in the seminar, in particular A. M. Zverkin, G. A. Kamenskiĭ, and A. B. Nersesjan for several discussions and many valuable remarks. The author is grateful to the editor of the book, I. A. Ožiganov, for a number of remarks which have improved the exposition.

The Author

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NUMERICAL INTEGRATION METHODS FOR DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENT

§1. Numerical integration methods for differential equations with deviating arguments

Differential equations with deviating arguments can be integrated in closed form only in a few exceptional cases. The step method is applicable for direct computation only under the hypotheses that the number of steps on the whole interval on which the solution is to be computed is not too large, not to mention the fact that the corresponding differential equation without delay obtained on each interval must also be integrable in closed form. Hence numerical methods of integration of equations with deviating argument are very important.

In this section we give a short survey of the work devoted to numerical integration methods for differential equations with deviating arguments, not limited to the types of equations considered in the basic text.

1. Euler's method and parabolic methods. Parabolic methods are methods for numerical integration of differential equations, based on approximation of the unknown function by a parabolic arc of order $n \geqq 2$. The Adams-Stömer methods, Milne's method, and so forth belong to this class. A qualitative basis for applications of Euler's method and parabolic methods for numerical integration of differential equations with deviating arguments was considered by L. È. Èl′sgol′c (see his books [**3**], [**5**]).

Calculations by means of these methods are carried out by the same schemes as for equations without deviating argument; however, since it is necessary to compute, along with $x(t_n)$, the values $x(t_n - \Delta(t_n))$ in case of a variable delay, it is required to use variable steps or interpolation.

The application of parabolic methods to equations with deviating argument is complicated by the fact that, with these methods, functions can be well approximated only if they are differentiable a sufficient number of times. Hence, parabolic methods give good results only beginning with those values of the argument for which the solution turns out to be already sufficiently smooth (in general, solutions of differential equations with retarded argument become smoother with increasing t, while for solutions of equations of neutral type the smoothing is absent, and for equations with advanced argument smoothness is lost). We note that since terms involving $x'(t-\Delta(t))$ are absent, the equations considered in the text have solutions which are continuous together with their first and second derivatives on all regions of variation of the independent variable t.

An attempt to improve parabolic methods has been made in the articles of T. S. Zverkina [1], [2]. Suppose that a piecewise smooth function $f(t)$ defined on $[A,B]$ has, on its intervals of smoothness, derivatives up to the nth order, while at the points t_k $(A<t_1<t_2<\cdots<t_m<B)$ the derivatives have jumps

$$\delta_k^r = f^{(r)}(t_k+0)-f^{(r)}(t_k-0) \quad (r=1,2,\cdots,n;\ k=1,2,\cdots,m).$$

Then the generalized Taylor's formula is deduced:

$$f(t)=\sum_{r=0}^{n}\frac{f^{(r)}(A)}{r!}(t-A)^r+\sum\nolimits_{k}\sum_{\substack{r=1\\ t_k<t}}^{n}\frac{\delta_k^r}{r!}(t-t_k)^r+R_n(t), \tag{1}$$

in which the summation on k is over those k for which $t_k<t$. An estimate is obtained for the remainder term, analogous to the estimate in the ordinary Taylor's formula. By means of formula (1), formulas are constructed for integration of differential equations, generalizing the well-known formulas of Adams, Milne, and so on.

The articles of B. M. Budak and A. D. Gorbunov [1] and N. V. Šarkova [1]-[3] also belong to the realm of questions under consideration.

2. Expansion in powers of the delay. In applied work, in order to approximate solutions, and sometimes also to investigate stability of solutions of differential equations with retarded argument with small delay, the method of expansion in powers of the retardation is widely applied. In connection with the equation

$$x'(t)=F(t,x(t),x(t-\tau)) \quad (A\leqq t<B\leqq\infty,\ x(t)\equiv\phi(t) \text{ on } E_A) \tag{2}$$

this method is based on the fact that equation (2) is replaced by the equation

$$x'(t)=F\left(t,x(t),x(t)-\tau x'(t)+\cdots+\frac{(-1)^m\tau^m}{m!}x^{(m)}(t)\right). \tag{3}$$

Usually such a substitution is carried out without any justification of its admissibility. In the article of Ja. Z. Cypkin [**1**], it is, for the first time, shown by examples that this method may lead to invalid results. Conditions for applicability of this method were investigated by L. È. Èls′gol′c (see his monograph [**3**]) by considering the theory of differential equations with small parameters in the highest derivative. After solving equation (3) for the derivative $x^{(m)}(t)$, one finds that equation (2) is contained in equation (3) provided one discards the term

$$\frac{\tau}{m+1} x^{(m+1)}(t - \theta\tau).$$

The theory of differential equations with small parameter in the highest derivative leads to the assertion that for $m > 1$ this term has order $1/\tau$, i.e. the discarded term for small τ is very large. For $m = 1$, the discarded term actually turns out to be small and the method under consideration gives good results.

We note that the method of expansion in powers of the small parameter until now has been recommended without any reservations in many quite serious texts. We mention, for example, the recently published book of W. J. Cunningham [**1**].

3. Asymptotic methods. We consider solutions of the equation

$$x'(t) = f(t, x(t), x(t-\tau)) \quad (\tau \leqq t < \infty) \tag{4}$$

with initial condition

$$x(t) \equiv \phi(t) \quad (0 \leqq t \leqq \tau). \tag{5}$$

In the article of A. D. Myškis [**1**] it was shown that if $\tau = 0$ in (4) then the solution of the equation without retardation

$$x'(t) = f(t, x(t), x(t)) \quad (0 \leqq t < \infty)$$

with initial condition $x(0) = \phi(0)$, under sufficiently general assumptions, at the endpoint of the interval of variation of t will be near to the solution of the initial value problem (4), (5) if τ is sufficiently small, and hence may be considered as the zero term of the asymptotic expansion for the latter solution.

The method of expansion in powers of the retardation considered in the preceding subsection yields an asymptotic formula of first order (in τ), and only in isolated cases of second order.

We shall find a method of obtaining an asymptotic formula of arbitrary order for solutions of the initial value problem (4), (5), the idea of which is due to A. B. Vasil′eva and A. M. Rodionov.

We denote the right side of (4) by $f(t,x,y)$, and, assuming that the solution of the initial value problem is sufficiently smooth (under evident assumptions on f and ϕ the derivative $x^{(k)}(t)$ will be continuous in t and uniformly bounded with respect to τ ($\tau \leqq \tau_0$) on the segment $k\tau \leqq t \leqq T$), we expand the right side of (4) in powers of τ:

$$\begin{aligned} x'(t) &= f\left(t, x(t), x(t) - \tau x'(t) + \frac{\tau^2}{2} x''(t) - \cdots\right) \\ &= f(t,x(t),x(t)) - \tau \frac{\partial}{\partial y} f(t,x(t),x(t)) \cdot x'(t) \\ &+ \frac{\tau^2}{2} \cdot \frac{\partial^2}{\partial y^2} f(t,x(t),x(t)) \cdot x'^2(t) \\ &+ \frac{\tau^2}{2} \cdot \frac{\partial}{\partial y} f(t,x(t),x(t)) \cdot x''(t) + \cdots, \end{aligned}$$

and we shall seek a formal solution of this equation in the form of an expansion in powers of τ:

$$x(t) = x_0(t) + \tau x_1(t) + \frac{\tau^2}{2} x_2(t) + \cdots$$

by equating coefficients of corresponding powers of τ.

It is essential that for the definition of $x_i(t)$ ($i \geqq 1$) we obtain a first order linear equation without delay. For obtaining a kth order asymptotic approximation (accuracy of τ^{k+1}) the initial values for $x_i(t)$ are given at the point $t_0 = (k+2)\tau$ (this guarantees the necessary smoothness of the solution). For this purpose, on the interval $[\tau, (k+2)\tau]$ one must calculate the solution of the problem (4), (5), for example, by the method of steps.

It has been shown that a solution $x(t)$ of equation (4), satisfying the initial conditions (5), has the asymptotic expansion

$$x(t) = x_0(t) + \tau x_1(t) + \cdots + \frac{\tau^k}{k!} x_k(t) + O(\tau^{k+1}),$$

uniformly in t and τ for $\tau(k+2) \leqq t \leqq T$, $\tau \leqq \tau_0$.

This method is explained in the articles of A. B. Vasil′eva and A. M. Rodionov [**1**] and A. B. Vasil′eva [**1**]. A. B. Vasil′eva in [**1**–**3**] generalizes this method to the case of equations of neutral type

$$x'(t) = f(t, x(t), x(t-\tau), x'(t-\tau)).$$

For this see also the article of V. I. Rožkov [**1**].

Similar problems are studied in the articles of Ju. A. Rjabov [**1**–**4**]. These articles are of interest since they not only treat asymptotic decompositions, but also prove convergence of the series obtained for $\tau \leqq \tau_0$ and obtain a method of bounding τ_0 and the remainder term of the series.

4. Iterative methods. We shall find first of all a method of successive approximations of Picard type. In connection with the differential equation with retarded argument

$$x'(t) = f(t, x(t), x(t - \Delta(t))), \qquad x(t) \equiv \phi(t) \text{ on } E_A, \tag{6}$$

or with the equivalent integral equation

$$x(t) = \phi(A) + \int_A^t f(\tau, x(\tau), x(\tau - \Delta(\tau)))\, d\tau$$

with the same initial conditions, the method of successive approximations implies that, starting from an arbitrary continuous function $x(t) = x_0(t)$ which satisfies the initial condition, one constructs the sequence of approximations

$$x_n(t) = \phi(A) + \int_A^t f(\tau, x_{n-1}(\tau), x_{n-1}(\tau - \Delta(\tau)))\, d\tau \quad (n = 1, 2, \cdots).$$

If f and ϕ are continuous and f satisfies a Lipschitz condition in its second and third arguments, then the sequence of approximations $\{x_n(t)\}$ converges uniformly to the unique solution of equation (6) which satisfies these conditions.

As also for equations without retardation, the method of successive approximations may be applied as an independent numerical method. Often it is combined with some method of interpolation.

The articles of È. I. Kljamko [**1**] and G. M. Ždanov [**1**] consider hypotheses for the applicability of the Čaplygin method to approximate a solution of an equation with retarded argument. G. M. Ždanov [**1**] considers the system of differential delay equations

$$x_i'(t) = f_i(t, x_1(t), \cdots, x_n(t), x_1(t - \Delta_1(t)), \cdots, x_n(t - \Delta_n(t)))$$

$(i = 1, 2, \cdots, n;\ t_0 \leqq t < t_1;\ \Delta_i(t) \geqq 0)$, where the right sides are continuous in all arguments and have continuous nonnegative partial derivatives with respect to all arguments beginning with the second. The fundamental initial value problem is considered with initial function continuous on the initial set. This article gives an algorithm for the construction of two-sided approximations.

A. D. Myškis [6] showed that the results of Ždanov, under certain hypotheses, are valid for a significantly broader class of equations, including an arbitrary number of concentrated and distributed delays, which in turn may depend on the unknown functions, and so on.

In the following sections we shall set forth one more iterative method, the method of moments, and we shall find conditions under which this method may be employed to compute eigenvalues and eigenfunctions of some of the boundary-value problems considered in this text.

We note that for solutions of boundary-value problems with delay it is also possible to apply other methods. For example, in the articles of S. S. Gaĭsarjan [1,2] a foundation is laid for application of Galerkin's method for solution of boundary-value problems for differential delay equations.

§2. The method of moments

The monograph of Ju. V. Vorob′ev [1] gives an account of the theory of the method of moments as applied to the approximate computation of eigenvalues and eigenfunctions of a nonselfadjoint completely continuous linear operator and solutions of nonhomogeneous operator equations.

Let A be a bounded linear operator defined on a Hilbert space H, and let z_0 be an arbitrary element of H. We construct the series of iterations

$$z_0, z_1 = Az_0, z_2 = A^2z_0, \cdots, z_n = A^n z_0, \cdots.$$

The problem of moments arises in the following way: it is required to construct a sequence of operators A_n, determined on H_n, the linear hull of the elements $z_0, z_1, \cdots, z_{n-1}$, such that

$$\begin{aligned} z_k &= A_n^k z_0 \quad (k = 0, 1, \cdots, n-1) \\ \bar{z}_n &= A_n^n z_0 \end{aligned} \tag{7}$$

where $\bar{z}_n$ is the projection of z_n into H_n. As was shown in the monograph of Vorob′ev, the relation (7) completely determines the sequence of operators A_n, i.e. it gives the solution of the problem of moments.

The eigenvalues of the operator A_n are the roots of the equation

$$P_n(\lambda) \equiv \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n = 0, \tag{8}$$

whose coefficients are defined by the system of linear equations

$$a_{i0}\alpha_n + a_{i1}\alpha_{n-1} + \cdots + a_{in-1}\alpha_1 + a_{in} = 0 \quad (i = 0, 1, \cdots, n-1), \tag{9}$$

where $a_{ik} = (z_i, z_k)$ $(i, k = 0, 1, \cdots, n-1)$, and $a_{in} = (z_i, \bar{z}_n) = (z_i, z_n)$.[1)]

Now let u_k be an eigenvector of the operator A_n corresponding to the eigenvalue λ_k. Since $u_k \in H_n$, we have $u_k = \xi_0 z_0 + \xi_1 z_1 + \cdots + \xi_{n-1} z_{n-1}$. The coefficients ξ_m are defined by the system of equations

$$- \alpha_0 \xi_{n-1} = \lambda_k \xi_0,$$
$$\xi_{i-1} - \alpha_i \xi_{n-1} = \lambda_k \xi_i \quad (i = 1, 2, \cdots, n-1),$$

whose determinant $D(\lambda) \equiv P_n(\lambda)$. By virtue of (8), $D(\lambda_k) = 0$.

If the elements $z_0, z_1, \cdots, z_{n-1}$ are linearly independent, then the determinant of the system (9) differs from 0. But if in the sequence of elements $z_0, z_1, \cdots, z_n, \cdots$, some are linearly dependent, for example, $z_m = \sum_{k=0}^{m-1} c_k z_k$, then $\bar{z}_m = z_m \in H_m$, i.e. the subspace H_m reduces A and the operator A_m simply coincides with A.

In the general case, for increasing values of n, the sequence of operators A_n converges strongly to A. Hence, in many problems one may replace A by A_n. If in this connection A is completely continuous, then the sequence of operators A_n converges uniformly to A and the eigenvalues and eigenvectors of the operators A_n converge correspondingly to the eigenvalues and eigenvectors of A.

Finally, let x_* be a solution of the linear nonhomogeneous equation

$$x = Ax + f, \tag{10}$$

where A is a linear operator with norm less than unity. Then, as was shown in the monograph of Vorob'ev [1], the sequence[2)] $x_n = (E - A_n)^{-1} f$ of solutions of the approximate equations $x = A_n x + f$ converges strongly to the solution x_* of equation (10) which is sought.

We consider the boundary-value problem for the equation

$$x''(t) + \lambda(x(t) + M(t)x(t - \Delta(t))) = 0 \tag{11}$$

with the boundary conditions

$$\begin{aligned} &x(0) = x(\pi) = 0, \\ &x(t - \Delta(t)) \equiv 0 \text{ if } t - \Delta(t) < 0. \end{aligned} \tag{12}$$

Here $M(t)$ and $\Delta(t) \geqq 0$ are continuous on $[0, \pi]$; λ is a parameter, in general complex. The BVP (VIII.2.18), (VIII.2.19) is a problem of the form (11), (12).

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1) (z_i, z_k) is the scalar product of z_i and z_k.

2) Here E is the identity operator.

Integrating equation (11) twice, we obtain, after using the boundary conditions (12),

$$\frac{1}{\lambda} x(t) = \int_0^\pi K(t,\tau)x(\tau)\,d\tau + \int_0^\pi K(t,\tau)M(\tau)x(\tau - \Delta(\tau))\,d\tau \tag{13}$$

(it is evident that the number $\lambda = 0$ cannot be an eigenvalue of the BVP (11), (12)), where

$$K(t,\tau) = \begin{cases} \dfrac{\tau(\pi - t)}{\pi}, & 0 \leqq \tau \leqq t, \\ \dfrac{t(\pi - \tau)}{\pi}, & t < \tau \leqq \pi. \end{cases} \tag{14}$$

The kernel $K(t,\tau)$ is evidently continuous.

Let $D_A(0,\pi)$ be the linear manifold whose elements are twice differentiable functions $x(t)$ on $[0,\pi]$ satisfying the boundary conditions (12). We consider on $\overline{D}_A(0,\pi)$[3] the linear operators A, P and Q defined by the relations

$$Ax = Q(E + P)x, \tag{15}$$

$$Px = M(t)x(t - \Delta(t)), \tag{16}$$

$$Qx = \int_0^\pi K(t,\tau)x(\tau)\,d\tau. \tag{17}$$

By virtue of (13) and (15) the boundary-value problem (11), (12) is equivalent to the operator equation $Ax = \mu x$. The eigenvalue λ_k of the boundary-value problem (11), (12) is defined by the equality $\lambda_k = 1/\mu_k$, where μ_k is an eigenvalue of the operator A. The eigenfunction of the boundary-value problem (11), (12) is an eigenvector of the operator A.

Corresponding to what was considered in the initial section, for applying the method of moments for computation of eigenvalues and eigenfunctions of the boundary-value problem (11), (12) it is sufficient that the operator A be completely continuous. We shall find conditions for complete continuity of A.

From the continuity of the kernel $K(t,\tau)$ follows the complete continuity of the operator Q, and, by virtue of (15), for the complete continuity of A it is sufficient that the operator P be bounded. By virtue of (16),

[3] $\overline{D}_A(0,\pi)$ is the closure of $D_A(0,\pi)$.

(18) $$\|Px\| = \left(\int_0^\pi |M(\tau)x(\tau - \Delta(\tau))|^2 d\tau \right)^{1/2}.$$

We let $M_0 = \max_{[0,\pi]} |M(t)|$.

Case I. $\Delta(t)$ is piecewise differentiable and

(19) $$\inf_{[0,\pi]} (1 - \Delta'(t)) = m > 0.$$

We take $t - \Delta(t) = s$; then $dt = ds/(1 - \Delta'(\kappa(s)))$, where $\kappa(s)$ is the inverse function of $F(t) = t - \Delta(t)$. From this, by virtue of (18),

(20) $$\|Px\| \leqq \frac{M_0}{\sqrt{m}} \left(\int_0^{\pi - \Delta(\pi)} |x(s)|^2 ds \right)^{1/2} \leqq \frac{M_0}{\sqrt{m}} \|x\|.$$

Consequently,

(21) $$\|P\| \leqq M_0/\sqrt{m},$$

and the boundedness of the operator P is proved in the case under consideration.

The hypothesis (19) excludes the case where on some interval $(\alpha, \beta) \subset [0, \pi]$, $t - \Delta(t)$ is constant. If the operator P is defined on $L^2(0, \pi)$ then, as was shown in the article of A. M. Zverkin [**3**], the presence of such intervals implies the unboundedness of the operator P.

For the problems considered in Chapters VII and VIII, it is of interest to find conditions under which, in spite of the presence of intervals on which $t - \Delta(t)$ is constant, the operator remains bounded on $\overline{D}_A(0, \pi)$.

Case II. Let

(22) $$\Delta(t) = \begin{cases} t, & 0 \leqq t \leqq a, \\ \Delta_*(t), & a < t \leqq \pi, \end{cases}$$

where the piecewise differentiable function $\Delta_*(t)$ is continuous on $a < t \leqq \pi$, $\Delta_*(a) = a$ and $\inf_{[a,\pi]} (1 - \Delta'_*(t)) = m_* > 0$.

Recalling that, by virtue of (12), for an arbitrary function $x(t) \in \overline{D}_A(0, \pi)$ we have $x(t - \Delta(t)) \equiv x(0) = 0$ $(0 \leqq t \leqq a)$,[4] by virtue of (18) we obtain, analogous to (20),

$$\|Px\| \leqq \left(\int_a^\pi |M(\tau)x(\tau - \Delta(\tau))|^2 d\tau \right)^{1/2}$$
$$\leqq \frac{M_0}{\sqrt{m_*}} \left(\int_0^{\pi - \Delta(\pi)} |x(s)|^2 ds \right)^{1/2} \leqq \frac{M_0}{\sqrt{m_*}} \|x\|,$$

[4] For the construction of the closure $\overline{D}_A(0, \pi)$ we require that the limit elements satisfy (12).

from which it follows that

$$\|P\| \leqq M_0/\sqrt{m_*}. \tag{23}$$

We now consider the boundary-value problem for the equation

$$x''(t) + \lambda x(t) + M(t)x(t - \Delta(t)) = 0 \tag{24}$$

with the boundary conditions (12). Here $M(t)$ and $\Delta(t) \geqq 0$ are continuous functions on $[0, \pi]$, and λ is a parameter, in general complex. A boundary-value problem of this type was studied in Chapter III.

Along with equation (24) we shall consider the equation

$$x''(t) + M(t)x(t - \Delta(t)) + f(t) = 0, \tag{25}$$

where $f(t)$ is a function integrable on $[0, \pi]$.

We shall show that if it is possible to find an exact solution of equation (25) (for example, if it is possible to make use of the step method), then to calculate eigenvalues and eigenfunctions of the boundary-value problem (24), (12), it is possible to apply the method of moments discussed at the beginning of this section.

We consider, on $D_A(0, \pi)$, defined above, the linear operator A defined by the relation

$$Ax = -x''(t) - M(t)x(t - \Delta(t)). \tag{26}$$

The eigenvalues and eigenvectors of A coincide with the eigenvalues and eigenfunctions of the boundary-value problems (24), (12).

The operator A is unbounded. However, if $\lambda = 0$ is not an eigenvalue of A, then there exists a bounded inverse operator A^{-1}. By virtue of Theorem III.4.1 and its Corollary, for this it is sufficient, for example, that on the interval $[0, \pi]$ one of the following conditions is fulfilled:

$$0 \leqq M(t) < \frac{(\pi/2 + \sqrt{2})^2}{\pi^2} \quad (= 0.9028 \cdots) \tag{27}$$

or

$$M(t) \leqq 0. \tag{28}$$

It will be shown below that under these hypotheses the operator A^{-1} is completely continuous.

We apply the method of moments to determine eigenvalues and eigenvectors of the operator A^{-1} (evidently its eigenvalues are the reciprocals of the eigenvalues of A, while the eigenvectors of A and A^{-1} coincide).

Starting with an arbitrary element $z_0 \in \overline{D}_A(0, \pi)$, we construct the set of iterates

$$z_0, z_1 = A^{-1}z_0, z_2 = (A^{-1})^2 z_0, \cdots, z_n = (A^{-1})^n z_0, \cdots. \tag{29}$$

Since we do not know the operator A^{-1}, to construct the sequence (29) we need to find solutions of the equation of the form (25) satisfying the boundary conditions (12):

$$\begin{aligned} x''(t) + M(t)x(t - \Delta(t)) + z_0(t) &= 0, \\ x''(t) + M(t)x(t - \Delta(t)) + z_1(t) &= 0, \end{aligned} \tag{30}$$

where $z_1(t)$ is the solution of equation (30) which has been found, and so on. In what follows, all will be done according to the scheme set out at the beginning of the section (for details see the monograph of Vorob′ev [1]).

However, in many important cases, the step method or any other method for obtaining an exact solution of an equation of the form (25) cannot be applied. In the following section a generalized method of moments is set forth, applicable also in this case, to construct a sequence of the form (29) and determine eigenvalues and eigenvectors of the operator A^{-1} in the scheme presented above.

§3. A generalization of the method of moments

Suppose that on the entire linear manifold D_A dense in H there is defined an unbounded linear operator A having a bounded inverse A^{-1} (which we do not know). We assume that in the space H it is possible to construct completely continuous linear operators B ($\|B\| < 1$) and B_0 such that on D_A

$$A = B_0^{-1}(E - B). \tag{31}$$

Beginning with an arbitrary element $z_0 \in H$, we construct the sequence of iterates

$$z_0, z_1 = A^{-1}z_0, z_2 = (A^{-1})^2 z_0, \cdots, z_n = (A^{-1})^n z_0, \cdots. \tag{32}$$

Since we do not know the operator A^{-1}, to define z_1 by means of z_0 it is required to solve the equation $Ax = z_0$, or, by virtue of (31), $x = Bx + z_{01}$, where $z_{01} = B_0 z_0 \in H$. Analogously, if the element z_k has already been determined by the sequence (32), the element z_{k+1} is defined from the equation

$$x = Bx + z_{k1}, \tag{33}$$

where $z_{k1} = B_0 z_k \in H$. We note that by virtue of the hypothesis that

$\|B\| < 1$, solutions of the equations of the form (33) exist and are determined uniquely for each $z_{k1} \in H$.

We construct the sequence of iterates

$$\tilde{z}_0 = z_0, \tilde{z}_1 = Bz_0, \tilde{z}_2 = B^2 z_0, \cdots, \tilde{z}_n = B^n z_0, \cdots \tag{34}$$

and let $\widetilde{H}_m$ be the linear hull of the elements $\tilde{z}_0, \tilde{z}_1, \cdots, \tilde{z}_{m-1}$, which belong to the sequence (34). Analogous to (7), we define the operator B_m on $\widetilde{H}_m$ by the relations

$$\begin{cases} \tilde{z}_k = B_m^k z_0 \quad (k = 0, 1, \cdots, m-1), \\ \bar{\tilde{z}}_m = B_m^m z_0, \end{cases}$$

where $\bar{\tilde{z}}_m$ is the projection of $\tilde{z}_m$ on $\widetilde{H}_m$. As was shown in the monograph of Vorob′ev [1], for arbitrary m

$$\|B_m\| \leqq \|B\| < 1 \tag{35}$$

and, by virtue of the complete continuity of the operator B, the sequence of operator B_m converges uniformly to B.

Equations (33) are of the form of equation (10), and, as was shown at the beginning of §2, for these the method of moments can be applied to approximate the solution. In correspondence with this we construct the sequence of elements

$$z_0^* = z_0, z_1^*, z_2^*, \cdots, z_n^*, \cdots \tag{36}$$

where z_k^* $(k = 1, 2, \cdots)$ is a solution of the equation

$$x = B_m x + z_{(k-1)1}^*, \quad z_{(k-1)1}^* = B_0 z_{k-1}^*. \tag{37}$$

Let H_n^* be the linear hull of the elements $z_0^*, z_1^*, \cdots, z_{n-1}^*$, which belong to the sequence (36). On H_n^* we construct the operator A_n^{-1}, satisfying the relations

$$\begin{aligned} z_k^* &= (A_n^{-1})^k z_0 \quad (k = 0, 1, \cdots, n-1), \\ \bar{z}_n^* &= (A_n^{-1})^n z_0, \end{aligned} \tag{38}$$

where $\bar{z}_n^*$ is the projection of z_n^* on H_n^*. It is easy to show that the operator A_n^{-1} is determined by (38) and gives a solution to the problem of moments for the sequence of iterates by the operator equation

$$A_m^{-1} = (E - B_m)^{-1} B_0. \tag{39}$$

We consider the subspace H_z, the closure of the linear manifold L_z of elements of the form $x = Q(A_n^{-1}) z_0$, where $Q(\lambda)$ is an arbitrary polynomial. By virtue of (39) the operator A_m^{-1} is completely continuous on H_z, and hence the sequence of operators A_n^{-1} converges uniformly to A_m^{-1}. Recalling that also the sequence of operators B_m converges uniformly to B, for each $\epsilon > 0$ there exist numbers m_0 and n_0 such that if $m > m_0$ and $n > n_0$ it follows that simultaneously

$$\| B - B_m \| < \frac{(1 - \| B \|)^2}{2 \| B_0 \|} \epsilon, \quad \| A_m^{-1} - A_n^{-1} \| < \frac{\epsilon}{2}.$$

Then, by virtue of (31), (39), and (35),

$$\begin{aligned} \| A^{-1} - A_n^{-1} \| &\leqq \| A^{-1} - A_m^{-1} \| + \| A_m^{-1} - A_n^{-1} \| \\ &< \| B_0 \| \, \| (E - B)^{-1} - (E - B_m)^{-1} \| + \frac{\epsilon}{2} \\ &\leqq \frac{\| B_0 \| \, \| B - B_m \|}{\| E - B \| \, \| E - B_m \|} + \frac{\epsilon}{2} \leqq \frac{\| B_0 \| \, \| B - B_m \|}{(1 - \| B \|)^2} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Thus, the operator A^{-1} is uniformly approximated by the sequence of operators A_n^{-1}, and hence the eigenvalues and eigenvectors of the operators A_n^{-1} converge to the eigenvalues and eigenvectors of A^{-1} respectively.

We now consider conditions which permit the application of such a "doubled" method of moments in order to compute the eigenvalues and eigenvectors of the boundary-value problem (24), (12).

We shall show that on $\overline{D}_A(0, \pi)$ it is possible to construct completely continuous operators B ($\| B \| < 1$) and B_0, which satisfy the condition (31).

We consider the equation

$$Ax = f, \tag{40}$$

where $f \in \overline{D}_A(0, \pi)$ and the operator A is determined by (26). The equation (40) is equivalent to the boundary-value problem

$$x''(t) + M(t) x(t - \Delta(t)) + f(t) = 0, \tag{25'}$$

$$\begin{aligned} &x(0) = x(\pi) = 0, \\ &x(t - \Delta(t)) \equiv 0 \text{ if } t - \Delta(t) < 0. \end{aligned} \tag{12}$$

We shall assume that one of the hypotheses (27) or (28) holds. Then $\lambda = 0$ is not an eigenvalue of the operator A and equation (40) (and hence also the boundary-value problem (25′), (12)) has a unique solution for arbitrary $f \in \overline{D}_A(0, \pi)$.

Integrating the equation $x''(t) = - M(t) x(t - \Delta(t)) - f(t)$ twice, and employing the boundary conditions (12), we obtain

$$x(t) = \int_0^\pi K(t,\tau)M(\tau)x(\tau - \Delta(\tau))\,d\tau + \int_0^\pi K(t,\tau)f(\tau)\,d\tau, \tag{41}$$

where the continuous kernel $K(t,\tau)$ is defined by (14).

For arbitrary $x \in \overline{D}_A(0,\pi)$, we set

$$\begin{aligned} B_0 x &= Qx, \\ Bx &= QPx, \end{aligned} \tag{42}$$

where the operators Q and P are defined by (17) and (16). In this connection, (17) and (14) imply that

$$\|Q\| = \left(\int_0^\pi \int_0^\pi K^2(t,\tau)\,d\tau dt\right)^{1/2} = \frac{\pi^2}{3\sqrt{10}}. \tag{43}$$

It then follows from (40) and (41) that for arbitrary $f \in \overline{D}_A(0,\pi)$,

$$x = A^{-1}f = QPx + Qf,$$

where x is a solution of (40). From this, by virtue of (40), $A^{-1}f = (QPA^{-1} + Q)f$ and hence,

$$A^{-1} = Q(E - QP)^{-1} = B_0(E - B)^{-1}, \tag{44}$$

i.e. the operators B_0 and B defined by (42) satisfy (31).

We shall find conditions under which the inequality

$$\|B\| < 1 \tag{45}$$

holds.

We again consider the cases I and II set forth in §2. For case I, by virtue of (42), (43) and (21),

$$\|B\| \leqq \|Q\|\,\|P\| \leqq \frac{\pi^2}{3\sqrt{10}} \cdot \frac{M_0}{\sqrt{m}},$$

and for (45) to hold it is sufficient that the inequality

$$\frac{M_0}{\sqrt{m}} < \frac{3\sqrt{10}}{\pi^2} \quad (= 0.9612\cdots)$$

be fulfilled.

Analogously, in case II, inequality (45), by virtue of (42), (43) and (23), is guaranteed by the inequality

$$\frac{M_0}{\sqrt{m_*}} < \frac{3\sqrt{10}}{\pi^2}$$

It remains to be shown that the operators B_0, B and A^{-1} are completely continuous. Indeed, the complete continuity of B_0 follows from the continuity of the kernel $K(t, \tau)$. In cases I and II, by virtue of the boundedness of the operator P, the complete continuity of B follows from the equality $B = B_0 P$.

In order that the operator $(E - B)^{-1}$ be bounded, it is sufficient that the number $\lambda = 0$ not be an eigenvalue of the operator $E - B$. Suppose the contrary; but then the equation $(E - B)x = 0$ has a nonzero solution $x(t) \in D_A(0, \pi)$, or, equivalently, the equation

$$x''(t) + M(t)x(t - \Delta(t)) = 0$$

has nontrivial solutions which satisfy the boundary conditions (12). But then the number $\lambda = 0$ is an eigenvalue of the boundary-value problem (24), (12), which is excluded by the hypothesis (27) or (28). The complete continuity of A^{-1} now follows from (44).

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