Foundations of a Structural Theory of Set Addition

by
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НАЧАЛА СТРУКТУРНОЙ ТЕОРИИ СЛОЖЕНИЯ МНОЖЕСТВ

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Под редакцией
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PREFACE

The present book is addressed to students of graduate courses and to doctoral candidates studying number theory as well as to the specialist working in this field. As an experiment, the material was treated by a seminar on "Supplementary Topics in Number Theory" held at Elabuga State Pedagogical Institute, and it gives a systematic exposition of a series of papers on additive number theory by the author. Only a basic knowledge of number theory is presupposed, such as contained, for example, in the books by I.M. Vinogradov or A.A. Buhštab. Many of the problems treated in the first chapter may be used as topics for term papers or masters' theses as well as for independent mathematical research.

The last decade has seen the beginning of the application in additive number theory of general principles arising from the addition of sets. The present book follows this trend.

From 1959 to 1964 a series of papers (cf. [15] to [23]) were published under the general title, "Inverse problems of additive number theory". In these papers the structure of sets A with small "double set" $A + A$ was investigated. More precisely, if we consider a set with a certain numerical parameter (the number of elements for finite sets of numbers or of residue classes, the density for sequences of integers, the measure for point sets) for which that numerical parameter of the corresponding double set is not too large in comparison with the original value, then the structure of the original set can be described in a certain sense. The content of the papers mentioned is, by and large, the subject of this book.

The book consists of three chapters. The first chapter contains an exposition of the basic concepts, general ideas and elementary results. In the second chapter a difficult specific inverse problem is solved. The third chapter is devoted to applications.

I wish to stress the importance of the concept of isomorphism between subsets of sets with an algebraic operation as introduced and applied in Chapter I.

The well-known significance of the concept of isomorphism between groups lies in the fact that it provides the possibility for disregarding incidental properties of each given specific group, and thus it leads to investigations of a more general kind. Furthermore it permits us to introduce an infinity of operations on the elements of a group.

In additive number theory, on the other hand, one often considers sums consisting of a bounded number of terms from suitable sets. Therefore we restrict the investigations in the present book to problems concerned with simple (i.e. noniterated) addition. It is natural to introduce a concept of isomorphism which is geared to this situation.

There is an analogy between a local isomorphism of topological groups and an isomorphism of subsets, a so-called "algebraically local isomorphism". In the first case the isomorphism is restricted by the boundary of a certain neighborhood, and in the second case, by the number of operations performed.

The concept of isomorphism between subsets has permitted us to express a general view on additive number theory as the theory which studies those properties of sets of numbers which are preserved by such isomorphisms.

The author has endeavored to make the presentation of the material in the first chapter as explicit and elaborate as possible in order to introduce the beginning student to a new field of ideas.

The exercises are, by and large, concentrated in the first chapter. Some of them are strictly for teaching purposes whereas others provide material for independent research.

The material of the second and third chapters is presented more briefly than that of Chapter I.

In Chapter II a fundamental theorem on the structure of finite sets of integers with small "double set" is proved. In this proof we use a modification of the method of trigonometric sums and also the language and the elementary methods of the geometry of numbers.

It would be highly desirable to obtain generalizations of the results of Chapters I and II for arbitrary abelian and especially for non-abelian groups. The choice of the title for the present work was prompted by the desire to stress such possibilities of developing the theory along with the prospects for its further number-theoretical extensions.

Chapter III is devoted to applications of the theory.

As a summary we give a revised form of the English text read in September 1965 at the Summer School on Number Theory held in Palanga, Latvian SSR. In that paper a general exposition was given on investigations in the field of addition of finite sets of numbers which are of special significance in additive number theory.

EXPLANATIONS FOR THE USE OF THE BOOK

1. Definitions are given in the text whenever a new concept occurs for the first time. They may be located by means of the Index.

2. Whenever a concept known from the literature occurs in this book for the first time, it is printed in italics (sometimes the italics are repeated further on). References to the appropriate literature are given in the Index.

3. Subsections are numbered consecutively in each chapter, without regard to

sections. They are designated by two numbers, the first of which is the chapter number. The sign § is used for both sections and subsections; since sections (rarely referred to) are designated by only one number, this should cause no confusion. Thus, e.g., §1.10 is the tenth subsection of Chapter I; it happens to be part of §2 of that chapter.

4. References to theorems, lemmas and definitions are listed by the numbers of the chapter and the subsection in which they are given. Thus, e.g., Theorem 1.9 is to be found in §1.9 of Chapter I. Formulas are denoted by three numbers, i.e. the number of the chapter, the number of the subsection and the number of the formula. Thus, e.g., formula (2.3.1) is the first formula in §2.3 of Chapter II. The first number is omitted if reference is made to a formula of the same chapter. The first two numbers are omitted if reference is made to a formula of the same subsection. Thus, the reference to formula (1) on page 49 refers to formula (2.3.1) of the same subsection.

5. A number of problems of the caliber of exercises are given without asterisks, whereas the remaining ones, which are of a more advanced nature, are marked with one, two or three asterisks, depending on their difficulty.

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SUMMARY
ON GENERAL INVESTIGATIONS IN ADDITIVE NUMBER THEORY

Report delivered at the Number Theory Summer School in Palanga on September 5, 1965.

The title of this report has been chosen as an allusion to Ostmann's survey of additive number theory ([35], [36]), which was published in two volumes in 1955/56. The first volume is devoted to general investigations, i.e. to questions of sums of sets of positive density. The second volume contains a review of special problems such as Goldbach's problem, Waring's problem, the theory of partitions, etc.

1. Mann's theorem (1942) was the main result in the theory of density.

The Šnirel′ man density of an increasing sequence (or set) $A = \{a_0, a_1, \ldots\}$ of nonnegative integers is defined as

$$\alpha'(A) = \inf_{x \in N} \frac{A(x)}{x},$$

where $A(x)$ is the number of positive elements in the set A which do not exceed x, and N is the set of natural numbers.

MANN'S THEOREM. *Let B and C be two sets of nonnegative integers with $0 \in B$ and $0 \in C$. Then*

$$\alpha'(B + C) \geqslant \min(1, \alpha'(B) + \alpha'(C)).$$

In recent years the evolution of the theory of density has been slow, due to the fact that both in methods and in results it essentially amounted to developments of Mann's Theorem and no new applications to classical problems had been obtained, though at the outset such applications were given by Šnirel′ man.

This report suggests a new approach to the study of general additive regularities.

2. The special problem which we shall now discuss will give us an opportunity to introduce gradually the necessary new notions.

Let K be a finite set of k integers:

$$K = \{a_0, a_1, \ldots, a_{k-1}\}, \qquad a_i < a_{i+1}, \, i = 0, 1, \ldots, k - 2.$$

Let $T(M)$ be the number of elements of the finite set M and let $T = T(2K)$.

Obviously, $T \geqslant 2k - 1$. Actually, in the set $a_0 + K$ which consists of k numbers, the maximum is $a_0 + a_{k-1}$, whereas a set $a_{k-1} + K$ has this number as its minimum.

Let $T < Ck$, where C is a positive number. The problem is to determine the structure of K.

Here are some examples to clarify the question. If $T = 2k - 1$, then K is an arithmetical progression containing k numbers. In fact, if for some j with $0 \leqslant j \leqslant k - 3$ the inequality

$$a_{j+1} - a_j \neq a_{j+2} - a_{j+1}$$

holds, then the set $2K$ contains in addition to $2a_0, a_0 + a_1, 2a_1, a_1 + a_2, 2a_2, \ldots$ and $2a_{k-1}$ the number $a_j + a_{j+2}$ which is distinct from them.

It is possible to prove the following theorem by induction on k:

THEOREM 1.9. *If* $0 \leqslant b < k - 2$ *and* $T = 2k - 1 + b$, *then* K *is contained in the set*

$$K_a = \{a, a + q, a + 2q, \ldots, a + (k - 1 + b)q\},$$

where a *is an integer and* q *a natural number.*

Thus, if $T < 3k - 3$, then K is a subset of an arithmetical progression whose number of elements is not larger than $T - k + 1$.

Now will it make any difference if $T \geqslant 3k - 3$? More specifically: If we consider all arithmetical progressions containing the set K and choose from them an arithmetical progression with the minimum number of elements, then we can call this minimum the "length" of K. The question arises whether it is possible to evaluate the lengths of all K in the class of sets with given k and T as a function which depends only on the parameters.

The example

$$K = \{0, 1, 2, \ldots, k - 2, u\}, \qquad u > 2k - 4,$$

shows that this is impossible even for $T = 3k - 3$.

Nevertheless, results rather close to those obtained in Theorem 1.9 are true even in the case when $T \geqslant 3k - 3$. To formulate these results some new notions will be necessary, the most important one of them being a generalization of the notion of isomorphism between sets with algebraic operations.

3. DEFINITION. The subsets B' and C' of the sets B and C, respectively, with an algebraic operation, written additively, are called *isomorphic* if there exists a one-to-one mapping $B' \to C'$ which induces naturally a one-to-one mapping $2B' \to 2C'$. The mapping $B' \to C'$ is then called an *isomorphic mapping*, or an *isomorphism*.

An isomorphic mapping of a set K does not change the value of T. This is the reason why the description of the structure of K must be invariant under isomorphic mappings.

If $B' = B$, $C' = C$ and the mappings $B' \to C'$ and $2B' \to 2C'$ correspond to each other, our definition becomes the usual definition of an isomorphism between sets with an algebraic operation. In this case the addition of any number of elements corresponding to summands results in an element corresponding to the sum.

Let us consider the two additive semigroups $B' = \{0, 1, 2, \ldots\}$ and $C' = \{10, 11, 12, \ldots\}$. Then the mappings $B' \to C'$, defined by $u \to u + 10$, $u = 0, 1, 2, \ldots$, and $2B' \to 2C'$, defined by $u \to u + 20$, $u = 0, 1, 2, \ldots$, correspond to each other, and therefore B' and C' are isomorphic. This example shows that correspondences $B' \leftrightarrow C'$ and $2B' \leftrightarrow 2C'$ are sometimes incompatible.

If two given subsets are isomorphic, such a correspondence exists in general if the addition $K + K$ is not iterated. Thus the concept of isomorphism of subsets seems especially fitting for the study of questions of sums of finitely many sets, and this is the very thing necessary for the purposes of additive number theory in the study of problems with a bounded number of summands.

An analogy may be drawn between a local isomorphism between topological groups and an isomorphism between subsets ("algebraically local isomorphisms"). In the first case the isomorphism is restricted to a certain neighborhood, and in the second case by the number of operations.

Let $B', C' \subset E_n$ (the euclidean space) and assume that there exists a one-to-one correspondence of B' and C' such that any equation of the form $\bar{b}_1 - \bar{b}_2 = \bar{b}_3 - \bar{b}_4$ implies the corresponding equation $\bar{c}_1 - \bar{c}_2 = \bar{c}_3 - \bar{c}_4$, and $\bar{b}_1 - \bar{b}_2 \neq \bar{b}_3 - \bar{b}_4$ implies $\bar{c}_1 - \bar{c}_2 \neq \bar{c}_3 - \bar{c}_4$, where $\bar{b}_i \in B'$, $\bar{c}_i \in C'$ and $\bar{b}_i \leftrightarrow \bar{c}_i$, $1 \leqslant i \leqslant 4$. The conditions formulated above are necessary and sufficient for the sets B' and C' to be isomorphic.

4. FUNDAMENTAL THEOREM. *If $T < Ck$, $C \geqslant 2$, then there exist constants c and k_0, depending only on C, and a number $n \leqslant [C - 1]$ such that if $k > k_0$, then K is contained in a certain set K_0 of integers which is isomorphic to the set of interior points of some convex set $D \subset E_n$ and which satisfies the condition $T(K_0) < ck$.*

Here are some examples illustrating the Fundamental Theorem. If $2 \leqslant C < 3$, then $n = 1$, and D is an interval whose number of integral points is not larger than ck. Theorem 1.9 gives an elementary proof of a more powerful, similar result. If $3 \leqslant C < 4$ then $n \leqslant 2$. If $n = 1$, then K is contained in an arithmetical progression whose number of elements is not larger than ck.

If $n = 2$ then it is possible to show that D can be taken as a rectangle. Thus in this case K is contained in the union of some arithmetical progression with identical differences whose first elements form an arithmetical progression with another difference.

In fact, the lattice points of the rectangle which are situated on any one line parallel to the axis of abscissas are transformed into arithmetical progressions with identical differences. The image of the set of lattice points in the rectangle which are located on a certain line parallel to the axis of ordinates will be the set of the first elements of these progressions.

The Fundamental Theorem was proved with the help of a modification of the method of trigonometric sums.

5. Now it is possible to express a certain general viewpoint concerning the study of additive regularities. For this purpose we shall recall Klein's view on geometry as the science studying those properties of geometrical objects which are invariant under certain groups of transformations.

We are now able to state (see §3) that *the goal of additive number theory, while studying sums of sets, consists in the investigation of those properties of these sets which are invariant under isomorphic transformations*. From this point of view the study of relations between invariants of isomorphisms (henceforth called additive characteristics) appears to be very important.

The most important additive characteristics are T, the number R of distinct positive differences $a_i - a_j$ and the number $M = \int_0^1 |S|^4 \, d\alpha$, where $S = \sum_{j=0}^{k-1} e^{2\pi i \alpha a_j}$ (if $b_1, \ldots, b_T$ are the elements of the set $2K$ and r_s is the number of representations $a_i + a_j = b_s$, then $M = \sum_{s=1}^{T} r_s^2$).

Inverse problems of additive number theory (§1.7) may be stated as problems of describing sets with given invariants.

The Fundamental Theorem is the first step towards realizing the ideas expressed above. In formulating and proving it, one of the important tools used consists in isomorphic mappings of certain sets of integers into sets of lattice points of euclidean spaces of higher dimension. By using them it was possible to establish a fact which had naturally remained unnoticed previously, i.e. dimension is not invariant under isomorphic transformations.

It is worth mentioning that density is not invariant inasmuch as it may be changed by an isomorphic transformation, as is illustrated by the example

$$A = \{0, 1, 4, 8, 12, \ldots\}, \qquad \alpha'(A) = 1/4,$$

$$B = \{0, 1, 3, 6, 9, \ldots\}, \qquad \alpha'(B) = 1/3.$$

This explains why the application of the concept of density has certain limitations.

6. The analogy between Klein's view of geometry and the view of additive number theory as the theory of isomorphic transformations goes deeper then it may seem at first sight. It turns out that an isomorphic transformation is an affinity in a suitable euclidean space.

An isomorphic transformation of $K \subset E_m$ into E_n is called *nonsingular* if no hyperplane from E_n contains the image of K.

Suppose there exists a nonsingular transformation of the set K into E_n but no such transformation of K into E_{n+1}. Let this value of n be denoted by r. Then r is an additive characteristic of the set K.

THEOREM. *Let K_1 and K_2 be finite sets in E_r neither of which is contained in any hyperplane from E_r. Then the sets K_1 and K_2 are isomorphic if there exists an affinity of E_r into itself which maps K_1 into K_2.*

7. *Applications.* Hinčin's Theorem is the special case of Mann's Theorem where two identical sets are added: If $\alpha'(A) = \alpha$ and $\alpha'(2A) = \gamma$, then $\gamma \geqslant \min(2\alpha, 1)$ provided that $a_0 = 0$.

An analog to Mann's Theorem for the addition of sets with positive asymptotic density was obtained by M. Kneser (1953). Expanding the theory of density, it became possible to show, roughly speaking, that if two identical sets are added then the density is at least doubled. It was found that, in general, the number 2 may be replaced by any positive constant C. This is the essence of the results derived from the Fundamental Theorem.

Let A be an increasing sequence of nonnegative integers:

$$A = \{a_0, a_1, \ldots, a_i, \ldots\}, \qquad a_{i+1} > a_i,\ i \geqslant 0,$$

$$d(A) = \lim_{k\to\infty} (a_1 - a_0, a_2 - a_0, \ldots, a_k - a_0).$$

Here $(a_1 - a_0, a_2 - a_0, \ldots, a_k - a_0)$ is the greatest common divisor of the numbers $a_1 - a_0$, $a_2 - a_0, \ldots, a_k - a_0$.

The number of positive integers in the set A (or $2A$) not exceeding x is denoted by $A(x)$ and $A_2(x)$, respectively. The asymptotic density $\alpha = \alpha(A)$ of the set A is defined by the equation

$$\alpha = \varliminf_{x\to\infty} \frac{A(x)}{x};$$

furthermore, we let $\gamma = \alpha(2A)$. The set of lattice points in the euclidean space E_n is denoted by Z_n. Let $\bar{e}_1, \ldots, \bar{e}_n$ be an orthonormal basis of the space E_n. Let φ: $Z_n \to Z_1$ be a homomorphism satisfying $\bar{e}_i\varphi = 0$, $1 \leqslant i \leqslant n-1$, and $e_n\varphi = 1$. Finally, let $N^{(x)}$ be the subset of the given set $N \subset E_n$ consisting of those elements for which $x_n \leqslant x$. As a consequence of the Fundamental Theorem the following theorem is proved in Chapter III:

THEOREM. *Suppose the set A satisfies the following conditions: $a_0 = 0$, $d(A) = 1$, $\alpha > 0$, $\gamma > C\alpha$ and $C > 3/2$. Furthermore, assume that there exists a constant C_1 such that the inequality $a_{i-1} > C_1 a_i$ is satisfied only for a finite number of indices i. Then there exist a cylinder $N \subset E_n$, $n \geqslant 2$, whose base is a convex set $P \subset L(\bar{e}_1, \ldots, \bar{e}_{n-1})$, and positive constants c and α_0, depending on C and C_1 only, such that for $\alpha < \alpha_0$ the following assertions are true:*

(1) $A \subset (N \cap Z_n)\varphi$.
(2) $N \cap Z_n$ *and* $(N \cap Z_n)\varphi$ *are isomorphic.*
(3) $V(P) < c\alpha$, $T(N^{(x)} \cap Z_n) < c\alpha x$.
(4) $n \leqslant [2C - 1]$.

For sequences of zero density an analog of Hinčin's Theorem was conjectured by P. Erdös [13]. His conjecture states that

$$\delta = \overline{\lim_{x\to\infty}} \frac{A_2(x)}{A(x)} \geqslant 3, \quad \text{if} \quad \lim_{x\to\infty} \frac{A(x)}{x} = 0.$$

By means of the Fundamental Theorem the following more general assertion was proved:

THEOREM. *If* $\underline{\lim}_{x\to\infty} A(x)/x = 0$ *and*

$$\overline{\lim_{x\to\infty}} \frac{A(x)}{x} < \frac{1}{2d(A)}$$

then $\delta \geqslant 3$.

The Fundamental Theorem also leads to applications to sum-sets of sets of residues modulo a prime, thus strengthening results received by Cauchy, Davenport and Vosper, and for point sets of positive measure (strengthening the Brunn-Minkowski Inequality).

8. In the Fundamental Theorem the structure of a set K with small double set (i.e. with $T < Ck$) is investigated. Further possible lines of investigation are the following ones:

Study the structure of K when $T/k \to \infty$.

Generalize the problem to the case where several summands are considered (iterated addition) or when the summands are different.

Investigate the structure of K when different additive characteristics of the set K or their combinations are given.

After dividing the sets consisting of a finite number of lattice points into classes of isomorphic sets, study the order of increase of the function $t(k)$, i.e. the number of such classes for given k.

All these problems may be studied for sets of a more general nature than sets of lattice points, especially in groups, both abelian and nonabelian, which would be of special interest.

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In this index reference is made to the following books:

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