Statistical Sequential Analysis

Optimal Stopping Rules

A. N. ŠIRJAEV

Volume 38

TRANSLATIONS OF
MATHEMATICAL MONOGRAPHS

American Mathematical Society

Statistical Sequential Analysis

Optimal Stopping Rules

10.1090/mmono/038

TRANSLATIONS OF MATHEMATICAL MONOGRAPHS

VOLUME 38

Statistical Sequential Analysis

Optimal Stopping Rules

A. N. ŠIRJAEV

American Mathematical Society · Providence · Rhode Island

СТАТИСТИЧЕСКИЙ ПОСЛЕДОВАТЕЛЬНЫЙ АНАЛИЗ
ОПТИМАЛЬНЫЕ ПРАВИЛА ОСТАНОВКИ

А. Н. ШИРЯЕВ

Издательство "Наука"
Главная Редакция
Физико-Математической Литературы
Москва 1969

Translated from the Russian by
Lisa and Judah Rosenblatt

AMS (MOS) subject classifications (1970).
Primary 62L12, 62L15; Secondary 60G40

Library of Congress Cataloging-in-Publication Data

Shiriaev, Al'bert Nikolaevich.
Statistical sequential analysis.
(Translations of mathematical monographs; v. 38)
Translation of Statisticheskiĭ posledovatel'nyĭ analiz.
1. Sequential analysis. I. Title. II. Series.
QA279.7.S5213 519.5'4 73-4451
ISBN 0-8218-1588-1

The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability. ♾
Printed in the United States of America.
10 9 8 7 6 5 4 3 2 95 94 93 92 91 90

TABLE OF CONTENTS

Notes

Chapter I

§ 1. The axiomatics of probability theory are presented in the fundamental work by Kolmogorov [**41**]. Proofs of the results given concerning measurability of random processes are contained in the monographs by Dynkin [**25**] and Meyer [**49**].

§ 2. Additional information about the properties of Markov times can be found in the monographs of Dynkin [**25**], Meyer [**49**] and [**50**], and Blumental and Getoor [**12**]. Theorem 1 is due to Courrège and Priouret [**20**]. The Remark to Theorem 1 was made by Hans-Jürgen Engelbert.

§ 3. The basic definitions given for the theory of Markov processes follow the monographs of Dynkin [**24**] and [**25**], and Blumental and Getoor [**12**].

§ 4. Proofs of the given theorems on martingales and super-martingales are given in Doob [**22**] and Meyer [**49**]. Generalized martingales and super-martingales are investigated in an article by Snell [**68**].

Chapter II

§ 1. The value $s(x)$ for the case of nonnegative functions was considered in the article by Dynkin [**26**], where he also gave properties of excessive functions and Lemma 2. A somewhat different proof of Lemma 1 is given in Meyer [**49**]. Lemma 3 is contained in work by Grigelionis and Širjaev [**37**]. The method for constructing a smallest excessive majorant, given in Lemma 4, was presented by A.D. Ventcel′ (see [**27**], Chapter III, Problem 21). In martingale theory Lemma 5 is known under the name of the theorem on transformation under a system of optional sampling (Doob [**22**], Theorem 2.2 in Chapter VII). The proof of Theorem 6 was adapted from Snell [**68**]. The method for constructing the s.e.m. of the function $g(x)$, given in Lemmas 7 and 9, is presented for the first time. Similar constructions are also contained in the article by Siegmund [**56**]. Another method of proof of Theorem 1, for the case $g(x) \geqq 0$, was given in the article by Dynkin [**26**].

§ 2. $(\varepsilon,\bar{s})$-optimal times are investigated for the first time. In the case $0 \leqq g(x) \leqq C < \infty$, (ε,s)-optimality of the time τ_ε was proved by Dynkin [**26**]. Assertions 2) and 3) of Theorem 2 are close to results in articles by Siegmund [**56**] and Chow and Robbins [**17**]. The problem of choosing the best object, also known as the "fastidious bride" problem, was investigated by Gardner [**29**] and Dynkin [**26**] (see also [**27**]). Similarly formulated problems were considered in articles by Chow, Moriguti, Robbins and Samuels [19], Gilbert and Mosteller [32], and Guseĭn-Zade [38].

§ 3. The example presented at the beginning of this section was given by Haggstrom [39]. The classes of Markov times $\mathfrak{N}$ and $\mathfrak{R}$ were considered in the article by Chow and Robbins [17]; their methods were also used in proving Lemma 10 and Theorems 3 and 4.

§ 4. Theorems 5 and 6 are contained in the work of Chow and Robbins [15], [16], [17], and Haggstrom [39].

§ 5. Questions of uniqueness of the solution of the recursion equations (2.85) were investigated by Bellman [9], Grigelionis and Širjaev [37] and Grigelionis [35]. Theorem 10 is due to Siegmund [56].

§ 6. The results given in this section were obtained in the article by Ray [53], and also by Grigelionis and Širjaev.

§ 7. Randomized and sufficient classes of Markov times were investigated in works by Siegmund [56], Širjaev [62], Dynkin [28], and Grigelionis [36].

§ 8. Functionals of the type (2.113) were studied in the article by Krylov [43]. The example given at the end of the section for the case $\alpha = 1$ was considered in articles by Chow and Robbins [15], [16] and Siegmund [56].

Chapter III

§ 1. The definitions and proofs of the given properties of excessive functions are due to Hunt [40] and Dynkin [25] (see also [12] and [49]).

§ 2. The method for constructing the smallest excessive majorant of the function $g(x)$ (Lemma 1) was given by Grigelionis and Širjaev [37]. Another method was presented earlier by Dynkin [26].

§ 3. Theorem 1 for the case $g(x) \geqq 0$ was presented by Dynkin [26]. The value $\bar{s}(x)$ for continuous Markov processes has not been investigated previously. The example given in § 3.3 is contained in the article by Taylor [57]. In the proof of Lemma 8 we used methods from the article by Dynkin [26]. Lemma 9 and Theorems 2 and 3 were obtained by the author.

§ 4. Assertion 1 of Theorem 4 was proved by Dynkin [26]. Results close to assertions 2) and 3) were also obtained by Siegmund [56]. Theorem 5 is given for the first time. Under special assumptions, Theorem 6 was proved by Taylor [70].

§ 5. Lemmas 11 and 12 are published for the first time. Theorems 7, 8 and 9 were obtained by Grigelionis and Širjaev [37].

§ 6. The "smooth pasting" conditions was used to solve concrete problems in work by Mihalevič [51], Chernoff [14], Lindley [46], Bather [6], Širjaev [61], Whittle [76] and Stratonovič [69]. Theorem 10 is due to Grigelionis and Širjaev [37]. Theorems 12 and 13 were obtained by Grigelionis [34].

Chapter IV

§ 1. The Bayesian and variational formulations of the problems of sequential testing of two simple hypotheses are due to Wald **[72]**. The properties of transitive statistics were investigated in work by Bahadur **[4]**, Širjaev **[62]**, **[65]**, and Grigelionis **[36]**.

Proofs of Lemma 3 and Theorem 1 are contained in the articles by Chow and Robbins **[16]** and Širjaev **[66]**.

Equations (4.36)–(4.37) were first obtained by Mihalevič **[51]**. Lemma 4 is contained in the article by Širjaev **[64]** (see also **[47]**). A somewhat different proof of Theorem 2 is given by Širjaev **[66]**. The idea of the proof of the equality $\rho(\pi) = f^*(\pi)$ used by us is due to B. Rozovskiĭ.

§ 2. Theorem 3 is due to Wald **[72]**. Lemma 8 and its proof were given by Shepp **[55]**. He is also credited with the results presented in Remarks 3 and 4 to this lemma. The comparison of the optimal properties of the Neyman-Pearson method and the sequential probability ratio test was given by Aĭvazjan **[1]**. An elegant proof of Theorem 4 is contained in the book by Lehmann **[45]**. The bounds (4.98) contained in Theorem 5 were presented by Wald **[72]**. Theorem 6 was obtained by Stein (see **[72]**).

Wald's identity (Lemma 10) has been the subject of investigation by many authors: Wald **[72]**, Blackwell **[10]**, Doob **[22]**, Chow, Robbins and Teicher **[18]**, Shepp **[55]**, and others.

Theorem 7 in the case $N = 2$ was obtained by Wald **[72]**. In the general case it was obtained by Hoeffding (communicated by him to the author in 1965). The proof given here is contained in the article by Simons **[57]**. An analogous proof is presented in the book by Bechhofer, Kiefer and Sobel **[8]** (Theorem 3.5.1), where formula (4.117) is also given.

§ 3. The disruption problem was first investigated in a report by A.N. Kolmogorov and the author at the Sixth All-Union Conference on Probability Theory and Mathematical Statistics (Vilnius, 1960). The results are contained in **[58]**, **[60]** and **[62]**.

§ 4. The disruption problem for a Wiener process was investigated by Širjaev in **[59]**,**[60]**, **[63]** and **[66]**. Other formulations of problems of quickest detection of the time of appearance of a disruption were also considered in these works. The disruption problem was also analyzed by Stratonovič **[69]** and Bather **[7]**.

Bibliography

[1] S. A. Aĭvazjan, *A comparison of the optimal properties of the Neyman-Pearson and the Wald sequential probability ratio tests,* Teor. Verojatnost. i Primenen. **4**(1959), 86–93 = Theor. Probability Appl. **4**(1959), 83–88. MR **21** #942.

[2] P. S. Aleksandrov, *Introduction to the general theory of sets and functions,* GITTL, Moscow, 1948; German transl., VEB Deutscher Verlag, Berlin, 1956. MR **12,** 682; **18,** 22.

[3] K. J. Arrow, D. Blackwell and M. A. Girshick, *Bayes and minimax solutions of sequential decision problems,* Econometrica **17** (1949), 213–214. MR **11,** 261.

[4] R. R. Bahadur, *Sufficiency and statistical decision functions,* Ann. Math. Statist. **25** (1954), 423–462. MR **16,** 154.

[5] A. E. Bašarinov and B. S. Fleĭšman, *Methods of statistical sequential analysis and its application to radio engineering,* Izdat. "Soviet Radio", Moscow, 1962. (Russian)

[6] J. A. Bather, *Bayes procedures for deciding the sign of a normal mean,* Proc. Cambridge Philos. Soc. **58** (1962), 599–620. MR **27** #869.

[7] _____, *On a quickest detection problem,* Ann. Math. Statist. **38** (1967), 711–724. MR **35** #2679.

[8] R. E. Bechhofer, J. Kiefer and M. Sobel, *Sequential identification and ranking procedures, with special reference to Koopman-Darmois populations,* Statistical Research Monographs, vol. 3, Univ. of Chicago Press, Chicago, Ill., 1968, MR **39** #6445.

[9] R. E. Bellman, *Dynamic programming,* Princeton Univ. Press, Princeton, N. J., 1957; Russian transl., IL, Moscow, 1960. MR **19,** 820; **22** #4898.

[10] D. H. Blackwell, *On an equation of Wald,* Ann. Math. Statist. **17** (1946), 84–87. MR **8,** 478.

[11] D. H. Blackwell and M. A. Girshick, *Theory of games and statistical decisions,* Wiley, New York; Chapman & Hall, London, 1954; Russian transl., IL, Moscow, 1958. MR **16,** 1135.

[12] R. M. Blumental and R. K. Getoor, *Markov processes and potential theory,* Pure and Appl. Math. vol. 29, Academic Press, New York and London, 1968, MR **41** #9348.

[13] D. L. Burkholder and R. A. Wijsman, *Optimum properties and admissibility of sequential tests,* Ann. Math. Statist. **34** (1963), 1-17. MR **26** #5668.

[14] H. Chernoff, *Sequential tests for the mean of a normal distribution,* Proc. Fourth Berkeley Sympos. Math. Statist. and Probability, vol. 1, Univ. of California Press, Berkeley, Calif., 1961, pp. 79-91. MR **24** #A1788.

[15] Y. S. Chow and H. Robbins, *A martingale system theorem and applications,* Proc. Fourth Berkeley Sympos. Math. Statist. and Probability, vol. 1, Univ. of California Press, Berkeley, Calif., 1961, pp. 93–104. MR **24** #A2433.

[16] _____, *On optimal stopping rules,* Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1963), 33–49. MR **28** #698.

[17] _____, *On values associated with a stochastic sequence,* Proc. Fifth Berkeley Sympos. Math. Statist. and Probability, vol. 1: Statistics, Univ. of California Press, Berkeley, Calif., 1967, pp. 427–440. MR **35** #6193.

[18] Y. S. Chow, H. Robbins and H. Teicher *Moments of randomly stopped sums,* Ann. Math. Statist. **36** (1965), 789–799. MR **31** #2777.

[19] Y. S. Chow, S. Moriguti, H. Robbins and S. M. Samuels, *Optimal selection based on relative rank (the "secretary problem")* Israel J. Math. **2** (1964), 81–90. MR **31** #855.

[20] P. Courrège and P. Priouret, *Temps d'arrêt d'une fonction aléatoire: Relations d'équivalence*

associeés et propriétés de décomposition, Publ. Inst. Statist. Univ. Paris **14** (1965), 245–274. MR **36** #4640.

[21] J. Dieudonné, *Fondements de l'analyse moderne*, Pure and Appl. Math., vol. 10, Academic Press, New York and London, 1960; Russian transl., "Mir", Moscow, 1964. MR **22** #11074.

[22] J. Doob, *Stochastic processes*, Wiley, New York; Chapman & Hall, London, 1953; Russian transl., IL, Moscow, 1956. MR **15**, 445; **19**, 71.

[23] A. Dvoretzky, J. Kiefer and J. Wolfowitz, *Sequential decision processes with continuous time parameter. Testing hypotheses*, Ann. Math. Statist. **24** (1953), 254–264. MR **14**, 997; 1279.

[24] E. B. Dynkin, *Foundations of the theory of Markov processes*, Fizmatgiz, Moscow, 1959; English transl., Prentice-Hall, Englewood Cliffs, N.J.; Pergamon Press, Oxford, 1961. MR **24** #A1745; #A1746; #A1747.

[25] _____, *Markov processes*, Fizmatgiz, Moscow, 1963; English transl., Die Grundlehren der math. Wissenschaften, Band 121, Academic Press, New York; Springer-Verlag, Berlin, 1965, MR **33** #1886; #1887.

[26] _____, *The optimum choice of the instant for stopping a Markov process*, Dokl. Akad. Nauk SSSR **150** (1963), 238–240 = Soviet Math. Dokl. **4** (1963), 627–629. MR **27** #4278.

[27] _____, *Sufficient statistics for the optimal stopping problem*, Teor. Verojatnost. i. Primenen. **13** (1968), 150–152 = Theor. Probability Appl. **13** (1968), 152–153. MR **36** #7227.

[28] E. B. Dynkin and A. A. Juškevič, *Theorems and problems in Markov processes*, "Nauka", Moscow, 1967; English transl., Plenum Press, New York, 1969. MR **36** #6006; 39 #3585 a.

[29] M. Gardner, *Mathematical games*, Sci. Amer. **202** (1960), no. 2, 150–156; ibid. **202** (1960), no. 3, 173–182.

[30] *I. I. Gihman and A. V. Skorohod, *Introduction to the theory of random processes*, "Nauka", Moscow, 1965; English transl., Saunders, Philadelphia, Pa., 1969. MR **33** #6689; **40** #923.

[31] _____, *Stochastic differential equations*, "Naukova Dumka", Kiev, 1968. (Russian) MR **41** 7777.

[32] J. P. Gilbert and F. Mosteller, *Recognizing the maximum of a sequence*, J. Amer. Statist. Assoc. **61** (1966), 35–73. MR **33** #6792.

[33] L. M. Graves, *The theory of functions of real variables*, McGraw-Hill, New York, 1946. MR **8**, 319.

[34] B. I. Grigelionis, *The optimal stopping of Markov processes*, Litovsk. Mat. Sb. **7** (1967), 265–279; English transl., Selected Transl., Math. Statist. and Probability, vol. 11, Amer. Math. Soc., Providence, R. I. (to appear). MR **38** #2838.

[35] _____, *Conditions for the uniqueness of the solution of Bellman's equation*, Litovsk. Mat. Sb. **8** (1968), 47–52. (Russian) MR **39** #5206.

[36] _____, *Sufficiency in optimal stopping problems*, Litovsk. Mat. Sb. **9** (1969), 471–480. (Russian)

[37] B. I. Grigelionis and A. N. Širjaev, *On the Stefan problem and optimal stopping rules for Markov processes*, Teor. Verojatnost. i Primenen. **11** (1966), 612–631 = Theor. Probability Appl. **11** (1966), 541–558. MR **35** #7538.

[38] S. M. Guseĭn-Zade, *The problem of choice and the optimal stopping rule for a sequence of independent trials*, Teor. Verojatnost. i Primenen. **11** (1966), 534–537 = Theor. Probability Appl. **11** (1966), 472–475. MR **34** #2129.

[39] G. W. Haggstrom, *Optimal stopping and experimental design*, Ann. Math. Statist. **37** (1966), 7–29. MR **33** #3424.

[40] J. A. Hunt, *Markov processes and potentials*. I, II, III, Illinois J. Math. **1** (1957), 44–93, 316–369; ibid. **2** (1958), 153–213; Russian transl., IL, Moscow, 1962. MR **19**, 951; **21** #5824.

[41] A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, Berlin, 1933; Russian transl., ONTI, Moscow, 1936; English transl., Chelsea, New York, 1948. MR **11**, 374; **18**, 155.

[42] A.N. Kolmogorov and S. V. Fomin, *Elements of the theory of functions and of functional analysis*, "Nauka", Moscow, 1968; English transl. of 1st ed., Graylock Press, Albany, N. Y.; Academic Press, New York, 1961. MR **38** #2559.

Editor's note*. All page references to **[30] are to the English translation.

[43] N. V. Krylov, *On optimal stopping of a control circuit*, Optimal Control and Information Theory, Abstracts of Reports at the Seventh All-Union Conf. Theory of Probability and Math Statist. (Tbilisi, 1963), Izdat. Inst. Mat. Akad. Nauk. Ukrain. SSR, Kiev, 1963, pp. 11–15. (Russian)

[44] S. Kullback, *Information theory and statistics*, Wiley, New York, 1959; Russian transl., "Nauka", Moscow, 1967, MR 21 #2325.

[45] E. Lehmann, *Testing statistical hypothesis*, Wiley, New York; Chapman & Hall, London, 1959; Russian transl., "Nauka", Moscow, 1964. MR **21** #6654.

[46] D. V. Lindley, *Dynamic programming and decision theory*, Appl. Statist. **10** (1961), 39–51. MR **23** #A740.

[47] R. Š. Lipcer and A. N. Širjaev *Nonlinear filtering of Markov diffusion processes*, Trudy Mat. Inst. Steklov. **104** (1968), 135–180 = Proc. Steklov Inst. Math. **104** (1968), 163–218. MR **41** #6342.

[48] M. Loève, *Probability theory. Foundations. Random sequences*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960; Russian transl., IL, Moscow, 1962. MR **23** A670; **25** #3551.

[49] P. A. Meyer, *Probabilités et potentiel*, Publ. Inst. Math. Univ. Strasbourg, no. 14, Actualités Sci. Indust., no. 1318, Hermann, Paris, 1966. MR **34** #5118

[50] _____, *Processus de Markov*: *La frontière de Martin*, Lecture Notes in Math., no. 77, Springer-Verlag, Berlin and New York, 1968. MR **39** #7669.

[51] V. S. Mihalevič, *Bayesian choice between two hypotheses for the mean value of a normal process*, Vīsnik Kiïv. Unīv. **1** (1958), no. 1, 101–104. (Ukrainian)

[52] I. P. Natanson, *Theory of functions of a real variable*, 2nd rev. ed., GITTL, Moscow, 1957; English transl., Ungar, New York, 1961. MR **26** #6309

[53] S. N. Ray, *Bounds on the maximum sample size of a Bayes sequential procedure*, Ann. Math. Statist. **36** (1965), 859–878. MR **30** #5446.

[54] L. I. Rubinšteĭn, *The Stefan problem*, Latvian State University Computing Center, Izdat. "Zvaĭgzne", Riga, 1967; English transl., Transl. Math. Monographs, vol. 27, Amer. Math. Soc., Providence, R. I., 1971. MR **36** #5488.

[55] L. A. Shepp, A *first passage problem for the Wiener process*, Ann. Math. Statist. **38** (1967), 1912–1914. MR **36** #968.

[56] D. O. Siegmund, *Some problems in the theory of optimal stopping rules*, Ann. Math. Statist. **38** (1967), 1627–1640. MR **36** #4718.

[57] G. Simons, *Lower bounds for average sample number of sequential multihypothesis tests*, Ann. Math. Statist. **38** (1967), 1343–1364. MR **36** #1037.

[58] A. N. Širjaev, *The detection of spontaneous effects*, Dokl. Akad. Nauk SSSR **138** (1961), 799–801 = Soviet Math. Dokl. **2** (1961), 740–743. MR **24** #A2504.

[59] _____, *The problem of the most rapid detection of a disturbance in a stationary process*, Dokl. Akad. Nauk SSSR **138** (1961), 1039–1042 = Soviet Math. Dokl. **2** (1961), 795–799. MR **24** #A2505.

[60] _____, *Optimal methods in quickest detection problems*, Teor. Verojatnost. i Primenen. **8** (1963), 26–51 = Theor. Probability Appl. **8** (1963), 22–45, MR **27** #5642.

[61] _____, *On the theory of decision functions and control of a process of observation based on incomplete information*, Trans. Third Prague Conf. Information Theory, Statistical Decision Functions, Random Processes (Liblice, 1962), Publ. House Czech. Acad. Sci., Prague, 1964, pp. 657–681; English transl., Selected Transl. Math. Statist. and Probability, vol. 6, Amer. Math. Soc., Providence, R. I., 1966, pp. 162–188. MR **30** #1591.

[62] _____, *On Markov sufficient statistics in non-additive Bayes problems of sequential analysis*, Thor. Verojatnost. i Primenen. **9** (1964), 670–686 = Theor. Probability Appl. **9** (1964), 604–618. MR **30** #350.

[63] _____, *Some explicit formulae in a problem on "disorder"*, Teor. Verojatnost. i Primenen. **10** (1965), 380–385 = Theor. Probability Appl. **10** (1965), 348–353. MR **32** #2263.

[64] _____, *Stochastic equations of non-linear filtering of jump-like Markov processes*, Problemy Peredači Informacii **2** (1966), vyp. 3, 3–22. (Russian) MR **34** #4045.

[65] _____, *Some new results in the theory of controlled random processes*, Trans. Fourth Prague

Conf. Information Theory, Statistical Decision Functions, Random Processes (Prague, 1965), Academia, Prague, 1967, pp. 131–203; English transl., Selected Transl. Math. Statist. and Probability, vol. 8, Amer. Math. Soc., Providence, R.I., 1970, pp. 49–130. MR **35** #4066; 37 #1186.

[66] _____, *On two problems of sequential analysis,* Kibernetika (Kiev) **1967**, no. 2, 79–80. (Russian)

[67] A. V. Skorohod, *Studies in the theory of random processes,* Izdat. Kiev. Univ., Kiev, 1961; English transl., Addison-Wesley, Reading, Mass., 1965. MR **32** #3082a,b.

[68] J. L. Snell, *Applications of martingale system theorems,* Trans. Amer. Math. Soc. **73** (1952), 293–312. MR **14**, 295.

[69] R. L. Stratonovič, *Conditional Markov processes and their application to the theory of automatic control,* Izdat. Moskov. Univ., Moscow, 1966; English transl., American Elsevier, New York, 1967. MR **33** #5391.

[70] H. M. Taylor, *Optimal stopping in a Markov process,* Ann. Math. Statist. **39** (1968), 1333–1344. MR **38** #769.

[71] A. Wald, *Statistical decision functions,* Wiley, New York; Chapman & Hall, London, 1950; Russian transl. in *Theory of games: Positional games,* "Nauka", Moscow, 1967, pp. 300–522. MR **12**, 193: **36** #4704.

[72] _____, *Sequential analysis,* Wiley, New York; Chapman & Hall, London, 1947; Russian transl., Fizmatgiz. Moscow, 1960. MR **8**, 593; **22** #7228.

[73] A. Wald and J. Wolfowitz, *Optimum character of the sequential probability ratio test,* Ann. Math. Statist. **19** (1948), 326–339 MR **10**, 201.

[74] _____, *Bayes solutions of sequential decision problems,* Ann. Math. Statist. **21** (1950), 82–99. MR **11**, 529.

[75] G. B. Wetherill, *Sequential methods in statistics,* Methuen, London, 1966. MR **35** #7504.

[76] P. Whittle, *Some general results in sequential design* (*with discussion*), J. Roy. Statist. Soc. Ser. B **27** (1965), 371–394. MR **34** #2133.

ISBN 0-8218-1588-1