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Providence, Rhode Island

УРАВНЕНИЯ В СВЕРТКАХ
И ПРОЕКЦИОННЫЕ МЕТОДЫ ИХ РЕШЕНИЯ

И. Ц. ГОХБЕРГ, И. А. ФЕЛЬДМАН

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PREFACE TO THE ENGLISH TRANSLATION

The present edition differs significantly in one part from the Russian one. We have in mind the end of the third chapter, where methods of inverting finite Toeplitz matrices and their continuous analogues are set forth. The last two sections of Chapter III (§§ 6 and 7) of the Russian edition are replaced by three new sections (§§ 6,7 and 8). The new presentation is more complete and is also distinguished by greater generality and simplicity. In addition, addenda are inserted in "Notes and guide to the literature" and in the bibliography. These addenda reflect publications appearing after the publication of the Russian edition.

We express thanks to the American Mathematical Society, to its Editor of Translations, Dr. S. H. Gould, and to the translator for translating the book into the English language.

We are sincerely grateful to Professor E. Hewitt for attention to the translation of our book.

The Authors

Kishinev
June 24, 1972

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Dedicated to
Mark Grigor'evič
KREĬN

PREFACE

The first profound results concerning integral equations on a semiaxis, with kernels depending on the difference of the arguments, were obtained in 1931 by N. Wiener and E. Hopf. After the appearance of their classical paper [1] such equations were given the name Wiener-Hopf equations.

By the present time the theory of Wiener-Hopf equations has been quite fully developed. A great many mathematicians (mainly Soviet) participated in its elaboration. We mention only M. G. Kreĭn's fundamental paper [4]. Beginning with that paper the ideas and methods of functional analysis have had extensive application in the analysis of Wiener-Hopf equations.

The presentation adopted in this book is also based on functional analysis. It has its origins in a distinctive operational calculus.

This route leads naturally to a definite class of convolution equations which contains in particular the original Wiener-Hopf integral equations. Besides that, discrete and difference analogues, the so-called pair equations, singular integral equations on the circle, etc. belong here.

Various projection methods of solving convolution equations make up a significant part of the book. Justification of these methods is also obtained within the framework of the general scheme.

To a considerable extent the presentation is based on papers of the authors which were published in 1963–1967.

The book was conceived in 1964. At that time the authors began reading special courses at Kishinev University. Preliminary publication of the mimeographed brochure "Projection Methods of Solving Wiener-Hopf Equations" (Kishinev, 1967; MR 37 # 1915) helped us substantially in the work on the book.

We assume that the reader is acquainted with the elements of the theory of operators in Hilbert and Banach spaces and the theory of Banach algebra.

The authors sincerely thank M. G. Kreĭn for support and abiding interest in this paper, as well as N. Ja. Krupnik, A. S. Markus, A. A. Semencul and I. B. Simonenko for discussion of various questions and valuable comments.

The authors express sincere gratitude to the book's editor, F. V. Širokov. His assistance contributed significantly to the simplicity and clarity of the presentation.

Kishinev
February 18, 1970

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Appendix

Asymptotics of Solutions of Homogeneous Convolution Equations

Here we take up the asymptotics of solutions of a system of homogeneous integral equations which is written in vector-matrix notation as

$$\begin{aligned}
\varphi(t) - \int_0^\infty k_{11}(t-s)\varphi(s)\,ds - \int_{-\infty}^0 k_{12}(t-s)\varphi(s)\,ds = 0 \qquad (0 < t < \infty),\\
\varphi(t) - \int_0^\infty k_{21}(t-s)\varphi(s)\,ds - \int_{-\infty}^0 k_{22}(t-s)\varphi(s)\,ds = 0 \qquad (-\infty < t < 0), \quad (1)
\end{aligned}$$

where $\varphi(t)$ is the n-dimensional vector-function sought, and the $k_{jr}(t), j, r = 1, 2$, are nth order matrix functions. System (1) is a generalization of both pair equation systems (the case when $k_{11} = k_{12}$ and $k_{21} = k_{22}$) and their transpose systems (the case when $k_{11} = k_{21}$ and $k_{12} = k_{22}$).

The first section is auxiliary; certain properties of the operator defined by the left side of equation (1) are established in it. By means of these properties we prove that equation (1) has the same solutions in each space (of some class).

Asympotic expansions of solutions of equation (1) in exponential functions with polynomial coefficients are found in § 2. Several general theorems concerning the operators and results of § 1 play a vital role in this connection. The asymptotics of solutions of a system of Wiener-Hopf equations, a system of pair integral equations, and its transpose system are obtained as a corollary.

More precise results are obtained in §§ 3 and 4 for the transpose equation of a pair and for a system of Wiener-Hopf equations. Here we manage, in addition, to determine the polynomial coefficients in the principal term of the asymptotics for functions forming a basis of the kernel of the corresponding operator.

All results of this appendix can be carried over to the discrete case.

§ 1. Some auxiliary propositions

1. In the sequel E and $\tilde{E}$ denote the Banach spaces introduced in § 8 of Chapter I. The notation $L = L_1(0, \infty)$ and $\tilde{L} = L_1(-\infty, \infty)$ is also utilized.

Let $\mathfrak{B}$ be some Banach space of functions defined on the real axis (semiaxis), and let $\sigma(t)$ be a continuous function which does not vanish on that axis (semiaxis).

Let us agree to write $f(t) \in \sigma(t)\mathfrak{B}$ if $\sigma^{-1}(t)f(t) \in \mathfrak{B}$. The set $\sigma(t)\mathfrak{B}$ is a Banach space with norm $|f| = |\sigma^{-1}f|_{\mathfrak{B}}$.

As in Chapter VIII, by $\mathfrak{B}_n$ and $\mathfrak{B}_{n\times n}$ we denote respectively the set of all n-dimensional vectors and the set of all nth order matrices with elements from $\mathfrak{B}$. The set $\mathfrak{B}_n$ becomes a Banach space if the norm of an element $f = \{f_1, \cdots, f_n\} \in \mathfrak{B}_n$ is defined by, for example, the equality

$$|f| = \sum_1^n |f_j|_{\mathfrak{B}}.$$

The spaces $\sigma(t)\mathfrak{B}_n$ and $\sigma(t)\mathfrak{B}_{n\times n}$ are defined analogously.

Everywhere in the sequel h denotes some fixed positive number. If an element of $e^{-h|t|}\tilde{L}_{n\times n}$ or $e^{-h|t|}\tilde{L}_n$ is denoted by a small letter, its Fourier transform is denoted by the corresponding capital letter.

The definitions and notation introduced in § 11 of Chapter I are also utilized in the sequel.

2. It follows easily from Theorem 6.1 of Chapter VIII that if the matrix function $k(t)$ belongs to $e^{-h|t|}\tilde{L}_{n\times n}$, then the condition

$$\det(I - K(\lambda + ih)) \neq 0 \qquad (-\infty < \lambda < \infty) \tag{1.1}$$

is necessary and sufficient for the operator

$$(I - \mathscr{K})\varphi = \varphi(t) - \int_0^\infty k(t-s)\varphi(s)\,ds \qquad (0 < t < \infty)$$

to be a Φ-operator in each of the spaces $e^{ht}E_n$. If condition (1.1) is fulfilled, the index[1] of the operator $I - \mathscr{K}$ in each of the spaces $e^{ht}E_n$ is calculated from the formula

$$\kappa(I - \mathscr{K}) = -\operatorname{ind}\det(I - K(\lambda + ih)). \tag{1.2}$$

Let us now consider in the space $e^{h|t|}\tilde{E}_n$ the equation

$$\begin{gathered}\varphi(t) - \int_0^\infty k_{11}(t-s)\varphi(s)\,ds - \int_{-\infty}^0 k_{12}(t-s)\varphi(s)\,ds = g(t) \\ (0 < t < \infty), \\ \varphi(t) - \int_0^\infty k_{21}(t-s)\varphi(s)\,ds - \int_{-\infty}^0 k_{22}(t-s)\varphi(s)\,ds = g(t) \\ (-\infty < t < 0),\end{gathered} \tag{1.3}$$

where $k_{jr}(t) \in e^{-h|t|}\tilde{L}_{n\times n}$.

The space $e^{h|t|}\tilde{E}_n$ is isomorphic to the space F consisting of all pairs $\hat{f}(t) = \{f_1(t), f_2(t)\}$ of vector functions belonging to $e^{ht}E_n$ ($f_j(t) \in e^{ht}E_n, j = 1, 2$) with norm $|\hat{f}| = |f_1| + |f_2|$.

Under this isomorphism a pair $\hat{f} = \{f_1, f_2\} \in F$ is put into correspondence with

[1] That is, dim Ker $(I - \mathscr{K})$ − dim Ker $(I - \mathscr{K}^*)$.

each vector $f(t) \in e^{h|t|}\tilde{E}_n$ in accordance with the rule $f_1(t) = f(t), f_2(t) = f(-t)$, $0 < t < \infty$. In this notation, equation (1.3) can be written in the form

$$\varphi_1(t) - \int_0^\infty k_{11}(t-s)\varphi_1(s)\,ds - \int_0^\infty k_{12}(t+s)\varphi_2(s)\,ds = g_1(t),$$
$$\varphi_2(t) - \int_0^\infty k_{21}(-t-s)\varphi_1(s)\,ds - \int_0^\infty k_{22}(s-t)\varphi_2(s)\,ds = g_2(t)$$

$(0 < t < \infty)$. Let us write the operator B defined by the left side of this system in the form of a matrix[2)]

$$B = \left\| \begin{matrix} I - \mathscr{K}_{11} & -\hat{\mathscr{K}}_{12} \\ -\hat{\mathscr{K}}_{21} & I - \mathscr{K}'_{22} \end{matrix} \right\|.$$

For any matrix function $k(t)$ of $L_{n\times n}$ the operator

$$(\hat{\mathscr{K}}\varphi)(t) = \int_0^\infty k(t+s)\varphi(s)\,ds$$

is completely continuous in each of the spaces E_n. It is sufficient to show this for the case $n = 1$. If a function $k(t) \in L_1(0, \infty)$ has the form $k(t) = e^{-t}p(t)$, $0 < t < \infty$, where $p(t)$ is a polynomial, then, as is easy to see, the operator $\hat{\mathscr{K}}$ is finite-dimensional in E. Since the set of all functions of the form $e^{-t}p(t)$ is dense in $L_1(0, \infty)$, the operator $\hat{\mathscr{K}}$ is completely continuous, whatever the function $k(t)$ may be.

It is not difficult to obtain by means of this the fact that the operator

$$T = \left\| \begin{matrix} 0 & -\hat{\mathscr{K}}_{12} \\ -\hat{\mathscr{K}}_{21} & 0 \end{matrix} \right\|$$

is completely continuous in the space F. Consequently the operator

$$B = \left\| \begin{matrix} I - \mathscr{K}_{11} & 0 \\ 0 & I - \mathscr{K}'_{22} \end{matrix} \right\| + T$$

is a Φ-operator if and only if the operators $I - \mathscr{K}_{11}$ and $I - \mathscr{K}'_{22}$ are. The latter is equivalent to fulfilling the conditions

$$\det(I - K_{11}(\lambda + ih)) \neq 0, \qquad \det(I - K_{22}(\lambda - ih)) \neq 0 \qquad (1.4)$$
$$(-\infty < \lambda < \infty).$$

If conditions (1.4) are fulfilled, then $\kappa(B) = \kappa(I - \mathscr{K}_{11}) + \kappa(I - \mathscr{K}'_{22})$. But by virtue of (1.2)

$$\kappa(I - \mathscr{K}_{11}) = -\operatorname{ind}\det(I - K_{11}(\lambda + ih)),$$
$$\kappa(I - \mathscr{K}'_{22}) = -\operatorname{ind}\det\left(I - \int_{-\infty}^{\infty} k_{22}(-t)e^{-ht}e^{i\lambda t}\,dt\right)$$
$$= \operatorname{ind}\det(I - K_{22}(\lambda - ih)).$$

2) For a function $k(t)$, $-\infty < t < \infty$, the integral operators on the semiaxis with kernels $k(t-s)$, $k(t+s)$ and $k(s-t)$ are denoted by the symbols $\mathscr{K}$, $\hat{\mathscr{K}}$ and $\mathscr{K}'$.

Thus the index of B is calculated from the formula

$$\kappa(B) = \text{ind det } (I - K_{22}(\lambda - ih)) - \text{ind det } (I - K_{11}(\lambda + ih)).$$

Since under the aforementioned isomorphism between the spaces $e^{h|t|}\tilde{E}_n$ and F the operator B corresponds to the operator A defined by the left side of equation (1.3), the following lemma has been established.

LEMMA 1.1. *Let the matrix functions $k_{jr}(t)$ belong to $e^{-h|t|}\tilde{L}_{n\times n}$ ($j, r = 1,2$). For an operator A defined by the left side of equation* (1.3) *to be a Φ-operator in each of the spaces $e^{h|t|}\tilde{E}_n$, it is necessary and sufficient that conditions* (1.4) *be fulfilled.*

Under fulfillment of these conditions the index of A in each of the spaces $e^{h|t|}\tilde{E}_n$ is calculated from the formula

$$\kappa(A) = \text{ind det}(I - K_{22}(\lambda - ih)) - \text{ind det } (I - K_{11}(\lambda + ih)). \tag{1.5}$$

REMARK. If the matrix functions $k_{jr}(t)$ of $e^{-h|t|}\tilde{L}_{n\times n}$ satisfy the conditions

$$\det(I - K_{rr}(\lambda \pm ih)) \neq 0 \qquad (-\infty < \lambda < \infty;\ r = 1, 2),$$

then

$$\kappa(A) = \kappa_{-h}(A) + m_1 = \kappa^h(A) + m_2, \tag{1.6}$$

where $\kappa_{\pm h}(A)$ is the index of A in the space $e^{\pm ht}\tilde{E}_n$, and m_r ($r = 1, 2$) is the number of zeros, with regard for their multiplicities, of the function det $(I - K_{rr}(\lambda))$ in the strip $|\text{Im } \lambda| < h$.

Indeed, formula (1.5) implies that

$$\kappa(A) = \text{ind } \frac{\det(I - K_{22}(\lambda - ih))}{\det(I - K_{11}(\lambda - ih))} + \text{ind } \frac{\det(I - K_{11}(\lambda - ih))}{\det(I - K_{11}(\lambda + ih))}.$$

It is not difficult to observe that the first addend of the right side of the latter equality equals $\kappa_{-h}(A)$, while the second, by virtue of Rouché's theorem, equals m_1 since the function det $(I - K_{11}(\lambda))$ is analytic in the strip $|\text{Im } \lambda| < h$ and does not vanish on the boundary. The second equality of (1.6) is established analogously.

We need the following proposition in the sequel.

1°. *Let the Banach space $\mathfrak{B}_1$ be contained in the Banach space $\mathfrak{B}$ and be dense in it, and let A_1 and A be bounded Φ-operators with equal indices which operate in $\mathfrak{B}_1$ and $\mathfrak{B}$, respectively. If A is a continuation of A_1, then* Ker A = Ker A_1.

PROOF. The inequality $\alpha(A_1) \leqq \alpha(A)$[3] holds since $\mathfrak{B}_1 \subset \mathfrak{B}$. We obtain the relation $\beta(A_1) \leqq \beta(A)$ from the inclusion $\mathfrak{B}^* \subset \mathfrak{B}_1^*$, which is a consequence of the fact that $\mathfrak{B}_1$ is dense in $\mathfrak{B}$. The inequalities just obtained, together with the equality $\kappa(A) = \kappa(A_1)$, reduce to the equality $\alpha(A) = \alpha(A_1)$, from which the assertion follows.

[3] For the notation, cf. § 11 of Chapter I.

THEOREM 1.1. *Let the matrix functions $k_{jr}(t)$ belong to $e^{-h|t|}\tilde{L}_{n\times n}$, $j, r = 1,2$, and satisfy condition* (1.4). *Then equation* (1) *has the same solutions in each of the spaces $e^{h|t|}\tilde{E}$.*

PROOF. Proposition 1° and Lemma 1.1 imply immediately that equation (1) has the same solutions in the spaces $e^{h|t|}(\tilde{L}_p)_n$ $(p \geqq 1)$ and $e^{h|t|}(\tilde{C}_0)_n$, since the intersection of these spaces is dense in each of them. Now let $\varphi(t)$ be a solution of equation (1) which belongs to $e^{h|t|}\tilde{M}_n$. Then each coordinate of the vector function

$$\psi(t) = \int_{-\infty}^{0} k_{12}(t-s)\varphi(s)\,ds$$

is the sum of n functions of the form

$$\omega(t) = \int_{-\infty}^{0} k(t-s)\chi(s)\,ds,$$

where $k(t) \in e^{-h|t|}\tilde{L}$ and $\chi(t) \in e^{h|t|}\tilde{M}$. Considering the function $\omega(t)$ for $t > 0$, we obtain $\omega(t) \in e^{-ht}C_0$. As a matter of fact, by setting $k(t) = \hat{k}(t)e^{-ht}$ and $\chi(t) = \hat{\chi}(t)e^{-ht}$, we obtain

$$\begin{aligned} \left|e^{ht}\omega(t)\right| &\leqq \int_{-\infty}^{0} \left|\hat{k}(t-s)\hat{\chi}(s)\right| ds \\ &\leqq \sup_{-\infty<s<0} \left|\hat{\chi}(s)\right| \int_{t}^{\infty} \left|\hat{k}(s)\right| ds \to 0 \qquad (t \to \infty), \end{aligned}$$

and the function $\omega(t)$ is continuous (cf. Gel'fand, Raĭkov and Šilov **[1]**, § 16, Lemma).

Thus the vector function $\psi(t)$ belongs to $e^{-ht}(C_0)_n$ $(t > 0)$, and a fortiori to the space $e^{ht}(C_0)_n$. Let us write the first equality of (1) in the form

$$\hat{\varphi}(t) - \int_{0}^{\infty} \hat{k}_{11}(t-s)\hat{\varphi}(s)\,ds = \hat{\psi}(t) \qquad (0 < t < \infty), \tag{1.7}$$

where $\hat{\psi}(t) = e^{-ht}\varphi(t)$, $\hat{k}_{11}(t) = e^{-ht}k_{11}(t)$ and $\hat{\chi}(t) = e^{-ht}\psi(t)$. Equation (1.7) is obviously solvable in the space M_n. Since $\hat{\psi}(t) \in (C_0)_n$, we obtain easily from Theorems 4.2 and 6.1, Chapter VIII, the fact that every solution of (1.7) which belongs to M_n also belongs to $(C_0)_n$. Therefore $\varphi(t) \in e^{ht}(C_0)_n$ $(t > 0)$.

The relation $\varphi(-t) \in e^{ht}(C_0)_n$ $(t > 0)$ is established analogously. Thus $\varphi(t) \in e^{h|t|}(\tilde{C}_0)_n$; consequently $\varphi(t)$ belongs to any space $e^{h|t|}\tilde{E}_n$.

§ 2. Asymptotic expansions of solutions of the general equation

1. Let the matrix functions $k_j(t)$ belong to $e^{-h|t|}\tilde{L}_{n\times n}$, $j = 1, 2$. Let us consider the equation

$$\varphi(t) - \int_0^\infty k_1(t-s)\varphi(s)\,ds - \int_{-\infty}^0 k_2(t-s)\varphi(s)\,ds \tag{2.1}$$

in the space $e^{h|t|}\tilde{L}_n$ $(-\infty < t < \infty)$.

Let us show that the equation (2.1) is equivalent to the boundary value problem

$$\left.\begin{aligned} [I - K_1(\lambda + ih)]\Phi_+(\lambda + ih) &= \Omega(\lambda + ih), \\ -[I - K_2(\lambda - ih)]\Phi_-(\lambda - ih) &= \Omega(\lambda - ih) \end{aligned}\right\} \quad (-\infty < \lambda < \infty), \tag{2.2}$$

where

$$\Phi_+(\lambda + ih) = \int_0^\infty \varphi(t)e^{-ht}e^{i\lambda t}dt, \qquad \Phi_-(\lambda - ih) = \int_{-\infty}^0 \varphi(t)e^{ht}e^{i\lambda t}dt, \tag{2.3}$$

and the vector function $\Omega(\lambda)$ is the Fourier transform of some vector function of $e^{-h|t|}\tilde{L}_n$.

Indeed, let us denote by P the projection defined in the space $e^{h|t|}\tilde{L}_n$ by the equality

$$(P\varphi)(t) = \begin{cases} \varphi(t), & 0 < t < \infty, \\ 0, & -\infty < t < 0, \end{cases}$$

and let us set $Q = I - P$. Let us rewrite equation (2.1) in the following form:

$$(P\varphi)(t) - \int_{-\infty}^\infty k_1(t-s)(P\varphi)(s)\,ds = -(Q\varphi)(t) + \int_{-\infty}^\infty k_2(t-s)(Q\varphi)(s)\,ds.$$

It is easy to verify that the left side of the latter equality is a vector function of $e^{ht}\tilde{L}_n$, and the right, of $e^{-ht}\tilde{L}_n$. Let us denote their common value by $\omega(t)$. The vector function $\omega(t)$ obviously belongs to $e^{-h|t|}\tilde{L}_n$. We obtain (2.2) by applying the Fourier transform to the equalities

$$e^{-ht}(P\varphi)(t) - e^{-ht}\int_{-\infty}^\infty k_1(t-s)(P\varphi)(s)ds = e^{-ht}\omega(t),$$

$$-e^{ht}(Q\varphi)(t) + e^{ht}\int_{-\infty}^\infty k_2(t-s)(Q\varphi)(s)ds = e^{ht}\omega(t).$$

The vector functions $\Phi_\pm(z)$ and $\Omega(z)$ admit continuation to vector functions which are holomorphic respectively in the regions $\operatorname{Im} z > h, \operatorname{Im} z < -h$ and $|\operatorname{Im} z| < h$ and continuous there, including the boundaries. Equalities (2.2) represent an n-dimensional homogeneous Hilbert problem with respect to the contour consisting of the pair of parallel lines $z = \lambda \pm ih$, $-\infty < \lambda < \infty$.

We shall henceforth write the vector functions $\{a_1(t),\cdots, a_n(t)\}$ concisely as $\{a_r(t)\}_1^n$.

THEOREM 2.1. *Let a matrix function $k(t) \in e^{-h|t|}\tilde{L}_{n\times n}$ possess the property*

$$\det (I - K(\lambda \pm ih)) \neq 0 \qquad (-\infty < \lambda < \infty) \tag{2.4}$$

and let the α_j, $j = 1,\cdots,m$, *be all of the distinct zeros of the function* $\det (I - K(\lambda))$ *in the strip* $|\operatorname{Im} \lambda| < h$, *and the* P_j, *their multiplicities. Then the vector functions*

$$\varphi_{jq}(t) = \{e^{-i\alpha_j t} P_{jqr}(t)\}_{r=1}^{n} \qquad (j = 1,\cdots, m; q = 1,\cdots,P_j),$$

where the $P_{jqr}(t)$ *are polynomials of degrees* $\leqq p_j - 1$, *form a basis of all solutions of the equation*

$$\varphi(t) - \int_{-\infty}^{\infty} k(t - s)\varphi(s)\, ds = 0 \qquad (-\infty < t < \infty) \tag{2.5}$$

in any space $e^{h|t|}\tilde{E}_n$.

PROOF. By virtue of Theorem 1.1, equation (2.5) has the same solutions in all of the spaces $e^{h|t|}\tilde{E}_n$. Therefore it is sufficient to establish the theorem for the space $e^{h|t|}\tilde{L}_n$. In this case equation (2.5) is equivalent to the boundary value problem

$$\left.\begin{array}{r} [I - K(\lambda + ih)]\, \Phi_+(\lambda + ih) = \Omega(\lambda + ih), \\ -[I - K(\lambda - ih)]\, \Phi_-(\lambda - ih) = \Omega(\lambda - ih) \end{array}\right\} \qquad (-\infty < \lambda < \infty). \tag{2.6}$$

The vector functions $\Phi_\pm(\lambda)$ and $\Omega(\lambda)$ which are a solution of problem (2.6) are uniquely determined by each of them. Therefore, when speaking of a solution of problem (2.6), we shall have one of these vector functions in mind. If $\Omega(\lambda)$ is a solution of problem (2.6), it is not difficult to ascertain that the vector function $[I - K(\lambda)]^{-1}\Omega(\lambda)$ is holomorphic in the entire plane except, perhaps, for the points α_j, $j = 1,\cdots,m$, which are poles for it of order $\leqq p_j$ and tend to zero at infinity. Consequently it is rational and has the form

$$[I - K(\lambda)]^{-1}\Omega(\lambda) = \left\{\sum_{j=1}^{m} \frac{Q_{jr}(\lambda)}{(\lambda - \alpha_j)^{p_j}}\right\}_{r=1}^{n},$$

where the $Q_{jr}(\lambda)$ are polynomials of degrees $\leqq p_j - 1$. Since $[I - K(\lambda)]^{-1}\Omega(\lambda)$ coincides with $\Phi_+(\lambda)$ for $\operatorname{Im} \lambda \geqq h$ and with $-\Phi_-(\lambda)$ for $\operatorname{Im} \lambda \leqq -h$, we obtain, by virtue of (2.3), the fact that every solution of equation (2.5) has the form

$$\varphi(t) = \left\{\sum_{j=1}^{m} e^{-i\alpha_j t} P_{jr}(t)\right\}_{r=1}^{n} \qquad (-\infty < t < \infty), \tag{2.7}$$

where the $P_{jr}(t)$ are polynomials of degrees $\leqq p_j - 1$.

Let us establish that $p = p_j$ linearly independent solutions of equation (2.5) correspond to each zero $\alpha = \alpha_j$. To this end let us transform the matrix $I - K(\lambda)$.

A constant nonsingular matrix A_1 exists such that the point $\lambda = \alpha$ is a zero of multiplicity $q_1 \geqq 1$ of the first column of the matrix $(I - K(\lambda))A_1^{-1}$, which we represent in the form $(I - K(\lambda))A_1^{-1} = G_1 D_1$. Here

$$D_1 = \begin{Vmatrix} \left(\dfrac{\lambda-\alpha}{\lambda-\beta}\right)^{q_1} & 0\cdots\cdots 0 \\ 0 & 1\cdots\cdots 0 \\ \cdots\cdots & \cdots\cdots \\ 0 & 0\cdots\cdots 1 \end{Vmatrix},$$

where β is an arbitrary number with $|\operatorname{Im}\beta| > h$, while the first column of the matrix G_1 does not vanish at the point $\lambda = \alpha$, (i.e. at least one of its elements does not vanish).

If $q_1 < p$, the determinant of G_1 vanishes at the point $\lambda = \alpha$, and we represent G_1 analogously: $G_1A_2^{-1} = G_2D_2$. Continuing this process, we obtain the following representation of the matrix $I - K(\lambda)$:

$$I - K(\lambda) = G_rD_rA_rD_{r-1}A_{r-1}\cdots D_1A_1,$$

where the determinant of G_r does not vanish at the point $\lambda = \alpha$, and $q_1 + \cdots + q_r = p$. The vector functions $F_+^{(lk)}(\lambda)$ defined by the equalities

$$A_kD_{k-1}A_{k-1}\cdots D_1A_1F_+^{(lk)} = \begin{pmatrix} (\lambda-\alpha)^{-l} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$(k = 1,\cdots,r;\ l = 1,\cdots, q_k)$ are linearly independent and are solutions of problem (2.6).

Thus we have found $p_1+\cdots+p_m$ linearly independent solutions of problem (2.6):

$$\Phi_+^{(jq)}(\lambda) = \left\{\frac{Q_{jqr}(\lambda)}{(\lambda-\alpha_j)^{p_j}}\right\}_{r=1}^n \qquad (j = 1,\cdots,m;\ q = 1,\cdots, p_j),$$

where the $Q_{jqr}(\lambda)$ are polynomials of degrees $\leqq p_j - 1$. Defining the vector functions $\varphi_{jq}(t)$ relative to them, we obtain $p_1 + \cdots + p_m$ linearly independent solutions of (2.5):

$$\begin{gathered} \varphi_{jq}(t) = \{e^{-i\alpha_jt}P_{jqr}(t)\}_{r=1}^n \\ (-\infty < t < \infty;\ j = 1,\cdots, m;\ q = 1,\cdots, p_j), \end{gathered} \tag{2.8}$$

where the P_{jqr} are polynomials of degrees $\leqq p_j - 1$.

In order to complete the theorem's proof, it remains for us to show that all linearly independent solutions of equation (2.5) are exhausted by the vector functions (2.8).

Lemma 1.1 implies that the index of the operator

$$(I - \mathscr{K})\varphi = \varphi(t) - \int_{-\infty}^{\infty} k(t-s)\varphi(s)ds$$

in the space $e^{h|t|}\tilde{L}_n$ is defined by the equality

$$\kappa(I - \mathscr{K}) = \operatorname{ind} \det (I - K(\lambda - ih)) - \operatorname{ind} \det (I - K(\lambda + ih)).$$

By virtue of Rouché's theorem, the right side of the latter equality equals $p_1 + \cdots + p_m$. But $\beta(I - \mathscr{K}) = \alpha(I - \mathscr{K})^* = 0$. As a matter of fact, the operator $(I - \mathscr{K})^*$ operating in the space $e^{-h|t|}\tilde{M}_n$ is of the same type as the operator $I - \mathscr{K}$. The part of the theorem which has been proved implies that all solutions of the equation $(I - \mathscr{K})^*\varphi = 0$ in the space $e^{h|t|}\tilde{M}_n$ have form (2.7) but none of them get into the space $e^{-h|t|}\tilde{M}_n$. Thus the equation $(I - \mathscr{K})^* \varphi = 0$ has only the trivial solution in the space $e^{-h|t|}\tilde{M}_n$; consequently

$$\alpha(I - \mathscr{K}) = \kappa(I - \mathscr{K}) = p_1 + \cdots + p_m.$$

The theorem is proved.

The theorem just proved implies that in the case $n = 1$ the functions $\varphi_{jq}(t) = e^{-i\alpha_j t} t^{q-1}$ $(j = 1,\cdots,m;\ q = 1,\cdots, p_j)$ form a basis of all solutions of equation (2.5). This result is well known (cf. E. Titchmarsh **[1]**, Theorem 146).

LEMMA 2.1. *Let the matrix functions* $k_r(t) \in e^{-h|t|}\tilde{L}_{n\times n}$ $(r = 1,2)$ *possess the property*

$$\det (I - K_r(\lambda \pm ih)) \neq 0 \qquad (-\infty < \lambda < \infty;\ r = 1,2), \tag{2.9}$$

and let B_1 *and* B_2 *be bounded linear operators in the space* $e^{h|t|}\tilde{E}_n$ *which map it into* $e^{-ht}\tilde{E}_n$ *and* $e^{ht}\tilde{E}_n$, *respectively. Then every solution* $\varphi(t) \in e^{h|t|}\tilde{E}_n$ *of the equation*

$$\left.\begin{aligned} &\varphi(t) - \int_0^\infty k_1(t-s)\varphi(s)\,ds - (B_1\varphi)(t) = 0 \qquad (0 < t < \infty),\\ &\varphi(t) - \int_{-\infty}^0 k_2(t-s)\varphi(s)\,ds - (B_2\varphi)(t) = 0 \qquad (-\infty < t < 0)\end{aligned}\right\} \tag{2.10}$$

admits the representations

$$\begin{aligned} \varphi(t) &= \left\{\sum_{j=1}^{m_1} e^{-i\alpha_{1j}t} P_{1jk}(t)\right\}_{k=1}^n + \varphi_1(t),\\ \varphi(t) &= \left\{\sum_{j=1}^{m_2} e^{-i\alpha_{2j}t} P_{2jk}(t)\right\}_{k=1}^n + \varphi_2(t),\end{aligned} \tag{2.11}$$

where the vector functions $\varphi_1(t)$ *and* $\varphi_2(t)$ *belong to the spaces* $e^{-ht}\tilde{E}_n$ *and* $e^{ht}\tilde{E}_n$ *respectively,* α_{rj} $(r = 1,2;\ j = 1,\cdots,m_r)$ *are all of the distinct zeros of the function* $\det (I - K_r(\lambda))$ *in the strip* $|\mathrm{Im}\lambda| < h$, p_{rj} *are their multiplicities, and* $P_{rjk}(t)$ *are polynomials of degrees* $\leqq p_{rj} - 1$.

PROOF. Let us observe first of all that if the condition

$$\det (I - K(\lambda - ih)) \neq 0 \qquad (-\infty < \lambda < \infty)$$

is fulfilled for a matrix function $k(t) \in e^{-h|t|}\tilde{L}_{n\times n}$, the operator

$$(I - \mathscr{K})f = f(t) - \int_{-\infty}^{\infty} k(t-s)f(s)\,ds$$

is invertible in the space $e^{-ht}\tilde{E}_n$. Indeed, by virtue of Wiener's theorem[4] a matrix function $q(t) \in L_{n\times n}$ exists such that $I - Q(\lambda) = [I - K(\lambda - ih)]^{-1}$. Now it is not difficult to verify that the operator inverse to $I - \mathscr{K}$ is defined by the equality

$$(I - \mathscr{K})^{-1}f = f(t) - \int_{-\infty}^{\infty} q(t-s)\, e^{h(s-t)} f(s)\,ds.$$

Now let the vector function $\varphi(t) \in e^{h|t|}\tilde{E}_n$ be a solution of equation (2.10). Let us write the first equality of (2.10) in the form

$$\varphi(t) - \int_{-\infty}^{\infty} k_1(t-s)\varphi(s)ds = (B_1\varphi)(t) - \int_{-\infty}^{0} k_1(t-s)\varphi(s)\,ds + \psi_-(t)$$

$(-\infty < t < \infty)$, where the vector function $\psi_-(t) = 0$ for $t > 0$. Thus $\varphi(t)$ satisfies the relation

$$\varphi(t) - \int_{-\infty}^{\infty} k_1(t-s)\varphi(s)ds = \psi(t) \qquad (-\infty < t < \infty),$$

where $\psi(t) \in e^{-ht}\tilde{E}_n$. The operator defined by the left side of the last equality (let us denote it by A) is invertible in the space $e^{-ht}\tilde{E}_n$. Therefore there exists a unique solution $\varphi_1(t) \in e^{-ht}\tilde{E}_n$ of the equation $A\varphi = \psi$. Thus $A(\varphi - \varphi_1) = 0$, and application of Theorem 2.1 leads to the first representation of (2.11). The second equality of (2.11) is established analogously.

THEOREM 2.2. *Let the matrix functions $k_{jr}(t) \in e^{-h|t|}\tilde{L}_{n\times n}$ $(j, r = 1,2)$ possess the property*

$$\det(I - K_{rr}(\lambda \pm ih)) \neq 0 \qquad (-\infty < \lambda < \infty;\ r = 1,2).$$

Then every solution $\varphi(t) \in e^{h|t|}\tilde{E}_n$ of equation (1) *admits the asymptotic expansions*

$$\left.\begin{aligned} \varphi(t) &= \left\{\sum_{j=1}^{m_1} e^{-i\alpha_{1j}t} P_{1jk}(t)\right\}_{k=1}^{n} + o(e^{-ht}) \qquad (t \to \infty),\\ \varphi(t) &= \left\{\sum_{j=1}^{m_2} e^{-i\alpha_{2j}t} P_{2jk}(t)\right\}_{k=1}^{n} + o(e^{ht}) \qquad (t \to -\infty), \end{aligned}\right\} \tag{2.12}$$

where α_{rj} $(r = 1,2;\ j = 1,\cdots,m_r)$ are all of the distinct zeros of the function $\det(I - K_{rr}(\lambda))$ *in the strip $|\operatorname{Im}\lambda| < h$, p_{rj} are their multiplicities, and $P_{rjk}(t)$ are polynomials of degrees $\leqq p_{rj} - 1$.*

PROOF. Lemma 2.1 is applicable to equation (1), since the operators

[4] Cf. Gel'fand, Raĭkov and Šilov [1], § 17.

$$(\mathscr{K}_{12}f)(t) = \int_{-\infty}^{0} k_{12}(t-s)f(s)ds$$

and

$$(\mathscr{K}_{21}f)(t) = \int_{0}^{\infty} k_{21}(t-s)f(s)ds$$

map the spaces $e^{h|t|}\tilde{E}_n$ into the spaces $e^{-ht}\tilde{E}_n$ and $e^{ht}\tilde{E}_n$, respectively. Theorem 1.1 implies that the vector functions $\varphi_1(t)$ and $\varphi_2(t)$ in representations (2.11) belong to the spaces $e^{-ht}(\tilde{C}_0)_n$ and $e^{ht}(\tilde{C}_0)_n$, respectively. The latter is equivalent to the asymptotic expansions (2.12).

COROLLARY 2.1. *Let the matrix function $k(t) \in e^{-h|t|}\tilde{L}_{n\times n}$ satisfy conditions* (2.4). *Then every solution $\varphi(t) \in e^{ht}E_n$ of the system of Wiener-Hopf equations*

$$\varphi(t) - \int_0^\infty k(t-s)\varphi(s)\,ds = 0 \qquad (0 < t < \infty)$$

admits the asymptotic expansion

$$\varphi(t) = \left\{\sum_{j=1}^{m} e^{-i\alpha_j t} P_{jr}(t)\right\}_{r=1}^{n} + o(e^{-ht}) \qquad (t \to \infty), \tag{2.13}$$

where α_j, $j = 1,\cdots,m$, are all the distinct zeros of $\det(I - K(\lambda))$ *in the strip* $|\operatorname{Im}\lambda| < h$, *$p_j$ are their multiplicities, and $P_{jr}(t)$ are polynomials of degrees* $\leqq p_j - 1$.

COROLLARY 2.2. *Let the matrix functions $k_j(t) \in e^{-h|t|}\tilde{L}_{n\times n}$, $j = 1,2$, satisfy conditions* (2.9). *Then every solution $\varphi(t) \in e^{h|t|}\tilde{E}_n$ of the system of pair integral equations*

$$\varphi(t) - \int_{-\infty}^{\infty} k_1(t-s)\varphi(s)ds = 0 \qquad (0 < t < \infty),$$

$$\varphi(t) - \int_{-\infty}^{\infty} k_2(t-s)\varphi(s)ds = 0 \qquad (-\infty < t < 0)$$

and the transpose system

$$\varphi(t) - \int_0^\infty k_1(t-s)\varphi(s)ds - \int_{-\infty}^{0} k_2(t-s)\varphi(s)ds = 0$$
$$(-\infty < t < \infty)$$

admits the asymptotic expansions (2.12). *where α_{rj} $(r = 1,2; j = 1,\cdots,m_r)$ are all of the distinct zeros of the function* $\det(I - K_r(\lambda))$ *in the strip* $|\operatorname{Im}\lambda| < h$, *$p_{rj}$ are their multiplicities, and $P_{rjk}(t)$ are polynomials of degrees* $\leqq p_{rj} - 1$.

2. As already noted, the results cited above remain valid for the discrete ana-

logues of the respective equations. Let us, for example, formulate the discrete analogue of Theorem 2.2. To this end let us agree on the following notation. Let F be one of the spaces $\tilde{l}_p$ $(p \geqq 1;\ \tilde{l}_1 = \tilde{l})$, $\tilde{m}$, $\tilde{c}$ or $\tilde{c}_0$ of sequences of complex numbers $\xi = \{\xi_m\}_{-\infty}^{\infty}$. By F_n we denote as usual the linear space of sequences of n-dimensional vectors whose coordinates with the same index k $(= 1, \cdots, n)$ form a sequence of F, and by $F_{n\times n}$ the set of sequences of square nth order matrices the corresponding elements of which form a sequence of F.

Let $\gamma = \{\gamma_m\}_{-\infty}^{\infty}$ be a numerical sequence such that $\gamma_m \neq 0$ for all m. We shall write $\xi \in \gamma F$ if $\{\gamma_m^{-1}\xi_m\}_{-\infty}^{\infty} \in F$. The spaces γF_n and $\gamma F_{n\times n}$ are defined analogously.

THEOREM 2.3. *Let the sequences of matrices* $\{A_m^{(jr)}\}_{m=-\infty}^{\infty}$ $(j, r = 1, 2)$ *belong to* $\{h^{-|m|}\}_{-\infty}^{\infty}\tilde{l}_{n\times n}$ *for some* $h > 1$, *and let the matrix functions*

$$U^{(rr)}(\zeta) = \sum_{j=-\infty}^{\infty} A_j^{(rr)}\zeta^j \qquad (r = 1, 2)$$

possess the property

$$\det U^{(rr)}(\zeta h^{\pm 1}) \neq 0 \qquad (|\zeta| = 1;\ r = 1, 2).$$

Then every solution $\xi = \{\xi_m\}_{-\infty}^{\infty}$ *of the infinite system of equations*

$$\sum_{j=0}^{\infty} A_{m-j}^{(11)}\xi_j + \sum_{j=-\infty}^{-1} A_{m-j}^{(12)}\xi_j = 0 \qquad (m = 0, 1, \cdots),$$
$$\sum_{j=0}^{\infty} A_{m-j}^{(21)}\xi_j + \sum_{j=-\infty}^{-1} A_{m-j}^{(22)}\xi_j = 0 \qquad (m = -1, -2, \cdots)$$

which is contained in $\{h^{|m|}\}_{-\infty}^{\infty}F_n$ *admits the asymptotic expansions*

$$\xi_m = \left\{\sum_{j=1}^{q_1} \alpha_{1j}^{-m}P_{1jk}(m)\right\}_{k=1}^{n} + o(h^{-m}) \qquad (m \to \infty),$$
$$\xi_m = \left\{\sum_{j=1}^{q_2} \alpha_{2j}^{-m}P_{2jk}(m)\right\}_{k=1}^{n} + o(h^{m}) \qquad (m \to -\infty),$$

where α_{rj} $(r = 1, 2;\ j = 1, \cdots, m_r)$ *are all of the distinct zeros of the function* $\det U^{(rr)}(\xi)$ *in the annulus* $h^{-1} < |\zeta| < h$, p_{rj} *are their multiplicities, and* $P_{rjk}(t)$ *are polynomials of degrees* $\leqq p_{rj} - 1$.

§ 3. Transpose equation of a pair

Let us denote by $\mathfrak{R}_h$ the algebra of all functions of the form

$$\mathscr{A}(z) = c + \begin{cases} \int_{-\infty}^{\infty} a_1(t)e^{izt}\,dt & (\operatorname{Im} z = h), \\ \int_{-\infty}^{\infty} a_2(t)e^{izt}\,dt & (\operatorname{Im} z = -h), \end{cases}$$

where $a_1(t) \in e^{ht}\tilde{L}$, $a_2(t) \in e^{-ht}\tilde{L}$ and c is an arbitrary complex number.

The functions $\mathscr{A}(z)$ of the algebra $\mathfrak{R}_h$ are defined on the contour consisting of the pair of straight lines $\operatorname{Im} z = \pm h$. Let us denote by $\mathfrak{R}_h^+$ the subalgebra of $\mathfrak{R}_h$ which consists of all functions $\mathscr{A}(z)$ for which $a_1(t) = 0$ for $t < 0$ and $a_2(t) = 0$ for $t > 0$, and let us denote by $\mathfrak{R}_h^-$ the subalgebra of $\mathfrak{R}_h$ which consists of all functions $\mathscr{A}(z)$ for which $a_1(t) = a_2(t) \in e^{-h|t|}\tilde{L}$. Every function of $\mathfrak{R}_h^+$ admits continuation to a function holomorphic in the region $|\operatorname{Im} z| > h$ (i.e. outside the strip $|\operatorname{Im} z| \leqq h$) and continuous right up to and including the boundary, while every function of $\mathfrak{R}_h^-$ admits continuation to a function holomorphic inside the strip $|\operatorname{Im} z| < h$ and continuous right up to and including the boundary.

Every function $\mathscr{A}(z) \in \mathfrak{R}_h$ is representable as the sum of two functions $\mathscr{A}_\pm(z) \in \mathfrak{R}_h^\pm$. As a matter of fact, it is sufficient to set

$$\mathscr{A}_+(z) = c + \begin{cases} \int\limits_0^\infty [a_1(t) - a_2(t)]\, e^{izt}\, dt & (\operatorname{Im} z = h), \\ \int\limits_{-\infty}^0 [a_2(t) - a_1(t)]\, e^{izt}\, dt & (\operatorname{Im} z = -h), \end{cases}$$

$$\mathscr{A}_-(z) = \int\limits_{-\infty}^0 a_1(t) e^{izt}\, dt + \int\limits_0^\infty a_2(t)\, e^{izt}\, dt \qquad (\operatorname{Im} z = \pm h).$$

It is easy to see that this representation is unique to within a constant addend.

Let the conditions

$$\mathscr{A}(z) \neq 0 \quad (\operatorname{Im} z = \pm h), \qquad \mathscr{A}(\infty) = 1 \tag{3.1}$$

be fulfilled for a function $\mathscr{A}(z) \in \mathfrak{R}_h$. Let us introduce the notation

$$\nu_1 = \operatorname{ind} \mathscr{A}(\lambda + ih)|_{\lambda=-\infty}^{\infty}, \qquad \nu_2 = \operatorname{ind} \mathscr{A}(\lambda - ih)|_{\lambda=-\infty}^{\infty},$$
$$\kappa = \nu_2 - \nu_1 \; (= \kappa(\mathscr{A})).$$

Let us consider an auxiliary function $\mathscr{A}_1(z)$ of $\mathfrak{R}_h$ which is defined by the equality

$$\mathscr{A}_1(z) = \begin{cases} \left(\dfrac{\lambda - a}{\lambda + a}\right)^\kappa & \text{for } \operatorname{Im} z = h, \\ 1 & \text{for } \operatorname{Im} z = -h, \end{cases}$$

where $\lambda = \operatorname{Re} z$ and $\operatorname{Im} a > 2h$. For it, $\kappa(\mathscr{A}_1) = -\kappa$; therefore $\kappa(\mathscr{A}_1, \mathscr{A}) = 0$, i.e. the function $\mathscr{A}_1(z)$ annihilates the index of the function $\mathscr{A}(z)$. From results of I.C. Gohberg [1] it follows that the function $\mathscr{A}_1(z)\mathscr{A}(z)$ admits the factorization

$$\mathscr{A}_1(z)\mathscr{A}(z) = G_+(z)G_-(z),$$

where $G_\pm(z) \in \mathfrak{R}_h^\pm$, $G_\pm^{-1}(z) \in \mathfrak{R}_h^\pm$ and $G_\pm(\infty) = 1$. The equalities

$$\left.\begin{aligned} \mathscr{A}(\lambda + ih) &= G_+(\lambda + ih)G_-(\lambda + ih)\left(\frac{\lambda + a}{\lambda - a}\right), \\ \mathscr{A}(\lambda - ih) &= G_+(\lambda - ih)G_-(\lambda - ih) \end{aligned}\right\} \qquad (-\infty < \lambda < \infty) \tag{3.2}$$

follow from this factorization.

Now let us consider in the space $e^{h|t|}\tilde{E}$ the homogeneous integral equation

$$\varphi(t) - \int_0^\infty k_1(t-s)\varphi(s)\,ds - \int_{-\infty}^0 k_2(t-s)\varphi(s)\,ds = 0 \qquad (3.3)$$
$$(-\infty < t < \infty),$$

which is the transpose of a pair, where the $k_j(t) \in e^{-h|t|}\tilde{L}$ $(j = 1, 2)$ and satisfy the conditions

$$1 - K_j(\lambda \pm ih) \neq 0 \qquad (-\infty < \lambda < \infty; j = 1, 2). \qquad (3.4)$$

Since, by virtue of Theorem 1.1, equation (3.3) has the same solutions in all spaces $e^{h|t|}E$, it is sufficient to consider this equation in the space $e^{h|t|}L$.

By virtue of (2.2), equation (3.3) is equivalent to the boundary value problem

$$\left.\begin{aligned} [1 - K_1(\lambda + ih)]\Phi_+(\lambda + ih) &= \Omega(\lambda + ih), \\ -[1 - K_2(\lambda - ih)]\Phi_-(\lambda - ih) &= \Omega(\lambda - ih) \end{aligned}\right\} \quad (-\infty < \lambda < \infty), \qquad (3.5)$$

where the functions

$$\Phi(z) = \begin{cases} \Phi_+(\lambda + ih), & \text{for } \operatorname{Im} z = h, \\ \Phi_-(\lambda - ih), & \text{for } \operatorname{Im} z = -h \end{cases}$$

and $\Omega(z)$ belong to the algebras $\mathfrak{R}_h^+$ and $\mathfrak{R}_h^-$, respectively.

The function

$$\mathscr{A}(z) = \begin{cases} 1 - K_1(\lambda + ih), & \operatorname{Im} z = h, \\ 1 - K_2(\lambda - ih), & \operatorname{Im} z = -h, \end{cases}$$

belongs to $\mathfrak{R}_h$ and satisfies conditions (3.1), and consequently admits factorization (3.2).

Factorization (3.2) implies that for $\kappa(\mathscr{A}) = \kappa = \nu_2 - \nu_1 > 0$ the general solution of problem (3.5) is given by the equalities

$$\left.\begin{aligned} \Omega(\lambda) &= \frac{P_{\kappa-1}(\lambda)}{(\lambda - a - ih)^\kappa} G_-(\lambda), \\ \Phi_+(\lambda + ih) &= \frac{P_{\kappa-1}(\lambda + ih)}{G_+(\lambda + ih)(\lambda + a)^\kappa}, \\ \Phi_-(\lambda - ih) &= -\frac{P_{\kappa-1}(\lambda - ih)}{G_+(\lambda - ih)(\lambda - a - 2ih)^\kappa}, \end{aligned}\right\} \qquad (3.6)$$

where $P_{\kappa-1}(z)$ is an arbitrary polynomial of degree $\leqq \kappa - 1$. If, however, $\kappa \leqq 0$, problem (3.5), and consequently also equation (3.3), has only the trivial solution.

LEMMA 3.1. *Let $a(t) \in e^{-ht}L_1(0, \infty)$, and let n be a nonnegative integer. Then for* $\operatorname{Im} \delta > -h$

$$\int_t^\infty (t - s)^n e^{-i\delta(t-s)} a(s)\, ds = o(e^{-ht}) \qquad (t \to \infty) \tag{3.7}$$

and for $\operatorname{Im} \delta < -h$

$$\int_0^t (t - s)^n e^{-i\delta(t-s)} a(s)\, ds = o(e^{-ht}) \qquad (t \to \infty). \tag{3.8}$$

PROOF. Let us note that for any $r > 0$ there is a constant $c > 0$ such that $t^n \leqq ce^{rt}$ $(0 \leqq t < \infty)$. In particular, the inequality

$$(s - t)^n \leqq ce^{(h+\operatorname{Im}\delta)(s-t)} \qquad (s \geqq t)$$

holds for $\operatorname{Im} \delta > -h$.

Relation (3.7) now follows from the following estimates:

$$\left| \int_t^\infty (t - s)^n e^{-i\delta(t-s)} a(s)\, ds \right| \leqq e^{\operatorname{Im}\delta t} \int_t^\infty (s - t)^n e^{-\operatorname{Im}\delta s} |a(s)| ds$$

$$\leqq e^{\operatorname{Im}\delta t} \int_t^\infty ce^{(h+\operatorname{Im}\delta)(s-t)} e^{-\operatorname{Im}\delta s} |a(s)| ds$$

$$= ce^{-ht} \int_t^\infty e^{hs} |a(s)| ds = o(e^{-ht}) \qquad (t \to \infty).$$

Turning to the proof of (3.8), we make use of the estimates

$$\left| \int_0^t (t - s)^n e^{-i\delta(t-s)} a(s)\, ds \right| \leqq e^{-ht} \int_0^t (t - s)^n e^{(h+\operatorname{Im}\delta)(t-s)} a^{hs} |a(s)| ds$$

$$= e^{-ht} \int_0^t s^n e^{-\varepsilon s} \rho(t - s) ds,$$

where $\rho(t) = e^{ht}|a(t)|$ and $\varepsilon = -(h + \operatorname{Im}\delta) > 0$.

Let us show that the last integral tends to zero as $t \to \infty$. Indeed, whatever the positive number $b < t$ may be, the relations

$$\int_0^t s^n e^{-\varepsilon s} \rho(t - s) ds = \int_0^b s^n e^{-\varepsilon s} \rho(t - s)\, ds + \int_b^t s^n e^{-\varepsilon s} \rho(t - s)\, ds$$

$$\leqq \sup_{0 \leqq s < \infty} s^n e^{-\varepsilon s} \int_{t-b}^t \rho(s)\, ds + \sup_{b \leqq s < \infty} s^n e^{-\varepsilon s} \int_b^t \rho(t - s)\, ds$$

are valid, which implies that

$$\lim_{t \to \infty} \int_0^t s^n e^{\varepsilon - s} \rho(t - s)\, ds = 0.$$

The lemma is proved.

As already noted, the solutions of equation (3.3) belong to each of the spaces

$e^{h|t|}\bar{E}$. In order to formulate a theorem concerning the asymptotics of these solutions, let us introduce in addition the following notation: the α_j, $j = 1, \cdots, n$, are all of the distinct zeros of the function $1 - K_1(\lambda)$ in the strip $|\mathrm{Im}\lambda| < h$, and are so numbered that $\mathrm{Im}\,\alpha_1 \geqq \mathrm{Im}\,\alpha_2 \geqq \cdots \geqq \mathrm{Im}\,\alpha_n$; the p_j are their multiplicities; and $m = p_1 + \cdots + p_n$. Let κ be some positive integer. If $\kappa < m$, let us denote by τ the largest integer such that $p_1 + \cdots + p_\tau \leqq k$ and $\mathrm{Im}\,\alpha_\tau > \mathrm{Im}\,\alpha_{\tau+1}$.

Let conditions (3.4) be fulfilled for the functions $k_j(t)$ of $e^{-h|t|}\bar{L}$ $(j = 1, 2)$. Let us set $\kappa = \nu_2 - \nu_1$, where

$$\nu_1 = \mathrm{ind}\,(I - K_1(\lambda + ih)), \qquad \nu_2 = \mathrm{ind}(1 - K_2(\lambda - ih)).$$

Let us set

$$G(\lambda) = 1 + \int_0^\infty \gamma(t) e^{i\lambda t}\, dt.$$

for a function $\gamma(t)$ prescribed on the semiaxis $0 < t < \infty$.

THEOREM 3.1 *Let $\kappa > 0$. Then the following assertions are true.*

a) *If $m \leqq \kappa$, a function $\gamma(t) \in e^{-ht}L_1(0, \infty)$ exists such that to each zero $\alpha = \alpha_j$, $j = 1, \cdots, n$, of the function $1 - K_1(\lambda)$ in the strip $|\mathrm{Im}\,\lambda| < h$ there correspond $p = p_j$ solutions of equation* (3.3) *with the asymptotic expansion*

$$\varphi_r(t) = e^{-i\alpha t} \sum_{j=0}^{r} \frac{(-i)^j\, t^{r-j}}{j!(r-j)!} G^{(j)}(\alpha) + o(e^{-ht}) \quad (r = 0, 1, \cdots, p-1) \tag{3.9}$$

as $t \to \infty$. The remaining $\kappa - m$ linearly independent solutions belong to any of the spaces $e^{-ht}\bar{E}$.

b) *If $m > \kappa$, a function $\gamma(t) \in e^{ct}L(0, \infty)$ $(\mathrm{Im}\,\alpha_{\tau+1} < c < \mathrm{Im}\,\alpha_\tau)$ exists such that to each "upper" zero $\alpha = \alpha_j$, $j = 1, \cdots, \tau$, there correspond $p = p_j$ solutions of equation* (3.3) *with the asymptotic expansion*

$$\varphi_r(t) = e^{-i\alpha t} \sum_{j=0}^{r} \frac{(-i)^j\, t^{r-j}}{j!(r-j)!} G^{(j)}(\alpha) + \sum_{j=\tau+1}^{n} P_{jr}(t)\, e^{-i\alpha_j t} + o(e^{-ht}) \tag{3.10}$$

$(r = 0, 1, \cdots, p-1)$ *as $t \to \infty$, where the $P_{jr}(t)$ are polynomials of degrees exactly $p_j - 1$ $(j = \tau + 1, \cdots, n,)$; the remaining $\kappa - p_1 - p_2 - \cdots - p_\tau$ linearly independent solutions admit asymptotic expansions of the form*

$$\sum_{j=\tau+1}^{n} P_j(t) e^{-i\alpha_j t} + o(e^{-ht})$$

as $t \to \infty$, where the $P_j(t)$ are polynomials of degrees $\leqq p_j - 1$ $(j = \tau + 1, \cdots, n)$.

PROOF. Let us note that if the function $\Phi_+(\lambda + ih)$ is the Fourier transform of the function $e^{-ht}\varphi(t) \in L_1(0, \infty)$ and

$$\Phi_+(\lambda + ih) = -(-i)^r G(\lambda + ih)/(\lambda - \delta)^r,$$

where $\operatorname{Im}\delta < 0$ and

$$G(\lambda) = 1 + \int_0^\infty \gamma(t)e^{i\lambda t}\,dt, \qquad \gamma(t) \in e^{ht}L_1(0, \infty),$$

then

$$\varphi(t) = \frac{t^{r-1}\,e^{-i(\delta+ih)t}}{(r-1)!} + \frac{1}{(r-1)!}\int_0^t (t-s)^{r-1}e^{-i(\delta+ih)(t-s)}\gamma(s)\,ds. \tag{3.11}$$

Equality (3.11) is obtained by applying the inverse Fourier transform to the function $\Phi_+(\lambda + ih)$.

The general solution of problem (3.5) is, by virtue of (3.6), defined by the equality

$$\Phi_+(\lambda + ih) = \frac{P_{\kappa-1}}{G_+(\lambda + ih)(\lambda + a)^\kappa}. \tag{3.12}$$

In the case $\kappa \geqq m$ let us rewrite the latter equality in the form

$$\Phi_+(\lambda + ih) = \frac{P_{\kappa-1}G(\lambda + ih)}{\prod_{j=1}^n (\lambda + ih - \alpha_j)^{p_j}(\lambda + a)^{\kappa-m}}, \tag{3.13}$$

where the function

$$G(\lambda) = \frac{\prod_{j=1}^n (\lambda - \alpha_j)^{p_j}}{G_+(\lambda)(\lambda + a - ih)^m},$$

which is holomorphic in the region $\operatorname{Im}\lambda > h$, admits, by virtue of the first equality of (3.2), continuation to a function holomorphic in the region $\operatorname{Im}\lambda > -h$. It is not difficult to infer from this that the function $G(\lambda)$ is representable in the form

$$G(\lambda) = 1 + \int_0^\infty \gamma(t)e^{i\lambda t}\,dt,$$

where $\gamma(t) \in e^{-ht}L_1(0, \infty)$.

By virtue of equality (3.13), the functions

$$\Phi_+^{(j,k)}(\lambda + ih) = \frac{-(-i)^k G(\lambda + ih)}{(\lambda + ih - \alpha_j)^k} \qquad (j = 1,\cdots,n;\ k = 1,\cdots, p_j) \tag{3.14}$$

and

$$\Phi_+^{(r)}(\lambda + ih) = \frac{-(-i)^r G(\lambda + ih)}{(\lambda + a)^r} \qquad (r = 1,\cdots, \kappa - m) \tag{3.15}$$

form a basis of all solutions of boundary value problem (3.5).

Equalities (3.14) and (3.11) imply that to each zero $\alpha = \alpha_j$, $j = 1,\cdots, n$, there correspond the $p = p_j$ solutions of equation (3.3)

$$\varphi_k(t) = \frac{t^k e^{-i\alpha t}}{k!} + \frac{1}{k!}\int_0^t (t-s)^k\, e^{-i\alpha(t-s)}\gamma(s)\,ds \qquad (t > 0;\ k = 0,\cdots, p-1).$$

They admit asymptotic expansions (3.9).

Indeed,

$$\varphi_k(t) = \frac{e^{-i\alpha t}}{k!}\left[t^k + \int_0^\infty (t-s)^k e^{i\alpha s}\gamma(s)\,ds\right] - \frac{1}{k!}\int_t^\infty (t-s)^k e^{-i\alpha(t-s)}\gamma(s)\,ds.$$

By virtue of (3.7), the last integral is $o(e^{-ht})$ as $t \to \infty$, and the expression in square brackets equals

$$\sum_{j=0}^{k} (-i)^j C_k^j t^{k-j} G^{(j)}(\alpha),$$

since

$$\int_0^\infty s^j e^{i\alpha s}\gamma(s)\,ds = (-i)^j G^{(j)}(\alpha).$$

Relative to the functions (3.15) we find $\kappa - m$ linearly independent solutions of equation (3.3), of the form

$$\varphi(t) = \frac{t^r e^{-i\beta t}}{r!} + \frac{1}{r!}\int_0^t (t-s)^r e^{-i\beta(t-s)}\gamma(s)\,ds,$$

where $\beta = -\alpha + ih$ ($\operatorname{Im}\beta < -h$). They all belong to any space $e^{-ht}\tilde{E}$. Indeed, this is obvious for the first addend and follows from estimates (3.8) for the second.

Let us observe that the existence of $\kappa - m$ linearly independent solutions of equation (3.3) which belong to any space $e^{-ht}\tilde{E}$ follows also from the equality $\kappa = \kappa_{-h} + m$, where κ_{-h} is the index of equation (3.3) in the space $e^{-ht}\tilde{E}$ (cf. (1.6)), and from the fact that in the case $\kappa_{-h} \geqq 0$ equation (3.3) has precisely κ_{-h} linearly independent solutions in the space $e^{-ht}\tilde{E}$.

Turning to consideration of the case $\kappa < m$, let us rewrite equality (3.12), which gives the general solution of problem (3.5), in the form

$$\Phi_+(\lambda + ih) = \frac{P_{\kappa-1}G(\lambda + ih)}{\prod_{j=1}^{\tau}(\lambda + ih - \alpha_j)^{p_j}(\lambda + a)^{\kappa - q}}, \tag{3.16}$$

where $q = p_1 + \cdots + p_\tau$, and the function

$$G(\lambda) = \frac{\prod_{j=1}^{\tau}(\lambda - \alpha_j)^{p_j}}{G_+(\lambda)(\lambda + a - ih)^q},$$

which is holomorphic in the region $\operatorname{Im}\lambda > h$, admits, by virtue of the first equality of (3.2), continuation to a function holomorphic in the region $\operatorname{Im}\lambda > -h$, with the exception of the points α_j, $j = \tau + 1, \cdots, n$, which are poles for it of order p_j. By detaching the sum of the principal parts of the function $G(\lambda)$ relative to its poles α_j, $j = \tau + 1, \cdots, n$, it is easy to obtain the fact that

$$G(\lambda) - \sum_{j=\tau+1}^{n} \frac{d_j(\lambda)}{(\lambda - \alpha_j)^{p_j}} = 1 + \int_0^\infty \rho(t)\, e^{i\lambda t}\,dt,$$

where $\rho(t) \in e^{-ht}L_1(0, \infty)$ and $d_j(\lambda)$, $j = \tau + 1,\cdots, n$, are polynomials of degrees $\leqq p_j - 1$. For $\operatorname{Im}\lambda > \operatorname{Im}\alpha_{\tau+1}$ the sum of the principal parts of the function $G(\lambda)$ admits the representation

$$\sum_{j=\tau+1}^{n} \frac{d_j(\lambda)}{(\lambda - \alpha_j)^{p_j}} = \int_0^\infty \omega(t)e^{i\lambda t}\,dt \qquad (\operatorname{Im}\lambda > \operatorname{Im}\alpha_{\tau+1}),$$

where

$$\omega(t) = \sum_{j=\tau+1}^{n} e^{-i\alpha_j t}\, Q_j(t),$$

and the $Q_j(t)$ are polynomials of degrees exactly $p_j - 1$.

Thus

$$G(\lambda) = 1 + \int_0^\infty \gamma(t)e^{i\lambda t}\,dt \qquad (\operatorname{Im}\lambda > \operatorname{Im}\alpha_{\tau+1}),$$

where $\gamma(t) = \rho(t) + \omega(t) \in e^{ct}L_1(0, \infty)$ $(\operatorname{Im}\alpha_{\tau+1} < c < \operatorname{Im}\alpha_\tau)$.

By virtue of (3.16), the functions

$$\Phi_+^{(j,k)}(\lambda + ih) = \frac{-(-i)^k G(\lambda + ih)}{(\lambda + ih - \alpha_j)_k} \qquad (j = 1,\cdots,\tau;\ k = 1,\cdots, p_j) \quad (3.17)$$

and

$$\Phi_+^{(r)}(\lambda + ih) = \frac{-(-i)^r G(\lambda + ih)}{(\lambda + a)^r} \qquad (r = 1,\cdots, \kappa - q) \quad (3.18)$$

form a basis of all solutions of problem (3.5).

Applying (3.11) to the functions (3.17), we obtain the fact that to each upper zero $\alpha = \alpha_j$, $j = 1,\cdots,\tau$, there correspond the $p = p_j$ solutions of (3.3):

$$\begin{aligned}\varphi_k(t) &= \frac{t_k e^{-i\alpha t}}{k!} + \frac{1}{k!}\int_0^t (t - s)^k e^{-i\alpha(t-s)}\gamma(s)\,ds \\ &= \left[\frac{t_k e^{-i\alpha t}}{k!} + \frac{1}{k!}\int_0^\infty (t - s)^k e^{-i\alpha(t-s)}\gamma(s)\,ds\right] \\ &\quad - \frac{1}{k!}\int_t^\infty (t - s)^k e^{-i\alpha(t-s)}\omega(s)\,ds - \frac{1}{k!}\int_t^\infty (t - s)^k e^{-i\alpha(t-s)}\rho(s)\,ds \\ &\qquad (t > 0;\ k = 1,\cdots, p - 1).\end{aligned}$$

As was shown above, the expression in square brackets equals

$$e^{-i\alpha t}\sum_{j=0}^{k} \frac{(-i)^j t^{k-j}}{j!\,(k - j)!}\, G^{(j)}(\alpha);$$

the last integral in the expression for $\varphi_k(t)$ is, by virtue of (3.7), $o(e^{-ht})$ as $t \to \infty$; and the penultimate integral, as is easy to see, represents a function of the same type as $\omega(t)$. Thus we obtain expansion (3.10).

To the functions (3.18) there correspond, by virtue of (3.11), $\kappa - q$ linearly independent solutions of equation (3.3) of the form

$$\varphi(t) = \frac{t^r e^{-i\beta t}}{r!} + \frac{1}{r!} \int_0^t (t-s)^r e^{-i\beta(t-s)} \rho(s)\, ds + \frac{1}{r!} \int_0^t (t-s)^r e^{-i\beta(t-s)} \omega(s)\, ds \qquad (t > 0),$$

where $\beta = -a + ih$ ($\operatorname{Im} \beta < -h$). We have already evaluated the sum of the first two terms. It is $o(e^{-ht})$ $(t \to \infty)$; the last integral, as is easy to see, equals

$$\sum_{j=\tau+1}^{n} P_j(t)\, e^{-i\alpha_j t} + o(e^{-ht}) \qquad (t \to \infty),$$

where the $P_j(t)$, $j = \tau + 1, \cdots, n$, are polynomials of degrees $\leqq p_j - 1$.

The theorem is proved.

Analogous asymptotic expansions hold as $t \to -\infty$. They are determined by the zeros of the function $1 - K_2(\lambda)$.

§ 4. Systems of Wiener-Hopf equations

The results of the present section represent refinements of the asymptotic expansions (2.13) of the solutions of the homogeneous system of Wiener-Hopf equations

$$\varphi(t) - \int_0^\infty k(t-s)\varphi(s)ds = 0 \qquad (0 < t < \infty), \tag{4.1}$$

where the matrix function $k(t)$ belongs to $e^{-h|t|} \tilde{L}_{n\times n}$ and its Fourier transform $K(\lambda)$ satisfies the conditions

$$\det (I - K(\lambda \pm ih)) \neq 0 \qquad (-\infty < \lambda < \infty). \tag{4.2}$$

Equation (4.1) is considered in the space $e^{ht}E_n$.

1. It follows easily from Theorems 2.4 and 6.1 of Chapter VIII that if conditions (4.2) are fulfilled, the matrix function $I - K(\lambda + ih)$ admits the factorization

$$I - K(\lambda + ih) = G_-(\lambda) \left\| \left(\frac{\lambda - a}{\lambda + a}\right)^{\kappa_j} \delta_{jk} \right\|_1^n G_+(\lambda) \qquad (-\infty < \lambda < \infty), \tag{4.3}$$

where $\operatorname{Im} a > 2h$, $\kappa_1 \geqq \kappa_2 \geqq \cdots \geqq \kappa_n$ are integers called the *right indices* of the matrix function $I - K(\lambda + ih)$, and $G_\pm(\lambda)$ are matrix functions respectively holomorphic in the upper and lower half-plane and continuous there right up to and including the boundary. Their determinants do not vanish in the respective half-planes. Moreover,

$$G_\pm(\lambda) = I + \int_0^\infty g_\pm(\pm t) e^{\pm i\lambda t} dt,$$

where the matrix functions $g_{\pm}(t)$ belong to $L_{n\times n}$ and in all of the spaces $e^{ht}E_n$ equation (4.1) has the same solutions, which form the linear manifold $\sigma = -(\kappa_\beta + \kappa_{\beta+1} + \cdots + \kappa_n)$ of finite dimension, where the κ_j are all of the negative right indices.

Let us introduce the following notation: the α_j, $j = 1,\cdots,r$, are all of the distinct zeros of the function $\det (I - K(\lambda))$ in the strip $|\mathrm{Im}\lambda| < h$, so numbered that $\mathrm{Im}\,\alpha_1 \geqq \mathrm{Im}\,\alpha_2 \geqq \cdots \geqq \mathrm{Im}\,\alpha_r$; the p_j are their multiplicities; and $m = p_1 + \cdots + p_r$. If the inequality $m > -\kappa_j$ is fulfilled for some negative right index κ_j (< 0), let us denote by τ_j the largest integer such that $p_1 + \cdots + p_\tau \leqq -\kappa_j$ and $\mathrm{Im}\ \alpha_{\tau_j} > \mathrm{Im}\ \alpha_{\tau_{j+1}}$.

THEOREM 4.1. *Let conditions* (4.2) *be fulfilled for a matrix function* $k(t)$ *of* $e^{-h|t|}\tilde{L}_{n\times n}$, *and let* $\kappa_\beta,\cdots,\kappa_n$ *be all of the negative right indices of the matrix function* $I - K(\lambda + ih)$. *Then the subspace of all solutions* $\varphi(t) \in e^{ht}E_n$ *of equation* (4.1) *can be decomposed into the direct sum of the subspaces* R_j, $j = \beta,\cdots,n$, *with bases*

$$\varphi_1^{(j)}(t),\cdots,\ \varphi_{-\kappa_j}^{(j)}(t) \tag{4.4}$$

of the following structure:

a) *If* $-\kappa_j \geqq m$, *the first* m *vector functions of the basis* (4.4) *admit the following asymptotic expansions as* $t \to \infty$:

$$\{e^{-i\alpha_\nu t}P_{\nu\mu l}^{(j)}\}_{l=1}^{n} + o(e^{-ht}) \qquad (\nu = 1,\cdots, r;\ \mu = 1,\cdots,p_\nu), \tag{4.5}$$

where the $P_{\nu\mu l}^{(j)}(t)$ *are polynomials of degrees* $\leqq \mu - 1$, *and the vector functions* $\varphi_{m+1}^{(j)}(t),\cdots,\ \varphi_{-\kappa_j}^{(j)}(t)$ *belong to any space* $e^{-ht}E_n$.

b) *If* $-\kappa_j < m$, *the first* $q_j = p_1 + \cdots + p_{\tau_j}$ *vector functions of the basis* (4.4) *admit the following asymptotic expansions as* $t \to \infty$:

$$\left\{e^{-i\alpha_\nu t}P_{\nu\mu l}^{(j)}(t) + \sum_{s=\tau_j+1}^{r} e^{-i\alpha_s t}\, P_{\nu\mu s l}^{(j)}(t)\right\}_{l=1}^{n} + o(e^{-ht})$$
$$(\nu = 1,\cdots,\tau_j;\ \mu = 1,\cdots, p_\nu), \tag{4.6}$$

where the $P_{\nu\mu l}^{(j)}(t)$ *are some polynomials of degrees* $\leqq \mu - 1$, *the* $P_{\nu\mu s l}^{(j)}(t)$ *are polynomials of degrees* $\leqq p_s - 1$ $(s = \tau_j + 1,\cdots, r)$, *and as* $t \to \infty$ *the vector functions* $\varphi_{q_j+1}^{(j)}(t),\cdots,\ \varphi_{-\kappa_j}^{(j)}(t)$ *admit asymptotic expansions of the form*

$$\left\{\sum_{s=\tau_j+1}^{r} e^{-i\alpha_s t}P_{sl}^{(j)}(t)\right\}_{l=1}^{n} + o(e^{-ht}), \tag{4.7}$$

where the $P_{sl}^{(j)}(t)$ *are polynomials of degrees* $\leqq p_s - 1$ $(s = \tau_j + 1,\cdots, r)$.

PROOF. Equation (4.1) considered in the space $e^{ht}L_n$ is equivalent to Hilbert's boundary value problem

$$[I - K(\lambda + ih)]\Phi_+(\lambda + ih) = B_-(\lambda + ih) \tag{4.8}$$
$$(-\infty < \lambda < \infty),$$

where

$$\left.\begin{aligned} \Phi_+(\lambda + ih) &= \int_0^\infty \varphi(t)e^{-ht}e^{i\lambda t}dt, \\ B_-(\lambda + ih) &= \int_{-\infty}^0 b(t)e^{-ht}e^{i\lambda t}dt, \\ b(t) = -\int_0^\infty k\,(t - s)\varphi(s)ds &\qquad (-\infty < t < 0). \end{aligned}\right\} \tag{4.9}$$

The vector functions $\Phi_+(\lambda)$ and $B_-(\lambda)$ admit continuation to vector functions which are respectively holomorphic in the half-planes Im $\lambda > h$ and Im $\lambda < h$ and continuous there, including the boundary.

The general solution of problem (4.8) is, by virtue of factorization (4.3), defined by the equality

$$\Phi_+(\lambda + ih) = G_+^{-1}(\lambda)\left\{0,\cdots, \frac{P_{-\kappa_\beta-1}(\lambda)}{(\lambda + a)^{-\kappa_\beta}},\cdots, \frac{P_{-\kappa_n-1}(\lambda)}{(\lambda + a)^{-\kappa_n}}\right\},$$

where $P_{-\kappa_j-1}(\lambda)$ is an arbitrary polynomial of degree $\leqq -k_j - 1$ ($j = \beta$, $\beta + 1,\cdots, n$), there being in correspondence with each right index $\kappa_j < 0$ the $-\kappa_j$ linearly independent solutions of problem (4.8) which are defined by the equality

$$\Phi_+^{(j)}(\lambda + ih) = G_+^{-1}(\lambda)\left\{0,\cdots, \frac{P_{-\kappa_j-1}(\lambda)}{(\lambda + a)^{-\kappa_j}}, 0, \cdots, 0\right\}. \tag{4.10}$$

In the case $-\kappa_j \geqq m$ let us rewrite the latter equality as

$$\Phi_+^{(j)}(\lambda + ih) = G(\lambda + ih)\left\{0,\cdots, \frac{P_{-\kappa_j-1}(\lambda)}{\prod_{s=1}^r(\lambda + ih - \alpha_s)^{p_s}(\lambda + a)^{-\kappa_j-m}}, 0, \cdots, 0\right\}, \tag{4.11}$$

where the matrix function

$$G(\lambda) = G_+^{-1}(\lambda - ih)\,\frac{\prod_{s=1}^r(\lambda - \alpha_s)^{p_s}}{(\lambda - ih + a)^m},$$

which is holomorphic in the region Im $\lambda > h$, admits, by virtue of (4.3), continuation to a matrix function which is holomorphic in the region Im $\lambda > -h$. Hence it is not difficult to infer that it is representable in the form

$$G(\lambda) = I + \int_0^\infty \gamma(t)e^{i\lambda t}\,dt,$$

where $\gamma(t) \in e^{-ht}L_{n\times n}$.

By virtue of (4.11) the vector functions

$$\left.\begin{aligned} &\Phi_+^{(j\nu\mu)}(\lambda + ih) = G(\lambda + ih)\left\{0,\cdots,\frac{1}{(\lambda + ih - \alpha_\nu)^\mu},\ 0,\cdots,0\right\} \\ &\qquad (\nu = 1,2,\cdots,r;\ \mu = 1,2,\cdots,p_\nu), \\ &\Phi_+^{(j\nu)}(\lambda + ih) = G(\lambda + ih)\left\{0,\cdots,\frac{1}{(\lambda + a)^\nu},\ 0,\cdots,0\right\} \\ &\qquad (\nu = 1,2,\cdots,-\kappa_j - m) \end{aligned}\right\} \tag{4.12}$$

form a basis of the subspace of those solutions of problem (4.8) which correspond to the right index κ_j.

Let us define relative to the vector functions $\Phi_+^{(j\nu\mu)}(\lambda + ih)$, and with the help of the first equality of (4.9), vector functions $\varphi_{\nu\mu}^{(j)}(t)$ which are solutions of equation (4.1). Each coordinate of one of these vector functions has the form

$$f(t) = ct^{\mu-1}e^{-i\alpha_\nu t} + \int_0^t (t-s)^{\mu-1}e^{-i\alpha_\nu(t-s)}a(s)\,ds,$$

where c is a number and $a(t) \in e^{-ht}L$ is a function depending on j, ν, μ and the subscript of the coordinate. It is easy to see that

$$f(t) = e^{-i\alpha_\nu t}P(t) + \int_t^\infty (t-s)^{\mu-1}e^{-i\alpha_\nu(t-s)}a(s)\,ds,$$

where $P(t)$ is a polynomial of degree $\leqq \mu - 1$. By virtue of equality (3.7), the latter integral is $o(e^{-ht})$ as $t \to \infty$.

Thus the vector functions $\varphi_{\nu\mu}^{(j)}(t)$ admit asymptotic expansions (4.5).

Relative to the vector functions $\Phi_+^{(j\nu)}$ we find $-\kappa_j - m$ linearly independent solutions of equation (4.1). By developing an analogous argument it is not difficult to obtain the fact that these solutions belong to any space $e^{-ht}E_n$.

Turning to consideration of the case $-\kappa_j < m$, let us rewrite equality (4.10) in the form

$$\Phi_+^{(j)}(\lambda + ih) = G_j(\lambda + ih)\left\{0,\cdots,\frac{P_{-\kappa_j-1}(\lambda)}{\prod_{s=1}^{\tau_j}(\lambda + ih - \alpha_s)^{p_s}(\lambda + a)^{-\kappa_j - q_j}},0,\cdots,0\right\} \tag{4.13}$$

where the matrix function

$$G_j(\lambda) = G_+^{-1}(\lambda - ih)\,\frac{\prod_{s=1}^{\tau_j}(\lambda - \alpha_s)^{p_s}}{(\lambda - ih + a)^{q_j}},$$

which is holomorphic in the region $\operatorname{Im}\lambda > h$, admits, by virtue of (4.3), continuation to a matrix function which is holomorphic in the region $\operatorname{Im}(\lambda) > -h$, with the exception of the points α_s, $s = \tau_j + 1,\cdots r$, which are poles for it of order $\leqq p_s$. By detaching the sum of the principal parts of the matrix function $G_j(\lambda)$ relative to its poles α_s, $s = \tau_j + 1,\cdots,r$, it is easy to obtain the fact that for $\operatorname{Im}\lambda > \operatorname{Im}\alpha_{\tau_j+1}$ it is representable in the form

$$G_j(\lambda) = I + \int_0^\infty \gamma_j(t)e^{i\lambda t}dt \qquad (\operatorname{Im}\lambda > \operatorname{Im}\alpha_{\tau_j+1}),$$

where $\gamma_j(t) = \rho_j(t) + \omega_j(t)$, $\rho_j(t) \in e^{-ht}L_{n\times n}$, and $\omega_j(t)$ is a matrix function whose elements have the form

$$\sum_{s=\tau_j+1}^{r} e^{-i\alpha_s t} Q_s(t),$$

where the $Q_s(t)$ are polynomials of degrees $\leqq p_s - 1$.

By virtue of equality (4.13), the vector functions

$$\Phi_+^{(j\nu\mu)}(\lambda + ih) = G_j(\lambda + ih)\left\{0,\cdots, \frac{1}{(\lambda + ih - \alpha_\nu)^\mu}, 0,\cdots, 0\right\}$$
$$(\nu = 1,2,\cdots,\tau_j,\ \mu = 1,2,\cdots, p_\nu)$$

and

$$\Phi_+^{(j\nu)}(\lambda + ih) = G_j(\lambda + ih)\left\{0,\cdots, \frac{1}{(\lambda + a)^\nu}, 0,\cdots, 0\right\}$$
$$(\nu = 1,2,\cdots, -\kappa_j - q_j)$$

form a basis of the subspace of those solutions of boundary value problem (4.8) which correspond to the right index κ_j.

By making use of the estimates obtained in the preceding section it is not difficult to establish the fact that the solutions of equation (4.1) which are defined by means of the vector functions $\Phi_+^{(j\nu\mu)}(\lambda + ih)$ and $\Phi_+^{(j\nu)}(\lambda + ih)$ admit asymptotic expansions (4.6) and (4.7).

In order to complete the proof of the theorem, let us observe that the solutions of equation (4.1) which correspond to distinct right indices are linearly independent.

REMARKS ON THE LITERATURE

Chapter I

Basically the results of I. C. Gohberg [1, 2, 3] are presented in this chapter. The presentation is developed in the setting of his article [3]. The abstract results are cited in somewhat less space than in the article cited above; however they are presented in more detail and are supplemented with several new ones.

§ 1. Propositions $5°-8°$ were established jointly with F. V. Širokov.

§ 4. The results of this section are published here for the first time.

§ 5. Theorem 5.1 in its sufficiency part generalizes a well-known theorem of M. G. Kreĭn [4] on the factorization of functions expanding into absolutely convergent Fourier series. The proof carried out here is simpler than in the cited article.

P. Masani [1] was the first to prove a lemma analogous to Lemma 5.1, by a method different from the one advanced here (cf. M. S. Budjanu and I.C. Gohberg [1]). The latter proof was presented earlier by H. Cartan [1] in a specific case. There are propositions similar in content and proof to Lemma 5.1 in G. Baxter [1], G. F. Mandžavidze [1], G. F. Mandžavidze and B. V. Hvedelidze, [1], I. B. Simonenko [2], F. B. Atkinson [1], Gohberg and Kreĭn ([5], Chapter IV, § 4), and Budjanu and Gohberg [1, 3]. In the proof we basically follow the book of Gohberg and Kreĭn [5], where this lemma is put forth in a more general setting.

§ 7. The results of this section are generalizations of M. G. Kreĭn's results [4] established for functions expanding into absolutely convergent Fourier series.

The concept of the symbol of a singular integral operator was introduced for the first time by S. G. Mihlin [2]. Concerning the relationship between the symbol introduced in this section and the symbol of a singular integral operator, cf. the Remarks on § 6 of Chapter V.

§ 8. Theorems 8.1 and 8.3 were established by M. G. Kreĭn [4]. The formula for the resolvent $\gamma(t, a)$ at the end of this section is also due to Kreĭn [8]. Theorem 8.2 was obtained in a somewhat different form by I. S. Čebotaru [1].

Equations with vanishing symbol are not considered in this book. A whole series of articles is devoted to these questions (for example, F. D. Gahov and V. I. Smagina [1], V. B. Dybin [1], V. B. Dybin and N. K. Karapetjanc [1], Z. Presdorf [1, 2], M. I. Haĭkin [1, 2], and G. N. Čebotarev [1, 2]).

Chapter II

§ 1. Theorem 1.1 was communicated to the authors by A. S. Markus.

§ 2. Theorem 2.1 was established by N. I. Pol'skiĭ [1].

§ 3. Theorem 3.1 was proved by S. G. Mihlin [1] for the case when the perturbed operator is positive definite and the projections $P_\tau = Q_\tau$ are orthogonal.

L. E. Lerer [1] recently proved a theorem for the Hilbert space case which in a certain sense is a converse of Theorem 3.1.

§ 4. Theorem 4.1 is due to A. S. Markus [1].

§ 5. All the results of this section were established by A. S. Markus [1]. Theorem 5.1 represents a refinement of a theorem of G. M. Vaĭnikko [1], in which it is proved that the operator A has form (5.2) if $A \in \Pi\{P_n, P_n\}$, where P_n $(n = 1, 2, \cdots)$ is an arbitrary sequence of finite-dimensional orthoprojections converging strongly to the identity operator. Vaĭnikko's method is utilized in the proof of Theorem 5.1.

§ 6. Theorem 6.2 was established by A. S. Markus [1]. Its proof is based on Theorem 6.1, which was obtained by A. Brown and C. Pearcy [1] in connection with a problem on commutators.

Chapter III

§ 1. The results of this section were obtained by the authors [1, 4].

Condition 2) in Theorem 1.1 is essential. This can be ascertained from the following example. Let an isometric operator V, a left inverse of it $V^{(-1)}$, and the projections P_n $(n = 1, 2, \cdots)$ be defined in l_2 by the equalities

$$V\{\xi_j\}_1^\infty = \{0, \xi_1, 0, \xi_2, \cdots\}, \qquad V^{(-1)}\{\xi_j\}_1^\infty = \{\xi_2, \xi_4, \xi_6, \cdots\},$$
$$P_n\{\xi_j\}_1^\infty = \{\xi_1, \cdots, \xi_n, 0, 0, \cdots\}.$$

Let us denote by A an operator in $\mathfrak{R}(V)$, by $A(\zeta)$ $(|\zeta| = 1)$ its symbol, and by a_j $(j = 0, \pm 1, \cdots)$ the Fourier coefficients of $A(\zeta)$. The following proposition holds.

For the projection method (P_n, P_n) to be applicable to an operator A of $\mathfrak{R}(V)$, it is necessary and sufficient that $A(\zeta) \neq 0$ $(|\zeta| = 1)$, that $\operatorname{ind} A(\zeta) = 0$, and that all the matrices $\|a_{j-k}\|_{j,k=1}^n$ $(n = 1, 2, \cdots)$ be invertible.

This result was established by A. S. Markus and I. A. Fel'dman.

§ 2. The second assertion of Theorem 2.1 was established for the first time by G. Baxter [2] in the space l_1 under the condition that the function $a(\zeta)$ expand into an absolutely convergent Fourier series. His proof, which is different from ours, is cited in this section.

Theorem 2.2 was proved by G. Szegö for the case when the function $a(\zeta)$ is nonnegative and the functions $a(\zeta)$ and $\log a(\zeta)$ are absolutely integrable. It was proved by G. Baxter [2] on the assumption that the functions $a(\zeta)$ and $\log a(\zeta)$ belong to a Wiener algebra with weight.

Theorems concerning projection methods for solving Wiener-Hopf equations with vanishing symbol have been proved by Gohberg and Levčenko [1, 2, 3].

§ 3. The second assertion of Theorem 3.1 was proved in the space L_1 by I. S. Čebotaru [1].

The second assertion of Theorem 3.2, as well as the remark concerning the Galerkin method for the nonzero index case, was established by V. V. Ivanov and E. A. Karagodova [1] (cf. also V. V. Ivanov [2]).

§ 4. Further development of the theory of multivariate Wiener-Hopf equations is contained in I. B. Simonenko's articles [4, 5].

§ 5. The method cited for calculating the index is due to V. L. Zaguskin and A. V. Haritonov [1].

The Schur-Kohn direct method of determining the index of a polynomial relative to the circle should also be noted. This method is presented in N. N. Meĭman's article [1] (cf. also M. G. Kreĭn and M. A. Naĭmark [1]).

§ 6. Theorem 6.1, in a different form, under the additional assumption that the polynomials $x(\zeta) = \sum_0^n x_j\zeta^j$ and $y(\zeta) = \sum_0^n y_{-j}\zeta^j$ do not vanish on the unit disk $|\zeta| \leqq 1$, was established by G. Baxter and I. Hirschman [1] (cf. also M. Shinbrot's article [1]). A. A. Semencul [1, 2] established formula (6.3) under the same assumption. A clearer proof of Baxter and Hirschman's theorem also belongs to this author. This proof, which makes use of a transition to infinite-dimensional discrete Wiener-Hopf equations, was cited in the Russian edition of the book.

The results of this section were established in Gohberg and Semencul's article [2].

The determinant of the matrix $\mathscr{P}$ in the formulation of Theorem 6.3 is the *resultant* of the polynomials $y(\zeta) = y_0\zeta^n + y_{-1}\zeta^{n-1} + \cdots + y_{-n}$ and $x(\zeta) = x_n\zeta^n + x_{n-1}\zeta^{n-1} + \cdots + x_0$; thus the condition of the nonsingularity of the matrix $\mathscr{P}$ can be replaced by the following: the polynomials $x(\zeta)$ and $y(\zeta)$ do not have common roots.

For a hermitian Toeplitz matrix $\mathscr{A}_n$ the latter condition is equivalent to the fact that the polynomial $y(\zeta)$ does not have roots on the unit circle and that among its roots there is no pair of roots specularly situated relative to that circle. M. G. Kreĭn [5] proves Theorem 6.3, so formulated, by a different method in the case when a Toeplitz matrix $\mathscr{A}_n$ is hermitian.

§ 7. The results of this section were established by I. C. Gohberg and N. Ja. Krupnik [8].

Comparing the results of §§ 6 and 7, we naturally arrive at the following question. Do there exist for every nonsingular Toeplitz matrix $\mathscr{A}_n = \|a_{j-k}\|_{j,k=0}^n$ numbers ν and μ $(0 \leqq \nu < \mu \leqq n)$ such that the matrix $\mathscr{A}_n^{-1}$ is uniquely determined by the solutions $s_0,\cdots, s_n$ and $t_0,\cdots, t_n$ of the equations

$$\sum_{k=0}^{n} a_{j-k}\, x_k = \delta_{j\nu}, \qquad \sum_{k=0}^{n} a_{j-k}\, z_k = \delta_{j\mu}, \qquad j = 0, 1, \cdots, n?$$

An affirmative answer to this question would mean that the numbers a_j $(j = 0, \pm 1, \cdots, \pm n)$ are uniquely determined by the equations

$$\sum_{k=0}^{n} a_{j-k} s_k = \delta_{j\nu}, \qquad \sum_{k=0}^{n} a_{j-k} t_k = \delta_{j\mu}, \qquad j = 0, 1, \cdots, n. \tag{$*$}$$

From the example of the matrix

$$\mathscr{A}_3 = \begin{Vmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{Vmatrix}$$

it can be ascertained that the latter does not hold; i. e., whatever the numbers ν and μ $(0 \leqq \nu \leqq \mu \leqq 3)$, the numbers a_j $(j = 0, \pm 1, \pm 2, \pm 3)$ are not determined uniquely by the equations $(*)$.

§ 8. It is possible to derive Theorem 8.1 in a different form, under some additional assumptions, from the abstract results of I. Hirschman **[1, 2]**. A. A. Semencul **[1]** established formula (8.4) under these additional assumptions. His proof of this theorem, which is based on the transition to Wiener-Hopf equations, was cited in the Russian edition of the book. The results of § 8 were established by Gohberg and Semencul **[1]**. Theorem 8.2, in a different form for hermitian kernels, was established by M. G. Kreĭn.

The results of §§ 6 and 8 have some points of contiguity with the fundamental research of M. G. Kreĭn **[1 – 3]** on the theory of inverse problems of the spectral analysis of differential equations (cf. also Gohberg and Kreĭn **[5]**). Unfortunately, this connection has remained beyond the limits of the book.

Questions on approximating the spectrum for the operators considered here are studied in L. E. Lerer's paper **[2]**.

Chapter IV

The basic results of this chapter with their detailed proofs are being published here for the first time.

§ 1. Lemma 1.2 is a special case of one of I. B. Simonenko's propositions **[2]**.

§ 2. The results of this section are formulated in I. C. Gohberg's article **[4]**. Part of them were obtained earlier by H. Widom **[1]**. The generalization of the results of this section for h_p spaces (consisting of the sequences of the Fourier coefficients of the functions in H_p) was obtained by Gohberg and Krupnik **[7]**. Recently P. V. Dudučava **[1]** generalized the results of this section to operators defined in l_p $(1 < p < \infty;\ p \neq 2)$ by Toeplitz matrices $\|a_{j-k}\|_{j,k=0}^{\infty}$ consisting of the Fourier coefficients of sectionally Wiener functions.

§ 4. The results of this section were established by Gohberg **[4]**. They have not yet been generalized to the spaces h_p and l_p.

§ 5. Theorem 5.1 is a special case of a theorem of I.B. Simonenko **[2]**. Theorems 5.2 and 5.3 were established by Gohberg **[4]**.

The class of functions M_R was introduced in Simonenko's article [2]. Another equivalent definition of this class is given there. Let us cite it.

A measurable function $f(\zeta)$ belongs to the class M_R if it satisfies the following two conditions:

a) $\operatorname{ess\,inf}_{|\zeta|=1} |f(\zeta)| > 0$ and $\operatorname{ess\,sup}_{|\zeta|=1} |f(\zeta)| < \infty$.

b) A covering of the circle $|\zeta| = 1$ by open arcs exists such that the values of the function $f(\zeta)$ on each of them lie inside some angle less than π, with its vertex at the point $\zeta = 0$.

Chapter V

The presentation in this chapter is conducted in the setting of I. C. Gohberg's article [3]. The general theorems for several classes of pair operators are first established in the abstract case, and then the basic statements concerning various concrete pair systems and their transposes are derived from them.

In contrast to Gohberg's article, here the simple relationship between pair equations and corresponding equations of the first chapter is exploited. This relationship noticeably simplifies the proofs of the necessity of the conditions in the basic theorems.

§ 1. The results of this section are being published here for the first time.

§ 2. Theorem 2.1 was established by Gohberg [3].

§ 3. Theorem 3.1 was obtained by Gohberg [3]. It is a generalization of earlier results of Gohberg and Kreĭn [3].

§ 4. Theorem 4.1 was established by Gohberg and Kreĭn [3].

§ 5. Theorem 5.2 was obtained by Gohberg and Krupnik [1]. There is a generalization of this theorem in articles [2, 3] of the same authors.

§ 6. Well-known propositions concerning singular integral equations (cf. N. I. Mushelišvili [1] and S. G. Mihlin [2]) can be obtained from the results of § 2. Theorem 6.1 was established by I. C. Gohberg [3], I. B. Simonenko [1], and B. V. Hvedelidze [1, 2]. Theorem 6.2 was established by Gohberg and Krupnik [1].

The definition of symbol cited here is due to S. G. Mihlin [2]. Natural considerations lead to the fact that the function $c(\zeta)$ should be regarded as the symbol of the operator $Pc(\zeta)P\,|H_p$. This agrees with the definition of the symbol of a Wiener-Hopf operator.

Chapter VI

The basic results of this chapter were established by the authors [1, 3].

§ 2. The last assertions in Theorems 2.1 and 2.4, as well as Theorems 2.2 and 2.3, were established by I. C. Gohberg and V. G. Čeban [1] under certain additional restrictions. Let us note that the conditions of applicability of the projection method analyzed in the case when the functions $a(\zeta)$ and $b(\zeta)$ are piecewise continuous still have not been determined.

§ 4. The sufficiency part of Theorem 4.1 under certain additional restrictions was established for the first time by V. V. Ivanov [**1, 2**].

Some of the results of this chapter have been generalized to the case of singular integral equations along an arbitrary closed contour (cf. Gohberg and Špigel′ [**1, 2**] and Špigel′ [**1, 2**]).

Chapter VII

The authors' results [**5, 6**] are presented in this chapter.

Part of the results of this chapter were obtained by R. G. Douglas and L. A. Coburn simultaneously with the authors. Further development of this chapter's results can be found in Semencul [**3**] and Gohberg and Semencul [**1**].

§ 6. The question concerning the conditions of the applicability of the projection method to pair integral difference equations and their transposes remains open.

Chapter VIII

§ 1. The results of this section generalize certain results of Gohberg and Kreĭn [**2**].

§ 2. Theorem 2.1 was established by Gohberg [**2, 3**]. (A generalization of it can be found in Budjanu and Gohberg [**3, 4**].) This theorem is a generalization of Theorems 2.2 and 2.4 established earlier by Gohberg and Kreĭn [**2**].

§ 3. Theorem 3.1 was established in another form by B. V. Hvedelidze and G. F. Mandžavidze [**1**], and by I. B. Simonenko [**1, 3**].

§ 4. Theorems 4.1 and 4.2 were established by Gohberg, and Theorem 4.3, by the authors [**2, 4**]. Lemma 4.1 is due to N. Ja. Krupnik [**1**].

§ 5. Theorem 5.1 was established by Gohberg [**3**]. It is a generalization of a result of Gohberg and Kreĭn [**2**]. Theorems 5.2, 5.3 and 5.4 were obtained by the authors [**2, 3, 4**]. The second assertion of Theorem 5.2 was proved independently of the authors for the case of the algebra $W_{n\times n}$ by I. I. Hirschman [**3**] (cf. also N. Bowers [**1**]). The generalization to the matrix case of the results of §§ 6 – 8 of Chapter III is contained in the same article [**3**] of I. I. Hirschman. There is a generalization of Theorem 5.1 to the case of a piecewise continuous matrix function in Gohberg and Krupnik's article [**4**]. The remaining theorems of this section have not yet found generalizations to the latter case.

A certain modification of the projection method for systems of Wiener-Hopf equations with nonzero partial indices is considered in I. S. Čebotaru's articles [**2, 3**]. The generalization of Theorems 5.1 and 5.3 to the case of systems of discrete Wiener-Hopf equations with operator coefficients was obtained by M. S. Budjanu [**1**].

§ 6. Theorem 6.1 was established by Gohberg and Kreĭn [**2**]. We note that this theorem has not yet found a generalization to the case of systems of integral-difference equations, i.e. equations of the form (1), Chapter VII. Theorems 6.2 and 6.3 were obtained by the authors **[2, 4]**.

I. A. Fel'dman **[1]** has generalized the results of this section to the case of Wiener-Hopf equations with operator kernels. He also considered applications to the radiative transfer equation.

§ 7. The results of this section are being published here for the first time.

§ 9. Results of the authors **[2, 4]** are presented in this section. All of the theorems in this section (except Theorem 9.3) are of a sufficiency character. The question of necessity of the conditions of these theorems remains open.

§ 10. The results of this section were established by Gohberg and Krupnik **[6]**.

Appendix

The results of the Appendix are due to I. A. Fel'dman **[1–3]**.

1. The complete continuity of an operator with kernel depending on the sum of the arguments was established by Gohberg and Kreĭn **[2]**.

2. The equivalence of equation (2.1) and boundary value problem (2.2) is established here analogously to the way it was done by F. D. Gahov and Ju. I. Čerskiĭ **[1]** in the one-dimensional case.

The asymptotic expansions obtained in §§ 3 and 4 are generalizations of M. G. Kreĭn's results **[4]** concerning the asymptotics of the solutions of a homogeneous Wiener-Hopf equation.

The theorems of this Appendix have been generalized by Fel'dman **[4, 5, 6]** to Wiener-Hopf integral equations with operator-valued kernels. Applications of these results to the radiative transfer equation are also obtained in those papers.

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